Solution of the Volterra Integral Equation in Orthogonal Partial Ordered Metric Spaces

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Abstract—**In this paper, we introduce a new concept of an orthogonal partial order metric space. We also establish some new fixed point theorems for an orthogonal rational type contraction mapping in orthogonal partial ordered metric space. Moreover, some examples and applications are provided to exhibit the utility of these obtained results.**

Index Terms—**Partially ordered metric space, orthogonal rational contractions, orthogonal Singh and Chatterjee contraction, fixed point.**

I. Introduction

THERE are several uses for Banach's contraction principle in fixed point and approximation theory. It has a significant impact on a wide range of mathematical issues, both theoretical and applied. Technical expansions and generalizations to the Banach contraction principle can be found in [[1], [2], [3], [4]], among other places. Nearly every discipline of practical mathematics uses iteration algorithms to prove convergence and estimate error processes, often by applying Banach's fixed point theorem. Using fixed points of mappings in ordered metric spaces to solve nonlinear equations has recently become quite popular in several areas of mathematical analysis. Wolk [5] and Monjardet [6] were the first to produce results in this direction in partially ordered sets.

Fixed points for specific mappings in partially ordered metric spaces were examined by Ran and Reurings [7], and their results were then applied to matrix equations as a result of their work. After that Nieto *et al.* [8] expand the non-descending map results for periodic boundary conditions (see [9], [10]). In 2008, Agarwal *et al.* [11] discussed the generalized contractions in partially ordered metric spaces results. Some of these generalization of fixed point and common fixed point results improvement for single and multi-valued operators in a variety of ordered spaces can be found in ([12], [13], [14], [15], [16] [17]). Paiwan [20] initiated SP-type extra-gradient iterative methods for finding fixed point problem. In 2021, Phannipa Worapun and Atid Kangtunyakarn [21] finding fixed point by using an approximation method. The solution of non-linear equations in a higher order iterative scheme method was introduced by Waikhom *et al.* [22]. In 2022, Karim Ivaz *et al.* [23] proved the Hilfer fractional Volterra-Fredholm Integro differential equation.

Gordji *et al.* [24] initiated an orthogonality notion in metric spaces. The fixed point results in generalized orthogonal metric space and various metric spaces were proved by many researchers (see [25], [26], [27], [28], [29], [30], [31] , [32], [33]).

In this paper, we prove some fixed point results of a mapping satisfying nonlinear orthogonal rational type contraction conditions in the context of a orthogonal complete partially ordered metric spaces. However, our results are suitably validated by constructive examples. Moreover, we investigate for existence and uniqueness of solution for a Volterra integral type equation.

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II. Preliminaries

Following are some definitions that appeared frequently throughout our results; we begin this section with them.

In 2004, partial ordered set concept was introduced by Ran and Reurings [7] as follows:

Definition 1. *[7] A $(\mathbf{K}, \partial, \preceq)$ is said to be partially ordered metric spaces, if $(\mathbf{K}, \preceq)$ is a partially ordered set in addition to $(\mathbf{K}, \partial)$ is a metric space.*

In 2004, Ran and Reurings [7] developed the concept of a complete metric space as follows:

Definition 2. *[7] If $(\mathbf{K}, \partial)$ is a complete metric space, then triplet $(\mathbf{K}, \partial, \preceq)$ is said to be complete partially ordered metric spaces.*

Arshad *et al.* [17] introduced the concept of an ordered complete as follows:

Definition 3. *[17] Let $(\mathbf{K}, \partial, \preceq)$ be a partially ordered metric spaces is said to be ordered complete, if for every convergent sequence $\{\nu_{\mathfrak{a}}\}_0^{\infty} \subset \mathbf{K}$, the following circumstance exists:*

1) *if $\{\nu_{\mathfrak{a}}\} \in \mathbf{K}$ be a non-descending sequence such that $\nu_{\mathfrak{a}} \to \nu \implies \nu_{\mathfrak{a}} \preceq \nu$, for all $\nu \in \mathbb{N}$ that is,*
   $\nu = \sup\{\nu_{\mathfrak{a}}\}$,
   (or)
2) *if $\{\nu_{\mathfrak{a}}\}$ is a non-increasing sequence in $\mathbf{K}$ such that $\nu_{\mathfrak{a}} \to \nu \implies \nu_{\mathfrak{a}} \preceq \nu$, for all $\nu \in \mathbb{N}$ that is,*
   $\nu = \inf\{\nu_{\mathfrak{a}}\}$.

In 1975, Wolk [5] develop the concept of converges as follows:

Definition 4. *[5] Let $(\mathbf{K}, \partial, \preceq)$ be a partially ordered metric spaces. And $\{\nu_{\mathfrak{a}}\}$ be any sequence in $\mathbf{K}$ is said to be convergent to a point $\nu \in \mathbf{K}$ if, for every $\epsilon > 0$ there* *exists* $\mathfrak{a}_0 \in \mathbb{N}$ *such that* $\partial(\nu_{\mathfrak{a}}, \nu) < \epsilon$ *for all* $\mathfrak{a} > \mathfrak{a}_0$. *The convergence is also represented as*

$$\lim_{\mathfrak{a}\to\infty} \nu_{\mathfrak{a}} = \nu \text{ or } \nu_{\mathfrak{a}} \to \nu \text{ as }, \mathfrak{a} \to \infty.$$

Gordji [24] proposed an orthogonal sets and generalized Banach fixed point theorems in 2017.

Definition 5. *[24] Let* $\mathbf{K}$ *be a non-void and* $\dashv \subseteq \mathbf{K}\times\mathbf{K}$ *be a binary relation. If* $\dashv$ *fulfill the below axiom:*

$$\exists \ \nu_0 \in \mathbf{K} : (\forall \ \nu \in \mathbf{K}, \nu \dashv \nu_0) \ \text{ or } \ (\forall \ \nu \in \mathbf{K}, \nu_0 \dashv \nu),$$

then $(\mathbf{K}, \dashv)$ *be an orthogonal set* $(\mathbf{O}_s)$.

The following orthogonal sequence definition was introduced by Gordji *et al.* [24] which will be utilized in this paper.

Definition 6. *[24] Let* $(\mathbf{K}, \dashv)$ *be an* $\mathbf{O}_s$. *A sequence* $\{\nu_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathbb{N}}$ *is called an orthogonal sequence* $(\mathbf{O}_{seq})$ *if*

$$(\forall \ \mathfrak{a} \in \mathbf{K}, \ \nu_{\mathfrak{a}} \dashv \nu_{\mathfrak{a}+1}) \text{ or } (\forall \ \mathfrak{a} \in \mathbf{K}, \ \nu_{\mathfrak{a}+1} \dashv \nu_{\mathfrak{a}}).$$

Again, the concepts of orthogonal continuous also introduced by Gordji *et al.* [24].

Definition 7. *[24] Let* $(\mathbf{K}, \partial_\dashv)$ *be an orthogonal metric space. Then a map* $\mathtt{Z}\colon \mathbf{K} \to \mathbf{K}$ *is called orthogonally continuous* $(\mathbf{O}_{con})$ *in* $\nu \in \mathbf{K}$ *if for each* $\mathbf{O}_{seq}$ $\{\nu_{\mathfrak{a}}\} \in \mathbf{K}$ *with* $\partial_\dashv(\nu_{\mathfrak{a}}, \nu) \to 0$, *we get* $\partial_\dashv(\mathtt{Z}\nu_{\mathfrak{a}}, \mathtt{Z}\nu) \to 0$ *as* $\mathfrak{a} \to \infty$.

Gordji *et al.* [24] introduced the concept of an orthogonal complete as follows:

Definition 8. *[24] Let* $(\mathbf{K}, \dashv, \partial_\dashv)$ *be an orthogonal set with the metric* ∂. *Then* $\mathbf{K}$ *says that an orthogonal complete if for each orthogonal Cauchy sequence is convergent.*

Definition 9. *[24] Let* $(\mathbf{K}, \dashv, \partial_\dashv)$ *be an orthogonal metric space and* $0 < \lambda < 1$. *A map* $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ *is said to be an orthogonal contraction* $(\mathbf{O}_{contr})$ *with Lipschitz constant* λ *if,* $\forall \ \nu, \mu \in \mathbf{K}$ *with* $\nu \dashv \mu$,

$$\partial(\mathtt{Z}\nu, \mathtt{Z}\mu) \le \lambda \partial(\nu, \mu).$$

Finally, the following orthogonal preserving concepts introduced by Gordji *et al.* [24] is of importance in this paper.

Definition 10. *[24] Let* $(\mathbf{K}, \dashv, \partial_\dashv)$ *be an orthogonal metric space. A map* $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ *is said to be orthogonal-preserving* $(\mathbf{O}_p)$ *if* $\mathtt{Z}\nu \dashv \mathtt{Z}\mu$, *whenever* $\nu \dashv \mu$.

In our main result, inspired by the notions of an rational type contraction mapping, Singh and Chatterjee contraction mapping defined Rao [19], we introduce a new orthogonal rational type contraction mapping, new orthogonal Singh and Chatterjee contraction and prove some fixed point theorems for these contraction mappings in orthogonal complete partially ordered metric spaces.

III. Main Results

Now, we generalize and improve our fixed point theorems from Rao [19] by introducing the concept of an orthogonal rational type contraction mapping in orthogonal complete partially ordered metric spaces.

Theorem 1. *Let* $(\mathbf{K}, \dashv, \partial_\dashv)$ *be an orthogonal complete partially ordered metric spaces. A function* $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ *be an* $\mathbf{O}_p$ *and* $\mathbf{O}_{con}$ *so that,* $\forall \ \nu, \mu \in \mathbf{K}$ *with* $\nu \dashv \mu$,

$$\partial_\dashv(\mathtt{Z}\nu, \mathtt{Z}\mu) \le \begin{cases} \Re\partial_\dashv(\nu, \mu) + \Im[\partial_\dashv(\nu, \mathtt{Z}\nu) \\ \quad +\partial_\dashv(\nu, \mathtt{Z}\mu)] \\ \quad +\wp\frac{\partial_\dashv(\nu,\mathtt{Z}\nu)\partial_\dashv(\nu,\mathtt{Z}\mu)+\partial_\dashv(\mu,\mathtt{Z}\nu)\partial_\dashv(\mu,\mathtt{Z}\mu)}{\partial_\dashv(\mu,\mathtt{Z}\nu)+\partial_\dashv(\nu,\mathtt{Z}\mu)} & \text{if } \mathtt{A} \neq 0 \\ 0, & \text{if } \mathtt{A} = 0, \end{cases} \tag{1}$$

where

$$\mathtt{A} = \partial_\dashv(\mu, \mathtt{Z}\nu) + \partial_\dashv(\nu, \mathtt{Z}\mu),$$

and there exists $\Re, \Im, \wp \in [0, 1)$ *such that* $0 \le \Re + 2\Im + \wp < 1$.

If $\exists \ \nu_0 \in \mathbf{K}$ *such that* $\nu_0 \preceq \mathtt{Z}\nu_0$, *then* $\mathtt{Z}$ *has a unique fixed point in* $\mathbf{K}$.

Proof: Since $(\mathbf{K}, \dashv)$ is an $\mathbf{O}_s$, there exists

$$\nu_0 \in \mathbf{K} : (\forall \ \nu \in \mathbf{K}, \ \nu \dashv \nu_0) \text{ (or) } (\forall \ \nu \in \mathbf{K}, \ \nu_0 \dashv \nu).$$

It follows that

$$\nu_0 \dashv \mathtt{Z}\nu_0 \text{ (or) } \mathtt{Z}\nu_0 \dashv \nu_0.$$

Let $\nu_{\mathfrak{a}} = \mathtt{Z}\nu_{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathbb{N} \cup \{0\}$. If $\nu_{\mathfrak{a}} = \nu_{\mathfrak{a}+1}$ for any $\mathfrak{a} \in \mathbb{N} \cup \{0\}$, then it is clear that $\nu_{\mathfrak{a}}$ is a fixed point of $\mathtt{Z}$.

Assume that

$$\nu_{\mathfrak{a}+1} \neq \nu_{\mathfrak{a}} \quad \forall \ \mathfrak{a} \in \mathbb{N} \cup \{0\}.$$

Since $\mathtt{Z}$ is $\mathbf{O}_p$, we have

$$\nu_{\mathfrak{a}+1} \dashv \nu_{\mathfrak{a}} \text{ (or) } \nu_{\mathfrak{a}} = \nu_{\mathfrak{a}+1} \ \ \forall \ \ \mathfrak{a} \in \mathbb{N} \cup \{0\}.$$

This implies that $\{\nu_{\mathfrak{a}}\}$ is an $\mathbf{O}_{seq}$.

As $\nu_{\mathfrak{a}}$ and $\nu_{\mathfrak{a}+1}$ are comparable for $\mathfrak{a} \ge 1$, we get the below cases:

Case 1 : If

$$\mathtt{A} = \partial_\dashv(\nu_{\mathfrak{a}-1}, \mathtt{Z}\nu_{\mathfrak{a}}) + \partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}-1}) \neq 0,$$

then from contraction condition (1), we have

$$\begin{aligned} \partial_\dashv(\nu_{\mathfrak{a}+1}, \nu_{\mathfrak{a}}) &= \partial_\dashv(\mathtt{Z}\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}-1}) \\ &\le \Re\partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}-1}) \\ &\quad + \Im[\partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}}) + \partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}-1})] \\ &\quad + \wp\left[\frac{\partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}})\partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}-1})}{\partial_\dashv(\nu_{\mathfrak{a}-1}, \mathtt{Z}\nu_{\mathfrak{a}}) + \partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}-1})}\right. \\ &\quad \left. + \frac{\partial_\dashv(\nu_{\mathfrak{a}-1}, \mathtt{Z}\nu_{\mathfrak{a}})\partial_\dashv(\nu_{\mathfrak{a}-1}, \mathtt{Z}\nu_{\mathfrak{a}-1})}{\partial_\dashv(\nu_{\mathfrak{a}-1}, \mathtt{Z}\nu_{\mathfrak{a}}) + \partial_\dashv(\nu_{\mathfrak{a}}, \mathtt{Z}\nu_{\mathfrak{a}-1})}\right], \end{aligned}$$

which implies

$$\begin{aligned} \partial_\dashv(\nu_{\mathfrak{a}+1}, \nu_{\mathfrak{a}}) &\le \Re\partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}-1}) + \Im\partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}+1}) \\ &\quad + \wp\left[\frac{\partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}+1})\partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}})}{\partial_\dashv(\nu_{\mathfrak{a}-1}, \nu_{\mathfrak{a}+1}) + \partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}})}\right. \\ &\quad \left. + \frac{\partial_\dashv(\nu_{\mathfrak{a}-1}, \nu_{\mathfrak{a}+1})\partial_\dashv(\nu_{\mathfrak{a}-1}, \nu_{\mathfrak{a}})}{\partial_\dashv(\nu_{\mathfrak{a}-1}, \nu_{\mathfrak{a}+1}) + \partial_\dashv(\nu_{\mathfrak{a}}, \nu_{\mathfrak{a}})}\right]. \end{aligned}$$

Finally, we get

$$\partial_{\dashv}(\nu_{\mathfrak{a}+1},\nu_{\mathfrak{a}}) \leq \Big(\frac{\Re+\wp}{1-\Im}\Big)^{\mathfrak{a}}\partial_{\dashv}(\nu_1,\nu_0).$$

Put $\mathtt{D}=\Big(\frac{\Re+\wp}{1-\Im}\Big)\in[0,1)$. Moreover, by triangular inequality for $\mathfrak{n}\geq\mathfrak{a}$, we have

$$\begin{aligned}\partial_{\dashv}(\nu_{\mathfrak{n}},\nu_{\mathfrak{a}}) &\leq \partial_{\dashv}(\nu_{\mathfrak{n}},\nu_{\mathfrak{n}-1})+\partial_{\dashv}(\nu_{\mathfrak{n}-1},\nu_{\mathfrak{n}-2})\\ &\quad+....+\partial_{\dashv}(\nu_{\mathfrak{a}+1},\nu_{\mathfrak{a}})\\ &\leq \frac{\mathtt{D}^{\mathfrak{a}}}{1-\mathtt{D}}\partial_{\dashv}(\nu_1,\nu_0),\end{aligned}$$

as $\mathfrak{n},\mathfrak{a}\to+\infty$, $\partial_{\dashv}(\nu_{\mathfrak{n}},\nu_{\mathfrak{a}})\to 0$. So, $\{\nu_{\mathfrak{a}}\}$ is a Cauchy $\mathbf{O}_{seq}$ in orthogonal complete metric space $\mathbf{K}$. Hence there exists $\mathfrak{v}\in\mathbf{K}$ such that

$$\lim_{\mathfrak{a}\to+\infty}\nu_{\mathfrak{a}}=\mathfrak{v}.$$

Further, the $\mathbf{O}_{con}$ of $\mathtt{Z}$ implies that

$$\begin{aligned}\mathtt{Z}\mathfrak{v} &= \mathtt{Z}\Big(\lim_{\mathfrak{a}\to+\infty}\nu_{\mathfrak{a}}\Big)\\ &= \lim_{\mathfrak{a}\to+\infty}\mathtt{Z}\nu_{\mathfrak{a}}\\ &= \lim_{\mathfrak{a}\to+\infty}\nu_{\mathfrak{a}+1}\\ &= \mathfrak{v},\end{aligned}$$

hence $\mathfrak{v}$ is a fixed point of $\mathtt{Z}\in\mathtt{F}$.

Case 2 : If

$$\mathtt{A}=\partial_{\dashv}(\nu_{\mathfrak{a}-1},\mathtt{Z}\nu_{\mathfrak{a}})+\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}\nu_{\mathfrak{a}-1})=0,$$

then

$$\partial_{\dashv}(\nu_{\mathfrak{a}+1},\nu_{\mathfrak{a}})=0,$$

from (1) $\implies \nu_{\mathfrak{a}}=\nu_{\mathfrak{a}+1}$, a contradiction as the $\mathbf{O}_{seq}$ points are comparable.

Thus $\exists\ \mathfrak{v}$ of $\mathtt{Z}$. Next, we demonstrate its uniqueness. Let $\nu_2\in\mathbf{K}$ be a fixed point of $\mathtt{Z}$. So, we have $\mathtt{Z}^{\mathfrak{a}}\nu^*=\nu^*$ and

$$\mathtt{Z}^{\mathfrak{a}}\nu_2^*=\nu_2^*,\ \forall\ \mathfrak{a}\in\mathbb{N}.$$

By an orthogonality definition, there is $\nu_1\in\mathbf{K}$ so that

$$\begin{aligned}&[\nu_1\dashv\nu^* \text{ and } \nu_1\dashv\nu_2^*]\\ \textit{or}\ \ &[\nu^*\dashv\nu_1 \text{ and } \nu_2^*\dashv\nu_1].\end{aligned}$$

Since $\mathtt{Z}$ is $\mathbf{O}_p$, one can write

$$\begin{aligned}&[\mathtt{Z}^{\mathfrak{a}}\nu_1\dashv\mathtt{Z}^{\mathfrak{a}}\nu^* \text{ and } \mathtt{Z}^{\mathfrak{a}}\nu_1\dashv\mathtt{Z}^{\mathfrak{a}}\nu_2^*]\\ \textit{or}\ \ &[\mathtt{Z}^{\mathfrak{a}}\nu^*\dashv\mathtt{Z}^{\mathfrak{a}}\nu_1 \text{ and } \mathtt{Z}^{\mathfrak{a}}\nu_2^*\dashv\mathtt{Z}^{\mathfrak{a}}\nu_1],\ \forall\ \ \mathfrak{a}\in\mathbb{N}.\end{aligned}$$

Therefore, by Definition 1, we have

$$\begin{aligned}\partial_{\dashv}(\nu_1^*,\nu_2^*) &= \partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\nu_1^*,\mathtt{Z}^{\mathfrak{a}}\nu_2^*)\\ &= \partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\nu_1^*,\mathtt{Z}^{\mathfrak{a}}\nu_1)+\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\nu_1,\mathtt{Z}^{\mathfrak{a}}\nu_2^*)\\ &\leq \partial_{\dashv}(\nu_1^*,\nu_1)+\partial_{\dashv}(\nu_1,\nu_2^*).\end{aligned}$$

Taking limit as $\mathfrak{a}\to\infty$, we get

$$\partial_{\dashv}(\nu_1^*,\nu_2^*)=0,$$

and so $\nu_1^*=\nu_2^*$. ■

Relaxing $\mathtt{Z}$ in Theorem 1 continuity condition yields the following theorem.

Theorem 2. *Let $(\mathbf{K},\dashv,\partial_{\dashv})$ be an orthogonal complete partially ordered metric spaces. Assume that $\mathbf{K}$ satisfies if a increasing $\mathbf{O}_{seq}$*

$$\{\nu_{\mathfrak{a}}\}\to\nu\in\mathbf{K},\ \textit{then}\ \nu=\sup\{\nu_{\mathfrak{a}}\}. \tag{2}$$

Let $\mathtt{Z}:\mathbf{K}\to\mathbf{K}$ be an $\mathbf{O}_p$ and $\mathbf{O}_{con}$ monotone increasing function holds the contraction condition 1. If $\exists\ \ \nu_0\in\mathbf{K}$ with $\nu_0\leq\mathtt{Z}\nu_0$, then $\mathtt{Z}$ has a unique fixed point.

Proof: We check the following condition only $\mathfrak{v}=\mathtt{Z}\mathfrak{v}$. Using Theorem 1, we get a increasing $\mathbf{O}_{seq}$, $\{\nu_{\mathfrak{a}}\}\in\mathbf{K}$ such that $\nu_{\mathfrak{a}}\to\mathfrak{v}\in\mathbf{K}$. Thus,

$$\mathfrak{v}=sup\{\nu_{\mathfrak{a}}\},\ \ \forall\ \ \mathfrak{a}\geq 0,$$

by (2). Since $\mathtt{Z}$ is a increasing mapping, then $\mathtt{Z}\nu_{\mathfrak{a}}\preceq\mathtt{Z}\mathfrak{v}$, $\forall\ \mathfrak{a}\in\mathbb{N}$, equivalently, $\nu_{\mathfrak{a}+1}\preceq\mathtt{Z}\mathfrak{v}$, $\forall\ \mathfrak{a}\in\mathbb{N}$. Further,

$$\nu_0<\nu_1\leq\mathtt{Z}\mathfrak{v}\ \ \textit{and}\ \ \mathfrak{v}=\sup\{\nu_{\mathfrak{a}}\},$$

we get $\mathfrak{v}\leq\mathtt{Z}\mathfrak{v}$.

Assume that $\mathfrak{v}<\mathtt{Z}\mathfrak{v}$. Proceeding like this way at proof of Theorem 1 for $\nu_0\leq\nu_0$, we have a increasing $\mathbf{O}_{seq}$ $\{\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}\}\in\mathbf{K}$ such that $\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}\to\mathfrak{u}\in\mathbf{K}$.

Again using (2), we get that $\mathfrak{u}=\sup\{\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}\}$. Moreover, from $\nu_0\preceq\mathfrak{v}$, we obtain that

$$\nu_{\mathfrak{a}}=\mathtt{Z}^{\mathfrak{a}}\nu_0\leq\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}\ \ \forall\ \ \mathfrak{a}\geq 1\ \ \textit{and}\ \ \nu_{\mathfrak{a}}<\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\ \ \forall\ \ \mathfrak{a}\geq 1$$

because $\nu_{\mathfrak{a}}\preceq\mathfrak{v}<\mathtt{Z}\mathfrak{v}\preceq\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}$, $\forall\ \ \mathfrak{a}\geq 1$.

As $\nu_{\mathfrak{a}}$ and $\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}$ are clearly different for $\mathfrak{a}\geq 1$, assume the below cases.

Case A : *Assume that*

$$\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\mathtt{Z}\nu_{\mathfrak{a}})+\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})\neq 0,$$

then from contraction condition (2), we have

$$\begin{aligned}&\partial_{\dashv}(\nu_{\mathfrak{a}+1},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})=\partial_{\dashv}(\mathtt{Z}\nu_{\mathfrak{a}},\mathtt{Z}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}))\\ &\leq\Re\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}}\mathfrak{v})+\Im[\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{a}+1})+\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})]\\ &\quad+\wp\Bigg[\frac{\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{a}+1})\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})}{\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\nu_{\mathfrak{a}+1})+\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})}\\ &\quad+\frac{\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\nu_{\mathfrak{a}+1})\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})}{\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\nu_{\mathfrak{a}+1})+\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})}\Bigg].\end{aligned}$$

On letting $\mathfrak{a}\to\infty$ in the above equation, we get

$$\partial_{\dashv}(\mathfrak{v},\mathfrak{u})\leq(\Re+\Im)\partial_{\dashv}(\mathfrak{v},\mathfrak{u}).$$

As $\Re+\Im<1$, we obtain $\partial_{\dashv}(\mathfrak{v},\mathfrak{u})=0$, thus $\mathfrak{v}=\mathfrak{u}$. Particularly, $\mathfrak{v}=\mathfrak{u}=sup\{\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}\}$ and consequently, $\mathtt{Z}\mathfrak{v}\leq\mathfrak{v}$, which contradicts that $\mathtt{Z}\mathfrak{v}<\mathfrak{v}$.

Hence, we have $\mathtt{Z}\mathfrak{v}=\mathfrak{v}$.

Case B : *If*

$$\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\mathfrak{v},\mathtt{Z}\nu_{\mathfrak{a}})+\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})=0,$$

then

$$\partial_{\dashv}(\nu_{\mathfrak{a}+1},\mathtt{Z}^{\mathfrak{a}+1}\mathfrak{v})=0.$$

Letting $\mathfrak{a}\to\infty$, we obtain that $\partial_{\dashv}(\mathfrak{v},\mathfrak{u})=0$. Then $\mathfrak{v}=\mathfrak{u}=\sup\{\mathtt{Z}^{\mathfrak{a}}\mathfrak{v}\}$, which implies that $\mathtt{Z}\mathfrak{v}\leq\mathfrak{v}$, this is contradiction. Therefore, $\mathtt{Z}\mathfrak{v}=\mathfrak{v}$.

Next, we demonstrate its uniqueness. Let $\nu_2 \in \mathbf{K}$ be a fixed point of $\mathtt{Z}$. So, we obtain

$$\mathtt{Z}^{\mathfrak{a}}\nu^* = \nu^* \ \text{ and } \ \mathtt{Z}^{\mathfrak{a}}\nu_2^* = \nu_2^*, \ \forall\ \mathfrak{a} \in \mathbb{N}.$$

By the definition of orthogonality, there is $\nu_1 \in \mathbf{K}$ so that

$$[\nu_1 \dashv \nu^* \text{ and } \nu_1 \dashv \nu_2^*] \quad \text{or} \quad [\nu^* \dashv \nu_1 \text{ and } \nu_2^* \dashv \nu_1].$$

Since $\mathtt{Z}$ is $\mathbf{O}_p$, we can write

$$[\mathtt{Z}^{\mathfrak{a}}\nu_1 \dashv \mathtt{Z}^{\mathfrak{a}}\nu^* \text{ and } \mathtt{Z}^{\mathfrak{a}}\nu_1 \dashv \mathtt{Z}^{\mathfrak{a}}\nu_2^*] \quad \text{or} \quad [\mathtt{Z}^{\mathfrak{a}}\nu^* \dashv \mathtt{Z}^{\mathfrak{a}}\nu_1 \text{ and } \mathtt{Z}^{\mathfrak{a}}\nu_2^* \dashv \mathtt{Z}^{\mathfrak{a}}\nu_1], \quad \forall\ \mathfrak{a} \in \mathbb{N}.$$

Therefore, by Definition 1, we have

$$\begin{aligned}\partial_\dashv(\nu_1^*, \nu_2^*) &= \partial_\dashv(\mathtt{Z}^{\mathfrak{a}}\nu_1^*, \mathtt{Z}^{\mathfrak{a}}\nu_2^*)\\ &= \partial_\dashv(\mathtt{Z}^{\mathfrak{a}}\nu_1^*, \mathtt{Z}^{\mathfrak{a}}\nu_1) + \partial_\dashv(\mathtt{Z}^{\mathfrak{a}}\nu_1, \mathtt{Z}^{\mathfrak{a}}\nu_2^*)\\ &\le \partial_\dashv(\nu_1^*, \nu_1) + \partial_\dashv(\nu_1, \nu_2^*).\end{aligned}$$

Taking limit as $\mathfrak{a} \to \infty$, we get

$$\partial_\dashv(\nu_1^*, \nu_2^*) = 0,$$

and so $\nu_1^* = \nu_2^*$. ∎

Now, we show that Theorems 1 and 2 do not ensure fixed point uniqueness.

Example 1. *Let $[(1,0),(0,1)] = \mathbf{K} \subseteq \mathbb{R}^2$ with $\partial_\dashv$. We consider an orthogonal partial order $\mathtt{U} \in \mathbf{K}$ as below:*
$\mathtt{U} : (\mathfrak{v},\mathfrak{u}) \le (\mathfrak{q},\mathfrak{r}) \iff \mathfrak{v} \le \mathfrak{q}$ and $\mathfrak{u} \le \mathfrak{r}$ with $\mathfrak{v} \dashv \mathfrak{q}$ and $\mathfrak{u} \dashv \mathfrak{r}$, $\forall\ \mathfrak{u},\mathfrak{v},\mathfrak{q},\mathfrak{r} \in \mathbf{K}$. Define the binary relation $\dashv$ on $\mathbf{K}$ by $\nu \dashv \mu$ if $\nu, \mu \ge 0$. Let $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ by $\mathtt{Z}(\nu,\mu) = (\nu,\mu)$ be a $\mathbf{O}_p$ and $\mathbf{O}_{con}$. Then $\mathtt{Z}$ has a fixed points in $\mathbf{K}$.

Proof: Given $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ be an $\mathbf{O}_p$ It is clearly $(\mathbf{K}, \dashv, \partial_\dashv)$ is a orthogonal complete partially ordered metric spaces. Besides, the identity mapping $\mathtt{Z}(\nu,\mu) = (\nu,\mu)$ is trivially $\mathbf{O}_{con}$ and holds the contraction condition.

$$\begin{aligned}&\partial_\dashv(\mathtt{Z}(\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{q},\mathfrak{r})) \le \Re\partial_\dashv((\mathfrak{v},\mathfrak{u}),(\mathfrak{q},\mathfrak{r}))\\ &\le \Re\partial_\dashv((\mathfrak{v},\mathfrak{u}),(\mathfrak{q},\mathfrak{r})) + \Im[\partial_\dashv((\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{v},\mathfrak{u}))\\ &\quad + \partial_\dashv((\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{q},\mathfrak{r}))]\\ &\quad + \wp\left[\frac{\partial_\dashv((\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{v},\mathfrak{u}))\partial_\dashv((\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{q},\mathfrak{r}))}{\partial_\dashv((\mathfrak{q},\mathfrak{r}), \mathtt{Z}(\mathfrak{v},\mathfrak{u})) + \partial_\dashv((\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{q},\mathfrak{r}))}\right.\\ &\quad \left. + \frac{\partial_\dashv((\mathfrak{q},\mathfrak{r}), \mathtt{Z}(\mathfrak{v},\mathfrak{u}))\partial_\dashv((\mathfrak{q},\mathfrak{r}), \mathtt{Z}(\mathfrak{q},\mathfrak{r}))}{\partial_\dashv((\mathfrak{q},\mathfrak{r}), \mathtt{Z}(\mathfrak{v},\mathfrak{u})) + \partial_\dashv((\mathfrak{v},\mathfrak{u}), \mathtt{Z}(\mathfrak{q},\mathfrak{r}))}\right],\end{aligned}$$

$\forall\ \mathfrak{v} \dashv \mathfrak{u}$ for any $\Re, \Im, \wp \in [0,1)$ with $0 \le \Re + 2\Im + \wp < 1$. $\mathbf{K}$ only compares to itself.

Moreover, $(1,0) \le \mathtt{Z}(1,0)$. Theorem 1 criteria is met, and $\mathtt{Z}$ has $(1,0)$ and $(0,1)$ are the fixed points. ∎

Example 2. *Let us consider in Example 1 and a increasing $\mathbf{O}_{seq}$ $\{(\nu_{\mathfrak{a}}, \mu_{\mathfrak{a}})\} \subseteq \mathbf{K}$ converging to (ν,μ). Then necessarily, $\{(\nu_{\mathfrak{a}}, \mu_{\mathfrak{a}})\}$ is a constant $\mathbf{O}_{seq}$ and $(\nu_{\mathfrak{a}}, \mu_{\mathfrak{a}}) = (\nu,\mu)$, $\forall\ \mathfrak{a} \in \mathbb{N}$.*

As a result, the upper bound for all terms in the $\mathbf{O}_{seq}$ is given by the limit (ν,μ). Theorem 2 hypotheses are fulfilled, and the fixed points of $\mathtt{Z}$ in $\mathbf{K}$ are (1, 0) and (0, 1).

Now, we give a hypotheses that is enough to show that the fixed point in Theorem 1 and Theorem 2 are unique.

Every pair of elements has a lower bound or an upper bound. (3)

In [8], it is proved that the condition is equivalent to for each $\nu, \mu \in \mathbf{K}$, $\exists\ \varsigma \in \mathbf{K}$ which is comparable to ν and μ.

Example 3. *Prove that the space $\mathtt{C}[0,1] = \{\nu : [0,1] \to \mathbb{R}, \mathbf{O}_{con}\}$ and define binary relation $\dashv$ on $\mathbf{K}$ by $\nu \dashv \mu$ if $\nu, \mu \ge 0$ with the orthogonal partially ordered metric spaces given by $\nu \le \mu \iff \nu(\mathfrak{z}) \le \mu(\mathfrak{z})$, for $\mathfrak{z} \in [0,1]$, and the metric given by*

$$\partial_\dashv(\nu,\mu) = \sup\{|\nu(\mathfrak{z}), \mu(\mathfrak{z})| : \mathfrak{z} \in [0,1]\}, \ \forall\ \nu,\mu \in [0,1]$$

holds (2). Moreover, as for $\nu, \mu \in [0,1]$, the function is $\mathbf{O}_p$ and $\max(\nu,\mu)(\mathfrak{z}) = \max\{\nu(\mathfrak{z}), \mu(\mathfrak{z})\}$ is $\mathbf{O}_{con}$. Also $(\mathtt{C}[0,1], \dashv)$ holds (3).

IV. Results for an Orthogonal Singh and Chatterjee contractions

The almost Singh and Chatterjee contractions introduced by Singh et. al. [18], we extend this into an orthogonal concept as follows.

Definition 11. *Let $(\mathbf{K}, \dashv, \partial_\dashv)$ be an orthogonal partially ordered metric spaces. A mapping $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ is said to be an orthogonal almost Singh and Chatterjee contractions, if there exist $\ell, \imath, \jmath \in [0,1)$ with $0 \le \ell + 2\imath + \jmath < 1$ and $\lambda \ge 0$ such that*

$$\begin{aligned}\partial_\dashv(\mathtt{Z}\nu, \mathtt{Z}\mu) &\le \ell\left(\frac{\partial_\dashv(\nu, \mathtt{Z}\nu) + \partial_\dashv(\mu, \mathtt{Z}\mu)}{\partial_\dashv(\nu,\mu)}\right) + \imath[\partial_\dashv(\nu, \mathtt{Z}\mu)\\ &\quad + \partial_\dashv(\mu, \mathtt{Z}\nu)] + \jmath\partial_\dashv(\nu,\mu)\\ &\quad + \lambda\min\{\partial_\dashv(\nu, \mathtt{Z}\mu), \partial_\dashv(\mu, \mathtt{Z}\nu), \partial_\dashv(\nu, \mathtt{Z}\nu)\},\end{aligned} \tag{4}$$

for all distinct $\nu, \mu \in \mathbf{K}$ with $\nu \le \mu$ and $\nu \dashv \mu$.

Now, we generalize and improve our fixed point theorem from Rao [19] by introducing the concept of an orthogonal Singh and Chatterjee contractions mapping in orthogonal complete partially ordered metric spaces.

Theorem 3. *Suppose that $(\mathbf{K}, \dashv, \partial_\dashv)$ be a orthogonal complete partially ordered metric spaces. Let $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ be an orthogonal almost Singh and Chatterjee contractions and, $\mathtt{Z}$ is non-decreasing $\mathbf{O}_p$ and $\mathbf{O}_{con}$. If $\exists\ \nu_0 \in \mathbf{K}$ such that $\nu_0 \le \mathtt{Z}\nu_0$, then $\mathtt{Z}$ has a unique fixed point in $\mathbf{K}$.*

Proof: Since $(\mathbf{K}, \dashv)$ is an $\mathbf{O}_s$, $\exists\ \nu_0 \in \mathbf{K} : (\forall\ \nu \in \mathbf{K},\ \nu \dashv \nu_0)$ or $(\forall\ \nu \in \mathbf{K},\ \nu_0 \dashv \nu)$

It follows that $\nu_0 \dashv \mathtt{Z}\nu_0$ or $\mathtt{Z}\nu_0 \dashv \nu_0$.

Let $\nu_{\mathfrak{a}} = \mathtt{Z}\nu_{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathbb{N} \cup \{0\}$. If $\nu_{\mathfrak{a}} = \nu_{\mathfrak{a}+1}$ for any $\mathfrak{a} \in \mathbb{N} \cup \{0\}$, then it is clear that $\nu_{\mathfrak{a}}$ is a fixed point of $\mathtt{Z}$.

Assume that $\nu_{\mathfrak{a}+1} \ne \nu_{\mathfrak{a}}\ \forall\ \mathfrak{a} \in \mathbb{N} \cup \{0\}$. Since $\mathtt{Z}$ is $\mathbf{O}_p$, we have

$$\nu_{\mathfrak{a}+1} \dashv \nu_{\mathfrak{a}} \ \text{ or } \ \nu_{\mathfrak{a}} = \nu_{\mathfrak{a}+1}, \ \forall\ \mathfrak{a} \in \mathbb{N} \cup \{0\}.$$

Since $\mathtt{Z}$ *is* $\mathbf{O}_p$, $\{\nu_{\mathfrak{a}}\}$ *is an* $\mathbf{O}_{seq}$.

$$\begin{aligned}\partial_{\dashv}(\nu_{\mathfrak{a}+1},\nu_{\mathfrak{a}}) &= \partial_{\dashv}(\mathtt{Z}\nu_{\mathfrak{a}},\mathtt{Z}\nu_{\mathfrak{a}-1})\\ &\le \ell\left(\frac{\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}\nu_{\mathfrak{a}})+\partial_{\dashv}(\nu_{\mathfrak{a}-1},\mathtt{Z}\nu_{\mathfrak{a}-1})}{\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{a}-1})}\right)\\ &\quad+\imath[\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}\nu_{\mathfrak{a}-1})+\partial_{\dashv}(\nu_{\mathfrak{a}-1},\mathtt{Z}\nu_{\mathfrak{a}})]\\ &\quad+\jmath\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{a}-1})\\ &\quad+\lambda\min\{\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}\nu_{\mathfrak{a}}),\partial_{\dashv}(\nu_{\mathfrak{a}-1},\mathtt{Z}\nu_{\mathfrak{a}}),\partial_{\dashv}(\nu_{\mathfrak{a}},\mathtt{Z}\nu_{\mathfrak{a}})\}.\end{aligned}$$

Which implies that

$$\begin{aligned}\partial_{\dashv}(\nu_{\mathfrak{a}+1},\nu_{\mathfrak{a}}) &\le \Big(\frac{\imath+\jmath}{1-\ell-\imath}\Big)\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{a}-1})\\ &\le \Big(\frac{\imath+\jmath}{1-\ell-\imath}\Big)^2\partial_{\dashv}(\nu_{\mathfrak{a}-1},\nu_{\mathfrak{a}-2})\\ &\ \ \vdots\\ &\le \left(\frac{\imath+\jmath}{1-\ell-\imath}\right)^{\mathfrak{a}}\partial_{\dashv}(\nu_1,\nu_0).\end{aligned}$$

Therefore, by the triangular inequality for $\mu\ge\nu$, *we have*

$$\begin{aligned}\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{n}}) &= \partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{a}+1})+\partial_{\dashv}(\nu_{\mathfrak{a}+1},\nu_{\mathfrak{a}+2})\\ &\quad+\ldots\ldots+\partial_{\dashv}(\nu_{\mathfrak{n}-1},\nu_{\mathfrak{n}})\\ &\le(\mathfrak{s}^{\mathfrak{a}}+\mathfrak{s}^{\mathfrak{a}+1}+\ldots\ldots+\mathfrak{s}^{\mathfrak{n}-1})\partial_{\dashv}(\nu_0,\mathtt{Z}\nu_0)\\ &\le\Big(\frac{\mathfrak{s}^{\mathfrak{a}}}{1-\mathfrak{s}}\Big)\partial_{\dashv}(\nu_1,\nu_0),\end{aligned}$$

where $\mathfrak{s}=\left(\frac{\imath+\jmath}{1-\ell-\imath}\right)\in[0,1)$. *Letting limit as* $\mathfrak{n},\mathfrak{a}\to+\infty$ *in the above equation, we get* $\partial_{\dashv}(\nu_{\mathfrak{a}},\nu_{\mathfrak{n}})=0$. *Thus,* $\{\nu_{\mathfrak{a}}\}$ *is a Cauchy* $\mathbf{O}_{seq}$ *in* $\mathbf{K}$. *Since* $\mathbf{K}$ *is a orthogonal complete partially ordered metric spaces, then* $\exists\ \mathfrak{v}\in\mathbf{K}$ *such that* $\lim_{\mathfrak{a}\to+\infty}\nu_{\mathfrak{a}}=\mathfrak{v}$. *From the* $\mathbf{O}_{con}$ *of* $\mathtt{Z}$, *we get*

$$\begin{aligned}\mathtt{Z}\mathfrak{v} &= \mathtt{Z}\big(\lim_{\mathfrak{a}\to+\infty}\nu_{\mathfrak{a}}\big)\\ &=\lim_{\mathfrak{a}\to+\infty}\mathtt{Z}\nu_{\mathfrak{a}}\\ &=\lim_{\mathfrak{a}\to+\infty}\nu_{\mathfrak{a}+1}=\mathfrak{v}.\end{aligned}$$

Hence, $\mathfrak{v}$ *is a fixed point of* $\mathtt{Z}\in\mathbf{K}$.

Next, we demonstrate its uniqueness. Let $\nu_2\in\mathbf{K}$ *be a fixed point of* $\mathtt{Z}$. *So, we obtain* $\mathtt{Z}^{\mathfrak{a}}\nu^*=\nu^*$ *and* $\mathtt{Z}^{\mathfrak{a}}\nu_2^*=\nu_2^*$, $\forall\ \mathfrak{a}\in\mathbb{N}$. *By the definition of orthogonality, there is* $\nu_1\in\mathbf{K}$ *so that*

$$[\nu_1\dashv\nu^*\ \textit{and}\ \nu_1\dashv\nu_2^*]$$
$$\textit{or}\ \ [\nu^*\dashv\nu_1\ \textit{and}\ \nu_2^*\dashv\nu_1].$$

Since $\mathtt{Z}$ *is* $\mathbf{O}_p$, *we can write*

$$[\mathtt{Z}^{\mathfrak{a}}\nu_1\dashv\mathtt{Z}^{\mathfrak{a}}\nu^*\ \textit{and}\ \mathtt{Z}^{\mathfrak{a}}\nu_1\dashv\mathtt{Z}^{\mathfrak{a}}\nu_2^*]$$
$$\textit{or}\ \ [\mathtt{Z}^{\mathfrak{a}}\nu^*\dashv\mathtt{Z}^{\mathfrak{a}}\nu_1\ \textit{and}\ \mathtt{Z}^{\mathfrak{a}}\nu_2^*\dashv\mathtt{Z}^{\mathfrak{a}}\nu_1],\ \forall\ \ \mathfrak{a}\in\mathbb{N}.$$

Therefore, by Definition 1, we have

$$\begin{aligned}\partial_{\dashv}(\nu_1^*,\nu_2^*) &= \partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\nu_1^*,\mathtt{Z}^{\mathfrak{a}}\nu_2^*)\\ &=\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\nu_1^*,\mathtt{Z}^{\mathfrak{a}}\nu_1)+\partial_{\dashv}(\mathtt{Z}^{\mathfrak{a}}\nu_1,\mathtt{Z}^{\mathfrak{a}}\nu_2^*)\\ &\le\partial_{\dashv}(\nu_1^*,\nu_1)+\partial_{\dashv}(\nu_1,\nu_2^*).\end{aligned}$$

Taking limit as $\mathfrak{a}\to\infty$, *we get*

$$\partial_{\dashv}(\nu_1^*,\nu_2^*)=0,$$

and so $\nu_1^*=\nu_2^*$. *Hence* $\mathtt{Z}$ *has a unique fixed point.* ∎

We prove the existence of a monotone sequence holds for contraction condition 1.

Theorem 4. *Let* $(\mathbf{K},\dashv,\partial_{\dashv})$ *be an orthogonal complete partially ordered metric spaces. Assume that* $\mathbf{K}$ *satisfies if a increasing* $\mathbf{O}_{seq}$ $\{\nu_{\mathfrak{a}}\}\to\nu\in\mathbf{K}$, *then*

$$\nu=\sup\{\nu_{\mathfrak{a}}\}. \tag{5}$$

Let $\mathtt{Z}:\mathtt{Z}\to\mathtt{Z}$ *be a* $\mathbf{O}_p$ *and* $\mathbf{O}_{con}$ *monotone increasing function holds the condition* (4). *If there exists* $\nu_0\in\mathbf{K}$ *with* $\nu_0\le\mathtt{Z}\nu_0$, *then* $\mathtt{Z}$ *has a unique fixed point in* $\mathbf{K}$.

Proof: The demonstration supports the Theorem 2. ∎

Now, we provide the example for Theorem 3.

Example 4. *Let* $\{(2,0),(0,2)\}=\mathbf{K}\subseteq\mathbb{R}^2$ *with* $\partial_{\dashv}$. *Define the binary relation* $\dashv$ *on* $\mathbf{K}$ *by* $\nu\dashv\mu$ *if* $\nu,\mu\ge0$. *We assume an orthogonal partial order in* $\mathbf{K}$ *as below:*

$$\begin{aligned}(\nu_1,\mu_1)\le(\nu_2,\mu_2)\iff&\ \nu_1\le\nu_2\ \ \textit{and}\ \ \mu_1\le\mu_2,\\ &\forall\ \ \nu_1\dashv\mu_1\ \ \textit{and}\ \ \nu_2\dashv\mu_2.\end{aligned}$$

Thus, $(\mathbf{K},\dashv,\partial_{\dashv})$ *is a orthogonal complete partially ordered metric spaces. The function* $\mathtt{Z}(\nu,\mu)=(\nu,\mu)$ *is an* $\mathbf{O}_{con}$, *non-decreasing and the contraction condition*

$$\begin{aligned}&\partial_{\dashv}(\mathtt{Z}(\nu_1,\mu_1),\mathtt{Z}(\nu_2,\mu_2))\le\jmath\partial_{\dashv}((\nu_1,\mu_1),(\nu_2,\mu_2))\\ &\quad\le\ell\Big(\frac{\partial_{\dashv}((\nu_1,\mu_1),\mathtt{Z}(\nu_1,\mu_1))\partial_{\dashv}((\nu_2,\mu_2),\mathtt{Z}(\nu_2,\mu_2))}{\partial_{\dashv}((\nu_1,\mu_1),(\nu_2,\mu_2))}\Big)\\ &\qquad+\imath[\partial_{\dashv}((\nu_1,\mu_1),\mathtt{Z}(\nu_2,\mu_2))\\ &\qquad+\partial_{\dashv}((\nu_2,\mu_2),\mathtt{Z}(\nu_1,\mu_1))]+\jmath\partial_{\dashv}((\nu_1,\mu_1),(\nu_2,\mu_2))\\ &\qquad+\lambda\min\{\partial_{\dashv}((\nu_1,\mu_1),\mathtt{Z}(\nu_2,\mu_2)),\partial_{\dashv}((\nu_2,\mu_2),\mathtt{Z}(\nu_1,\mu_1)),\\ &\qquad\quad\partial_{\dashv}(\nu_1,\mu_1),\mathtt{Z}(\nu_1,\mu_1)\},\ \forall\ \ \nu_1\ \dashv\ \mu_1,\end{aligned}$$

holds for any $\ell,\imath,\jmath\in[0,1)$ *with* $0\le\ell+2\imath+\jmath<1$ *and for any* $\lambda\ge0$. *Clearly* $\mathbf{K}$ *is an* $\mathbf{O}_p$ *and* $\mathbf{O}_{con}$.
$\mathbf{K}$ *elements are solely similar to themselves. Moreover,* $(0,2)\preceq\mathtt{Z}(0,2)$. *Here all the axioms of Theorem 3 are hold, (2, 0) and (0, 2) are the fixed points of* $\mathtt{Z}$.

In 1988 Singh and Chatterjee [1] introduced the concept of Singh and Chatterjee contractions. We modified that contraction as below:

Definition 12. *Let* $(\mathbf{K},\dashv,\partial_{\dashv})$ *be an orthogonal partially ordered metric spaces. A self-mapping* $\mathtt{Z}$ *on* $\mathbf{K}$ *is said to be an orthogonal Singh and Chatterjee contractions, if* $\exists\ \ \ell,\imath,\jmath\in[0,1)$ *with* $0\le\ell+2\imath+\jmath<1$ *such that*

$$\begin{aligned}\partial_{\dashv}(\mathtt{Z}\nu,\mathtt{Z}\mu)\le\ell\Big(&\frac{\partial_{\dashv}(\nu,\mathtt{Z}\nu)\partial_{\dashv}(\mu,\mathtt{Z}\mu)}{\partial_{\dashv}(\nu,\mu)}\Big)\\ &+\imath[\partial_{\dashv}(\nu,\mathtt{Z}\mu)+\partial_{\dashv}(\mu,\mathtt{Z}\nu)]+\jmath\partial_{\dashv}(\nu,\mu),\end{aligned}\tag{6}$$

for all distinct $\nu,\mu\in\mathbf{K}$ *with* $\nu\dashv\mu$ *and* $\nu\le\mu$.

Corollary 1. *Let* $(\mathbf{K},\dashv,\partial_{\dashv})$ *be an orthogonal partially ordered metric spaces. A self-mapping* $\mathtt{Z}$ *on* $\mathbf{K}$ *be a orthogonal Singh and Chatterjee contractions, non-decreasing and* $\mathbf{O}_{con}$. *If* $\exists\ \ \nu_0\in\mathbf{K}$ *such that* $\nu_0\le\mathtt{Z}\nu_0$, *then* $\mathtt{Z}$ *has a unique fixed point in* $\mathbf{K}$.

Proof: Set $\lambda=0$ *in Theorem 3.* ∎

If $\nu_0\ge\mathtt{Z}\nu_0$, in Theorem 3, then we obtain the following result:

Theorem 5. *Let* $(\mathbf{K}, \dashv, \partial_\dashv)$ *be an orthogonal partially ordered metric spaces. Assume that either* $\mathtt{Z}$ *is* $\mathbf{O}_{con}$ *or* $\mathbf{K}$ *is such that if a decreasing* $\mathbf{O}_{seq}$ $\{\nu_\mathfrak{a}\} \to \nu \in \mathbf{K}$*, then* $\nu = \inf\{\nu_\mathfrak{a}\}$*. Let* $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ *be a* $\mathbf{O}_p$ *and monotone increasing function holds the contraction* (4) *(or)* (6)*. If* $\exists\ \nu_0 \in \mathbf{K}$ *with* $\nu_0 \geq \mathtt{Z}\nu_0$*, then* $\mathtt{Z}$ *has a unique fixed point in* $\mathbf{K}$.

Proof: The proof follows a similar pattern to prior Theorem 3, with the exception of a few minor differences. ■

Theorem 6. *Let* $(\mathbf{K}, \dashv, \partial_\dashv)$ *be an orthogonal partially ordered metric spaces. Suppose that* $\mathtt{Z} : \mathbf{K} \to \mathbf{K}$ *be an orthogonal almost Singh and Chatterjee contractions and increasing. Also, suppose* $\exists\ \nu_0 \in \mathbf{K}$ *such that* $\nu_0 \leq \mathtt{Z}\nu_0$*. If the function* $\mathtt{Z}^\mathfrak{q}$ *is* $\mathbf{O}_{con}$ *for some non-negative integer* $\mathfrak{q}$*, then* $\mathtt{Z}$ *has a unique fixed point in* $\mathbf{K}$.

Proof: Based on the proof of Theorem 3, we construct a increasing $\mathbf{O}_{seq}$ $\{\nu_\mathfrak{a}\} \in \mathbf{K}$ such that $\nu_\mathfrak{a} \to \nu_1$, for some $\nu_1 \in \mathbf{K}$. Also, its subsequences $\nu_{\mathfrak{a}_\xi}(\mathfrak{a}_\xi = \xi\mathfrak{q})$ converges to the same point ν_1.
Therefore,

$$\begin{aligned}\mathtt{Z}^\mathfrak{q}\nu_1 &= \mathtt{Z}^\mathfrak{q}\Big(\lim_{\mathfrak{a}\to+\infty} \nu_{\mathfrak{a}_\xi}\Big)\\ &= \lim_{\mathfrak{a}\to+\infty} \nu_{\mathfrak{a}_{\xi+1}}\\ &= \nu_1.\end{aligned}$$

Thus, ν_1 is a fixed point of $\mathtt{Z}^\mathfrak{q}$.
Next, we prove ν_1 is a fixed point of $\mathtt{Z}$. Let $\mathfrak{n}$ be the small non-negative integer such that $\mathtt{Z}^\mathfrak{n}\nu_1 = \nu_1$ and $\mathtt{Z}^\mathfrak{s}\nu_1 \neq \nu_1$ $(\mathfrak{s} = 1, 2, ..., \mathfrak{n}-1)$. If $\mathfrak{n} > 1$, then

$$\begin{aligned}\partial_\dashv(\mathtt{Z}\nu_1, \nu_1) &= \partial_\dashv(\mathtt{Z}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1)\\ &\leq \ell\Big(\frac{\partial_\dashv(\nu_1, \mathtt{Z}\nu_1) + \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1)}{\partial_\dashv(\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1)}\Big)\\ &\quad + \imath[\partial_\dashv(\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1) + \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}\nu_1)]\\ &\quad + \jmath\partial_\dashv(\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1)\\ &\quad + \lambda\min\{\partial_\dashv(\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1), \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}\nu_1),\\ &\qquad \partial_\dashv(\nu_1, \mathtt{Z}\nu_1)\},\end{aligned}$$

which implies that

$$\partial_\dashv(\mathtt{Z}\nu_1, \nu_1) \leq \Big(\frac{\imath + \jmath}{1 - \ell - \imath}\Big)\partial_\dashv(\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1).$$

Again from contraction condition (4), we have

$$\begin{aligned}\partial_\dashv(\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1) &= \partial_\dashv(\mathtt{Z}^\mathfrak{n}\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1)\\ &\leq \ell\Big(\frac{\partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1) + \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-2}\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1)}{\partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^{\mathfrak{n}-2}\nu_1)}\Big)\\ &\quad + \imath[\partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1)\\ &\quad + \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-2}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1)] + \jmath\partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^{\mathfrak{n}-2}\nu_1)\\ &\quad + \lambda\min\{\partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1), \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1),\\ &\qquad \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-2}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1), \partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^\mathfrak{n}\nu_1)\}.\end{aligned}$$

Inductively, we get

$$\begin{aligned}\partial_\dashv(\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1) &= \partial_\dashv(\mathtt{Z}^\mathfrak{n}\nu_1, \mathtt{Z}^{\mathfrak{n}-1}\nu_1)\\ &\leq \mathfrak{s}\partial_\dashv(\mathtt{Z}^{\mathfrak{n}-1}\nu_1, \mathtt{Z}^{\mathfrak{n}-2}\nu_1)\\ &\leq .... \leq \mathfrak{s}^{\mathfrak{n}-1}\partial_\dashv(\nu_1, \mathtt{Z}\nu_1),\\ \text{where } \ \mathfrak{s} &= \frac{\imath + \jmath}{1 - \ell - \imath} \in [0, 1).\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_\dashv(\mathtt{Z}\nu_1, \nu_1) &\leq \mathfrak{s}^\mathfrak{n}\partial_\dashv(\mathtt{Z}\nu_1, \nu_1)\\ &< \partial_\dashv(\mathtt{Z}\nu_1, \nu_1),\end{aligned}$$

this is contradiction. Hence, $\mathtt{Z}\nu_1 = \nu_1$.

Next, we prove uniqueness. Let $\nu_2 \in \mathbf{K}$ be a fixed point of $\mathtt{Z}$. So, we get $\mathtt{Z}^\mathfrak{a}\nu^* = \nu^*$ and $\mathtt{Z}^\mathfrak{a}\nu_2^* = \nu_2^*$, $\forall\ \mathfrak{a} \in \mathbb{N}$. According to the notion of orthogonality, there is $\nu_1 \in \mathbf{K}$ so that

$$\begin{aligned}&[\nu_1 \dashv \nu^* \text{ and } \nu_1 \dashv \nu_2^*]\\ \textit{or} \quad &[\nu^* \dashv \nu_1 \text{ and } \nu_2^* \dashv \nu_1].\end{aligned}$$

Since $\mathtt{Z}$ is $\mathbf{O}_p$, we can write

$$\begin{aligned}&[\mathtt{Z}^\mathfrak{a}\nu_1 \dashv \mathtt{Z}^\mathfrak{a}\nu^* \text{ and } \mathtt{Z}^\mathfrak{a}\nu_1 \dashv \mathtt{Z}^\mathfrak{a}\nu_2^*]\\ \text{or } \ &[\mathtt{Z}^\mathfrak{a}\nu^* \dashv \mathtt{Z}^\mathfrak{a}\nu_1 \text{ and } \mathtt{Z}^\mathfrak{a}\nu_2^* \dashv \mathtt{Z}^\mathfrak{a}\nu_1],\\ &\forall\ \ \mathfrak{a} \in \mathbb{N}.\end{aligned}$$

Therefore, because of Definition 1, we get

$$\begin{aligned}\partial_\dashv(\nu_1^*, \nu_2^*) &= \partial_\dashv(\mathtt{Z}^\mathfrak{a}\nu_1^*, \mathtt{Z}^\mathfrak{a}\nu_2^*)\\ &= \partial_\dashv(\mathtt{Z}^\mathfrak{a}\nu_1^*, \mathtt{Z}^\mathfrak{a}\nu_1) + \partial_\dashv(\mathtt{Z}^\mathfrak{a}\nu_1, \mathtt{Z}^\mathfrak{a}\nu_2^*)\\ &\leq \partial_\dashv(\nu_1^*, \nu_1) + \partial_\dashv(\nu_1, \nu_2^*).\end{aligned}$$

Taking limit as $\mathfrak{a} \to \infty$, we get

$$\partial_\dashv(\nu_1^*, \nu_2^*) = 0,$$

and so $\nu_1^* = \nu_2^*$. Hence $\mathtt{Z}$ has a unique fixed point. ■

V. Partial ordered Application

Here, we assume the Volterra integral type equation:

$$\nu(\mathfrak{r}) = \rho(\mathfrak{r}) + \xi\int_0^1 \mathfrak{v}(\mathfrak{r}, \tau)\mathfrak{g}(\tau, \nu(\tau))\partial\tau, \mathfrak{r} \in [0, 1], \xi \geq 0. \tag{7}$$

Take $\mathbf{K} = \mathcal{C}(\mathtt{I})$ is continuous map defined on $\mathtt{I}$ endowed with a metric as below

$$\mathtt{P}(\nu, \mu) = \sup_{\mathfrak{r}\to\mathtt{I}} \big|\nu(\mathfrak{r}) - \rho(\mathfrak{r})\big|,\ \forall\ \nu, \mu \in \mathtt{I}.$$

Let υ be the class of map $\upsilon\colon [0, +\infty) \to [0, +\infty)$ so that $(\upsilon(\wp))^\mathfrak{s} \leq \upsilon(\wp^\mathfrak{s})$, $\forall\ \ \mathfrak{s} \geq 1$ and $\wp \geq 0$.

We assume the following conditions

1) $\mathfrak{g}\colon \mathtt{I} \times (-\infty, +\infty) \to (-\infty, +\infty)$ is non-descending continuous with respect to second variable so that there is $0 \leq \mathtt{L} \leq 1$:
$$\begin{aligned}&\Big|\mathfrak{g}(\mathfrak{r}, [u_1]) - \mathfrak{g}(\mathfrak{r}, [u_2])\Big| \leq \mathtt{L}\upsilon([u_1] - [u_2]),\\ &\quad \forall\ \ [u_1], [u_2] \in \mathbb{R} \text{ with } \ [u_1] \geq [u_2].\end{aligned}$$
2) $\rho\colon \mathtt{I} \to \mathbb{R}$ is continuous on $\mathtt{I}$.
3) $\mathfrak{v} : \mathtt{I} \times \mathtt{I} \to (-\infty, +\infty)$ is continuous with respect to its first variable, and its second variable can be measured such that for every $\mathfrak{r} \in \mathtt{I}$,
$$\int_0^1 \mathfrak{v}(\mathfrak{r}, \tau)\partial\tau \leq \omega.$$
4) $\mathtt{L}^\mathfrak{s}\xi^\mathfrak{s}\omega^\mathfrak{s} \leq \dfrac{1}{2^{4\mathfrak{s}-4}}$.

We assume on $\mathbf{K}$ the following $\nu, \mu \in \mathcal{C}(\mathtt{I})$ and $\nu \dashv \mu \iff \nu \leq \mu$.

Now, for $\mathfrak{s}=1$,

$$\begin{aligned}\partial(\nu,\mu) &= (\mathtt{P}(\nu,\mu)) \\ &= \sup_{\mathfrak{r}\to\mathtt{I}}\big|\nu(\mathfrak{r})-\rho(\mathfrak{r})\big|,\ \forall\ \nu,\mu\in\mathcal{C}(\mathtt{I}).\end{aligned}$$

We conculde that $(\mathbf{K},\partial,\dashv)$ is a orthogonal partially ordered metric spaces.

Theorem 7. *Under the assumptions (1)-(4), then Equation* (7) *has a unique solution in* $\mathcal{C}(\mathtt{I})$.

Proof: By the definition of $\mathbf{O}_p$, for all $\nu,\mu\in\mathbf{K}$, $\nu\dashv\mu \implies \mathtt{Z}\nu\dashv\mathtt{Z}\mu$.

We assume that $\Psi\colon\mathbf{K}\to\mathbf{K}$ defined by

$$\Psi\nu(\mathfrak{r})=\rho(\mathfrak{r})+\xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathfrak{g}(\tau,\nu(\tau))\partial\tau,\ \ \mathfrak{r}\in\mathtt{I},\xi\geq 0.$$

Let Ψ is described as if $\nu\in\mathbf{K}$, then $\Psi(\nu)\in\mathbf{K}$. It is easy to see that, $(\mathbf{K},\partial_{\dashv},\dashv)$ is orthogonal partially ordered metric spaces.

For $\nu,\mu\in\mathbf{K}$ with $\nu\leq\mu$ and $\mathfrak{r}\in\mathtt{I}$, we have

$$\begin{aligned}\Psi\nu(\mathfrak{r})-\Psi\mu(\mathfrak{r}) &= \rho(\mathfrak{r})+\xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathfrak{g}(\tau,\nu(\tau))\partial\tau-\rho(\mathfrak{r}) \\ &\quad -\xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathfrak{g}(\tau,\mu(\tau))\partial\tau \\ &= \xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)[\mathfrak{g}(\tau,\nu(\tau))-\mathfrak{g}(\tau,\mu(\tau))]\partial\tau \\ &\leq 0.\end{aligned}$$

Therefore, Ψ has the nondescending property. Also, Ψ is $\mathbf{O}_p$, we get

$$\begin{aligned}|\Psi\nu(\mathfrak{r})-\Psi\mu(\mathfrak{r})| &= \Bigg|\rho(\mathfrak{r})+\xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathfrak{g}(\tau,\nu(\tau))\partial\tau \\ &\quad -\rho(\mathfrak{r})-\xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathfrak{g}(\tau,\mu(\tau))\partial\tau\Bigg| \\ &\leq \xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)|\mathfrak{g}(\tau,\nu(\tau))-\mathfrak{g}(\tau,\mu(\tau))|\partial\tau \\ &\leq \xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathrm{L}\jmath|\nu-\mu|.\end{aligned}$$

Since $\nu\dashv\mu$, we get

$$\begin{aligned}\jmath(\mu(\mathfrak{r})-\nu(\mathfrak{r})) &\leq \big(\sup_{\mathfrak{r}\in\mathtt{I}}|\mu(\mathfrak{r})-\nu(\mathfrak{r})|\big) \\ &= \jmath(\mathtt{P}(\nu,\mu)),\end{aligned}$$

hence

$$\begin{aligned}|\Psi\nu(\mathfrak{r})-\Psi\mu(\mathfrak{r})| &\leq \xi\int_0^1\mathfrak{v}(\mathfrak{r},\tau)\mathrm{L}\jmath(\mathtt{P}(\nu,\mu))\partial\tau \\ &\leq \xi\omega\mathrm{L}\jmath(\mathtt{P}(\nu,\mu)).\end{aligned}$$

Then, we obtain

$$\begin{aligned}\partial_{\dashv}(\Psi(\nu),\Psi(\mu)) &= \sup_{\mathfrak{r}\in\mathtt{I}}|\Psi\nu(\mathfrak{r})-\Psi\mu(\mathfrak{r})| \\ &\leq \xi\omega\mathrm{L}\jmath\mathtt{P}(\nu,\mu) \\ &= \xi\omega\mathrm{L}\partial_{\dashv}(\nu,\mu) \\ &\leq \partial_{\dashv}(\nu,\mu).\end{aligned}$$

This shows that the operator Ψ satisfying the contraction requirement. So, (7) has a unique solution. ■

Example 5. *Solve the integral equation and discuss all its possible cases*

$$u(\nu)=\rho(\nu)+\beta\int_0^1(1-3\nu\tau)u(\tau)\partial\tau. \tag{8}$$

Solution:
From Equation (8) *implies that*

$$u(\nu)=\rho(\nu)+\beta\int_0^1(u(\tau)-3\nu\tau u(\tau))\partial\tau$$

$$u(\nu)=\rho(\nu)+\beta[\mathcal{D}_1-3\nu\mathcal{D}_2], \tag{9}$$

$$\textit{where}\ \ \mathcal{D}_1=\int_0^1 u(\tau)\partial\tau, \tag{10}$$

$$\mathcal{D}_2=\int_0^1\tau u(\tau)\partial\tau, \tag{11}$$

$\mathcal{D}_1$ and $\mathcal{D}_2$ are constants to be determined.

Equation (9) *is an orthogonal continuous and integrating with respect to ν over the limit o to 1.*

$$\begin{aligned}\int_0^1 u(\nu)\partial\nu &= \int_0^1 f(\nu)\partial\nu+\beta(\mathcal{D}_1-3\nu\mathcal{D}_2)\partial\nu \\ (10)\implies \mathcal{D}_1 &= \int_0^1 f(\nu)\partial\nu+\beta(\mathcal{D}_1-\frac{3}{2}\nu\mathcal{D}_2) \\ \int_0^1 f(\nu)\partial\nu &= (1-\beta)\mathcal{D}_1+\frac{3}{2}\mathcal{D}_2\beta\end{aligned}$$

$$f_1=(1-\beta)\mathcal{D}_1+\frac{3}{2}\mathcal{D}_2\beta. \tag{12}$$

Now multiplying (9) *with ν and integrating with respect to ν between 0 and 1. We get*

$$\begin{aligned}\int_0^1\nu u(\nu)\partial\nu &= \int_0^1\nu f(\nu)\partial\nu+\beta\int_0^1(\mathcal{D}_1\nu-3\nu^2\mathcal{D}_2)\partial\nu \\ (11)\implies \mathcal{D}_2 &= f_2+\beta[\frac{\mathcal{D}_1}{2}-\mathcal{D}_2]\end{aligned}$$

$$f_2=-\frac{\beta}{2}\mathcal{D}_1+\mathcal{D}_2(1+\beta), \tag{13}$$

where $f_2=\int_0^1\nu f(\nu)\partial\nu$. From (12) *and* (13)*, we get*

$$\begin{aligned}\Delta(\beta) &= \begin{bmatrix}1-\beta & \frac{3}{2}\beta \\ -\frac{\beta}{2} & 1+\beta\end{bmatrix} \\ &= 1-\beta^2+\frac{3}{4}\beta^2 \\ &= 1-\frac{\beta^2}{4} \\ \Delta(\beta) &= \frac{4-\beta^2}{4}.\end{aligned}$$

Now (12) *and* (13) *can be written as*

$$(I-\beta\mathcal{A})\mathcal{D}=f,$$

where

$$\mathcal{D}=\begin{bmatrix}\mathcal{D}_1\\ \mathcal{D}_2\end{bmatrix},\mathcal{F}=\begin{bmatrix}f_1\\ f_2\end{bmatrix}.$$

Also, $\big|1-\beta\mathcal{A}\big|=\Delta(\beta)$.

Table I

S.No	β	$\Delta(\beta)$
1	0.0000	1.0000
2	0.1000	0.9975
3	0.2000	0.9900
4	0.3000	0.9775
5	0.4000	0.6900
6	0.5000	0.9375
7	0.6000	0.9100
8	0.7000	0.8775
9	0.8000	0.8400
10	0.9000	0.7975
11	1.0000	0.7500
12	2.0000	0.0000

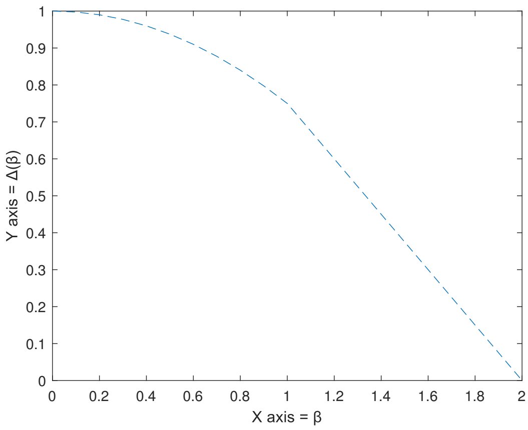

Figure 1. Graph of $|1-\beta\mathcal{A}| = \Delta(\beta)$ for Example 5.

Case A: *If $f(\nu) \neq 0$ and $\mathcal{F} \neq 0$ then (12) and (13) has a unique solution, if $\Delta(\beta) \neq 0$, that is, $\beta = -2, 2$. When $\beta = 2$ or $\beta = 2$, then these equations have either no solution or infinite many solutions.*

Sub case A-1: *If $\beta = 2$, then, (12) and (13) reduce to*

$$-\mathcal{D}_1 + 3\mathcal{D}_2 = f_1$$
$$-\mathcal{D}_1 + 3\mathcal{D}_2 = f_2.$$

These equations have no solution if $f_1 \neq f_2$ and have infinitely many solutions when $f_1 = f_2$, that is

$$\int_0^1 f(\nu)\partial\nu = \int_0^1 \nu f(\nu)\partial\nu$$

or

$$\int_0^1 (1-\nu) f(\nu)\partial\nu = 0$$

Thus, the solution of given integral equation is

$$\begin{aligned} u(\nu) &= f(\nu) + 2[\mathcal{D}_1\mathfrak{a}_1(\nu) + \mathcal{D}_2\mathfrak{a}_2(\nu)] \\ &= f(\nu) + 2[\mathcal{D}_1 \,.\, 1 + \mathcal{D}_2(-3\nu)] \\ &= f(\nu) + 2[3\mathcal{D}_2 - f_1 - 3\nu\mathcal{D}_2] \\ &= f(\nu) + 6\mathcal{D}_2(1-\nu) - 2f_1, \end{aligned}$$

or

$$u(\nu) = f(\nu) + 6\mathcal{D}_2(1-\nu) - 2\int_0^1 f(\nu)\partial\nu,$$

where $\mathcal{D}_2$ is arbitrary.

Sub case A-2: *If $\beta = -2$. As done above, the solution is given by*

$$u(\nu) = f(\nu) - 2\mathcal{D}_2(1-3\nu) - 2\int_0^1 \nu f(\nu)\partial\nu.$$

Case B: *When $f(\nu) = 0, \mathcal{F} = 0$.*
In this case, the Equations (12) and (13) becomes;

$$(1-\beta)\mathcal{D}_1 + \frac{3\beta}{2}\mathcal{D}_2 = 0, \; -\frac{\beta}{2} + (1+\beta)\mathcal{D}_2 = 0. \qquad (14)$$

If $\beta \neq 2$, and -2, then the system has only trivial solution $\mathcal{D}_1 = 0 = \mathcal{D}_2$.

Thus $u(\nu) = 0$ is the solution of given integral equation.

Sub case B-1: *If $\beta = 2$ then (14) becomes*

$$-\mathcal{D}_1 + 3\mathcal{D}_2 = 0 \implies \mathcal{D}_1 = 3\mathcal{D}_2.$$

Thus the solution of given integral equation is

$$u(\nu) = 0 + 2(3\mathcal{D}_2 - 3\nu\mathcal{D}_2)$$
$$u(\nu) = 6\mathcal{D}_2(1-\nu).$$

Sub case B-2: *If $\beta = -2$ then Equation (14) becomes*

$$\mathcal{D}_1 - \mathcal{D}_2 = 0 \implies \mathcal{D}_1 = \mathcal{D}_2.$$

Thus, the solution is

$$u(\nu) = 0 - 2(\mathcal{D}_2 - 3\nu\mathcal{D}_2)$$
$$u(\nu) = 2\mathcal{D}_2(3\nu - 1).$$

Case C: *When $f(\nu) \neq 0$ and $\mathcal{F} = 0$.*
If $\beta \neq 2, -2$ then the system (14) has only trivial solution $\mathcal{D}_1 = \mathcal{D}_2 = 0$ and therefore $u(\nu) = f(\nu)$ is the solution.

Sub case C-1: *If $\beta = 2$, then $\mathcal{D}_1 = 3\mathcal{D}_2$ and the solution is*

$$u(\nu) = f(\nu) + 2(3\mathcal{D}_2 - 3\nu\mathcal{D}_2)$$
$$u(\nu) = f(\nu) + 6\mathcal{D}_2(1-\nu).$$

Sub case C-2: *If $\beta = -2$, then $\mathcal{D}_1 = \mathcal{D}_2$ and the solution is*

$$u(\nu) = f(\nu) - 2(\mathcal{D}_2 - 3\nu\mathcal{D}_2)$$
$$u(\nu) = f(\nu) + 2\mathcal{D}_2(3\nu - 1).$$

Hence the solution is complete.

VI. Conclusion

In this paper, we proved fixed point theorems using an orthogonal rational type contraction and an orthogonal Singh and Chatterjee contraction in orthogonal complete partially ordered metric spaces. Furthermore, we presented some examples to strengthen our main results. Also, we provided an application to the Volterra integral type equation.

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