Interior and Closure Operators on Quasi-pseudo-BL Algebras

Jinfan Xu, Guangling Cao, and Wenjuan Chen

Abstract—**In this paper, interior and closure operators on quasi-pseudo-BL algebras are introduced and investigated. First, we give the definitions of multiplicative interior operators (mi-operators, for short), weak mi-operators, wmi-operators and weak wmi-operators on a quasi-pseudo-BL algebra and discuss the relationship among them. Meanwhile, we study the related properties of these operators on a quasi-pseudo-BL algebra and discuss the operators on the quotient algebra with respect to a normal weak filter. Second, we introduce the concepts of additive closure operators (ac-operators, for short), weak ac-operators, sac-operators and weak sac-operators on a good quasi-pseudo-BL algebra. We study the relations among them and discuss the related properties. Moreover, we present the connections between (weak) wmi-operators and (weak) ac-operators on a good quasi-pseudo-BL algebra. Finally, we investigate the properties of the induced operators on some quasi-pseudo-MV algebras.**

Index Terms—**quasi-pseudo-BL algebras, good quasi-pseudo-BL algebras, interior operators, closure operators, weak filters.**

I. Introduction

RECENTLY, the algebras based on quantum computational logic have been received more and more attention [1], [2], [3], [4], [5], [6], [7]. In [6], quasi-pseudo-BL algebras (qpBL algebras, for short) were introduced which can be regarded as generalizations of quasi-pseudo-MV algebras and pseudo-BL algebras. Quasi-pseudo-MV algebras were studied by Chen and Dudek in [2] as a generalization of pseudo-MV algebras and quasi-MV algebras, while pseudo-BL algebras were investigated by Di Nola et al. which were the non-commutative generalization of BL algebras [8], [9]. Since qpBL algebras form a larger class and have a vital role in connection with quantum computational logic and fuzzy logic, it makes sense to generalize and extend the known results to qpBL algebras in order to study the common properties and provide a more general algebraic foundation.

The study of interior and closure algebras originated from topological Boolean algebras which generalized topological spaces given by topological interior and closure operators [10]. As the generalization of topological Boolean algebras, Rachunek had introduced interior and closure MV-algebras through the so-called multiplicative interior and additive closure operators [11]. Since the multiplicative operation and additive operation were dual in an MV-algebra, we have that there is the dual relation between multiplicative interior operators and additive closure operators on an MV-algebra. Subsequently, the concepts of multiplicative interior operators and additive closure operators were extended to commutative bounded integral residuated lattices satisfying divisibility [12], commutative residuated ℓ-monoids [13], bounded integral residuated lattices [14] and so on. As we have seen, multiplicative interior operators and additive closure operators are successfully investigated in the algebras which are related to fuzzy logic. In this paper, we want to generalize and extend the multiplicative interior operators and additive closure operators to some algebraic systems in the setting of quantum computational logic.

This paper is arranged as follows: In Section 2, some properties and results in qpBL algebras are recalled which will be used in the following. In Section 3, we give the definitions of multiplicative interior operators (mi-operators, for short), weak mi-operators, wmi-operators and weak wmi-operators on a qpBL algebra and then investigate the relations among them. Meanwhile, we discuss the related properties of these operators on a qpBL algebra and study the operators on the quotient algebra with respect to a normal weak filter. In Section 4, the concepts of additive closure operators (ac-operators, for short), weak ac-operators, sac-operators and weak sac-operators on a good qpBL algebra are introduced. The relationships among them are studied and the related properties are discussed. Moreover, we present the connections between (weak) wmi-operators and (weak) ac-operators on a good qpBL algebra. In Section 5, we discuss the properties of the induced operators on some quasi-pseudo-MV algebras.

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II. Preliminary

In this section, some definitions and results of quasi-pseudo-BL algebras are recalled.

Definition 1. [15] An algebra $(S;\Cup,\Cap)$ of type $(2,2)$ is called a *quasi-lattice*, if it satisfies the following conditions for any $\kappa,\vartheta,\tau\in S$,

(1) $\kappa\Cup\vartheta=\vartheta\Cup\kappa$ and $\kappa\Cap\vartheta=\vartheta\Cap\kappa$;

(2) $\kappa\Cup(\vartheta\Cup\tau)=(\kappa\Cup\vartheta)\Cup\tau$ and $\kappa\Cap(\vartheta\Cap\tau)=(\kappa\Cap\vartheta)\Cap\tau$;

(3) $\kappa\Cup(\vartheta\Cap\kappa)=\kappa\Cup\kappa$ and $\kappa\Cap(\vartheta\Cup\kappa)=\kappa\Cap\kappa$;

(4) $\kappa\Cup\vartheta=\kappa\Cup(\vartheta\Cup\vartheta)$ and $\kappa\Cap\vartheta=\kappa\Cap(\vartheta\Cap\vartheta)$;

(5) $\kappa\Cup\kappa=\kappa\Cap\kappa$.

On a quasi-lattice $(S;\Cup,\Cap)$, one can define a relation $\kappa\preceq\vartheta$ by $\kappa\Cup\vartheta=\vartheta\Cup\vartheta$, or equivalently, $\kappa\Cap\vartheta=\kappa\Cap\kappa$. In [15], Chajda showed that the relation $\preceq$ is *quasi-ordering*.

Definition 2. [6] An algebra $\mathbf{S}=(S;\Cup,\Cap,\boxdot,\rightharpoonup,\rightarrowtail,0,1)$ of type $(2,2,2,2,2,0,0)$ is called a *quasi-pseudo-BL algebra* (*qpBL algebra*, for short), if it satisfies the following conditions for any $\kappa,\vartheta,\tau\in S$,

(QPBL1) $(S;\Cup,\Cap,0,1)$ is a bounded quasi-lattice, i.e., $(S;\Cup,\Cap)$ is a quasi-lattice and it has the least element 0 and the largest element 1 (with respect to the quasi-ordering $\preceq$);

(QPBL2) $(S;\boxdot,1)$ is a quasi-monoid, i.e., $(\kappa\boxdot\vartheta)\boxdot\tau=\kappa\boxdot(\vartheta\boxdot\tau)$, $1\boxdot\kappa=\kappa\boxdot 1$ and $1\boxdot 1=1$;

(QPBL3) $\kappa\Cap\kappa=\kappa\boxdot 1$ and $0\Cap 0=0$;

(QPBL4) $\kappa\preceq\vartheta\rightharpoondown\tau$ iff $\kappa\boxdot\vartheta\preceq\tau$ iff $\vartheta\preceq\kappa\rightarrowtail\tau$;

(QPBL5) $(\kappa\rightharpoondown\vartheta)\boxdot 1=\kappa\rightharpoondown\vartheta$ and $(\kappa\rightarrowtail\vartheta)\boxdot 1=\kappa\rightarrowtail\vartheta$;

(QPBL6) $\kappa\Cap\vartheta=(\kappa\rightharpoondown\vartheta)\boxdot\kappa=\kappa\boxdot(\kappa\rightarrowtail\vartheta)$;

(QPBL7) $(\kappa\rightharpoondown\vartheta)\Cup(\vartheta\rightharpoondown\kappa)=(\kappa\rightarrowtail\vartheta)\Cup(\vartheta\rightarrowtail\kappa)=1$.

In a qpBL algebra **S**, we denote $R(S)=\{\kappa\in S|\kappa\boxdot 1=\kappa\}$ the set of regular elements in **S**. Then $\mathbf{R}(\mathbf{S})=(R(S);\Cup^{R(S)},\Cap^{R(S)},\boxdot^{R(S)},\rightharpoondown^{R(S)},\rightarrowtail^{R(S)},0,1)$ is a pseudo-BL subalgebra of **S**, where the operations are those of **S** restricted to $R(S)$ ([6]). Moreover, two unary operations $^{\ulcorner}$ and $^{\urcorner}$ are defined on $R(S)$: for any $\kappa\in R(S)$, $\kappa^{\ulcorner}\triangleq\kappa\rightharpoondown 0$ and $\kappa^{\urcorner}\triangleq\kappa\rightarrowtail 0$. Then the operations can be extended on **S** as follows: for any $\kappa\in S,\kappa^{\ulcorner}\in S$ with $\kappa^{\ulcorner}\boxdot 1=(\kappa\boxdot 1)^{\ulcorner}=\kappa\rightharpoondown 0$ and $\kappa^{\urcorner}\in S$ with $\kappa^{\urcorner}\boxdot 1=(\kappa\boxdot 1)^{\urcorner}=\kappa\rightarrowtail 0$.

Following from [6], a qpBL algebra **S** is,

- a *quasi-BL algebra* iff the operations "$\rightharpoondown$" and "$\rightarrowtail$" coincide iff the operation "$\boxdot$" in **S** is commutative;
- a *pseudo-BL algebra* iff $(S;\boxdot,1)$ is a monoid iff $(S;\Cup,\Cap,0,1)$ is a bounded lattice;
- a *quasi-pseudo-MV algebra* iff $\kappa^{\ulcorner\urcorner}=\kappa=\kappa^{\urcorner\ulcorner}$ for any $\kappa\in S$.

Proposition 1. [6] *Let* **S** *be a qpBL algebra. Then the following results hold for any* $\kappa,\vartheta,\tau\in S$,

(P1) *If* $\kappa\preceq\vartheta$ *and* $\vartheta\preceq\kappa$, *then* $\kappa\boxdot 1=\vartheta\boxdot 1$;

(P2) $\kappa\preceq\kappa\boxdot 1$ *and* $\kappa\boxdot 1\preceq\kappa$;

(P3) $(\kappa\rightharpoondown\vartheta)\boxdot\kappa\preceq\kappa\preceq\vartheta\rightharpoondown(\kappa\boxdot\vartheta)$ *and* $(\kappa\rightharpoondown\vartheta)\boxdot\kappa\preceq\vartheta\preceq\kappa\rightharpoondown(\vartheta\boxdot\kappa)$;

(P4) $\kappa\boxdot(\kappa\rightarrowtail\vartheta)\preceq\kappa\preceq\vartheta\rightarrowtail(\vartheta\boxdot\kappa)$ *and* $\kappa\boxdot(\kappa\rightarrowtail\vartheta)\preceq\vartheta\preceq\kappa\rightarrowtail(\kappa\boxdot\vartheta)$;

(P5) *if* $\kappa\preceq\vartheta$, *then* $\tau\rightharpoondown\kappa\preceq\tau\rightharpoondown\vartheta$ *and* $\tau\rightarrowtail\kappa\preceq\tau\rightarrowtail\vartheta$;

(P6) *if* $\kappa\preceq\vartheta$, *then* $\tau\boxdot\kappa\preceq\tau\boxdot\vartheta$ *and* $\kappa\boxdot\tau\preceq\vartheta\boxdot\tau$;

(P7) *if* $\kappa\preceq\vartheta$, *then* $\vartheta\rightharpoondown\tau\preceq\kappa\rightharpoondown\tau$ *and* $\vartheta\rightarrowtail\tau\preceq\kappa\rightarrowtail\tau$;

(P8) $\kappa\boxdot\vartheta\preceq\kappa,\vartheta$ *and* $\kappa\boxdot\vartheta\preceq\kappa\Cap\vartheta$;

(P9) $\kappa\preceq\vartheta$ *iff* $\kappa\rightharpoondown\vartheta=1$ *iff* $\kappa\rightarrowtail\vartheta=1$;

(P10) $\kappa\rightharpoondown\vartheta=(\kappa\rightharpoondown\vartheta)\boxdot 1=(\kappa\boxdot 1)\rightharpoondown\vartheta=\kappa\rightharpoondown(\vartheta\boxdot 1)$;

(P11) $\kappa\rightarrowtail\vartheta=(\kappa\rightarrowtail\vartheta)\boxdot 1=(\kappa\boxdot 1)\rightarrowtail\vartheta=\kappa\rightarrowtail(\vartheta\boxdot 1)$;

(P12) $1\rightarrowtail\kappa=\kappa\boxdot 1=1\rightharpoondown\kappa$;

(P13) $1^{\ulcorner}=1^{\urcorner}=0$ *and* $0^{\ulcorner}=0^{\urcorner}=1$;

(P14) $\kappa\preceq\kappa^{\urcorner}\rightharpoondown\vartheta$ *and* $\kappa\preceq\kappa^{\ulcorner}\rightarrowtail\vartheta$;

(P15) $\kappa\preceq\kappa^{\ulcorner\urcorner}$ *and* $\kappa\preceq\kappa^{\urcorner\ulcorner}$;

(P16) *if* $\kappa\preceq\vartheta$, *then* $\vartheta^{\ulcorner}\preceq\kappa^{\ulcorner}$ *and* $\vartheta^{\urcorner}\preceq\kappa^{\urcorner}$;

(P17) $\kappa^{\urcorner\ulcorner\urcorner}\boxdot 1=\kappa^{\urcorner}\boxdot 1$ *and* $\kappa^{\ulcorner\urcorner\ulcorner}\boxdot 1=\kappa^{\ulcorner}\boxdot 1$, *if* $\kappa\in R(S)$, *then* $\kappa^{\urcorner\ulcorner\urcorner}=\kappa^{\urcorner}$ *and* $\kappa^{\ulcorner\urcorner\ulcorner}=\kappa^{\ulcorner}$;

(P18) $\vartheta^{\urcorner}\rightharpoondown\kappa^{\urcorner}=\kappa\rightarrowtail\vartheta^{\urcorner\ulcorner}=\kappa^{\urcorner\ulcorner}\rightarrowtail\vartheta^{\urcorner\ulcorner}$ *and* $\vartheta^{\ulcorner}\rightarrowtail\kappa^{\ulcorner}=\kappa\rightharpoondown\vartheta^{\ulcorner\urcorner}=\kappa^{\ulcorner\urcorner}\rightharpoondown\vartheta^{\ulcorner\urcorner}$;

(P19) $(\kappa\boxdot\vartheta)^{\ulcorner}=\kappa\rightharpoondown\vartheta^{\ulcorner}$ *and* $(\kappa\boxdot\vartheta)^{\urcorner}=\vartheta\rightarrowtail\kappa^{\urcorner}$.

Let **S** be a qpBL algebra. If $\kappa^{\ulcorner\urcorner}=\kappa^{\urcorner\ulcorner}$ for any $\kappa\in S$, then **S** is called *good*. Given a good qpBL algebra **S**, the binary operation "$\boxplus$" can be defined on **S** as follows: $\kappa\boxplus\vartheta=\left(\kappa^{\urcorner}\boxdot\vartheta^{\urcorner}\right)^{\ulcorner}$ for any $\kappa,\vartheta\in S$.

Proposition 2. *Let* **S** *be a good qpBL algebra. Then the following results hold for any* $\kappa,\vartheta,\tau\in S$,

(G1) $(\kappa\rightharpoondown\vartheta)^{\urcorner\ulcorner}=\kappa^{\urcorner\ulcorner}\rightharpoondown\vartheta^{\urcorner\ulcorner}=\kappa\rightharpoondown\vartheta^{\urcorner\ulcorner}$ *and* $(\kappa\rightarrowtail\vartheta)^{\ulcorner\urcorner}=\kappa^{\ulcorner\urcorner}\rightarrowtail\vartheta^{\ulcorner\urcorner}=\kappa\rightarrowtail\vartheta^{\ulcorner\urcorner}$;

(G2) $(\kappa\rightharpoondown\vartheta^{\ulcorner\urcorner})^{\ulcorner\urcorner}=\kappa\rightharpoondown\vartheta^{\ulcorner\urcorner}$ *and* $(\kappa\rightarrowtail\vartheta^{\urcorner\ulcorner})^{\urcorner\ulcorner}=\kappa\rightarrowtail\vartheta^{\urcorner\ulcorner}$;

(G3) $(\kappa^{\ulcorner\urcorner}\rightharpoondown\kappa)^{\ulcorner\urcorner}=1=(\kappa^{\urcorner\ulcorner}\rightarrowtail\kappa)^{\urcorner\ulcorner}$;

(G4) $\left(\kappa^{\ulcorner}\boxdot\vartheta^{\ulcorner}\right)^{\urcorner}=\left(\kappa^{\urcorner}\boxdot\vartheta^{\urcorner}\right)^{\ulcorner}$;

(G5) $(\kappa\boxdot\vartheta)^{\urcorner\ulcorner}=\kappa^{\urcorner\ulcorner}\boxdot\vartheta^{\urcorner\ulcorner}$ *and* $(\kappa\boxdot\vartheta)^{\ulcorner\urcorner}=\kappa^{\ulcorner\urcorner}\boxdot\vartheta^{\ulcorner\urcorner}$;

(G6) $\kappa\boxplus\vartheta=\kappa^{\ulcorner\urcorner}\boxplus\vartheta^{\ulcorner\urcorner}$;

(G7) $(\kappa\boxdot\vartheta)^{\ulcorner}=\kappa^{\ulcorner}\boxplus\vartheta^{\ulcorner}$ *and* $(\kappa\boxdot\vartheta)^{\urcorner}=\kappa^{\urcorner}\boxplus\vartheta^{\urcorner}$;

(G8) $(\kappa\boxplus\vartheta)^{\ulcorner}=\kappa^{\ulcorner}\boxdot\vartheta^{\ulcorner}$ *and* $(\kappa\boxplus\vartheta)^{\urcorner}=\kappa^{\urcorner}\boxdot\vartheta^{\urcorner}$;

(G9) *if* $\kappa\preceq\vartheta$, *then* $\tau\boxplus\kappa\preceq\tau\boxplus\vartheta$ *and* $\kappa\boxplus\tau\preceq\vartheta\boxplus\tau$.

Proof: We only prove (G6), (G7), (G8) and (G9). The rest can be seen in [6].

(G6) For any $\kappa,\vartheta\in S$, we have $\kappa^{\ulcorner\urcorner}\boxplus\vartheta^{\ulcorner\urcorner}=(\kappa^{\ulcorner\urcorner\ulcorner}\boxdot\vartheta^{\ulcorner\urcorner\ulcorner})^{\urcorner}=((\kappa^{\ulcorner\urcorner\ulcorner}\boxdot 1)\boxdot(\vartheta^{\ulcorner\urcorner\ulcorner}\boxdot 1))^{\urcorner}=((\kappa^{\ulcorner}\boxdot 1)\boxdot(\vartheta^{\ulcorner}\boxdot 1))^{\urcorner}=(\kappa^{\ulcorner}\boxdot\vartheta^{\ulcorner})^{\urcorner}=\kappa\boxplus\vartheta$ by (P17).

(G7) Since $\kappa\boxdot\vartheta\in R(S)$, we have $(\kappa\boxdot\vartheta)^{\ulcorner}=(\kappa\boxdot\vartheta)^{\ulcorner\urcorner\ulcorner}$ by (P17). Thus $\kappa^{\ulcorner}\boxplus\vartheta^{\ulcorner}=(\kappa^{\ulcorner\urcorner}\boxdot\vartheta^{\ulcorner\urcorner})^{\ulcorner}=(\kappa\boxdot\vartheta)^{\ulcorner\urcorner\ulcorner}=(\kappa\boxdot\vartheta)^{\ulcorner}$ by (G5) and (P17). Similarly, we have $(\kappa\boxdot\vartheta)^{\urcorner}=\kappa^{\urcorner}\boxplus\vartheta^{\urcorner}$.

(G8) We have $(\kappa\boxplus\vartheta)^{\ulcorner}=(\kappa^{\ulcorner}\boxdot\vartheta^{\ulcorner})^{\urcorner\ulcorner}=\kappa^{\ulcorner\urcorner\ulcorner}\boxdot\vartheta^{\ulcorner\urcorner\ulcorner}=(\kappa^{\ulcorner\urcorner\ulcorner}\boxdot 1)\boxdot(\vartheta^{\ulcorner\urcorner\ulcorner}\boxdot 1)=(\kappa^{\ulcorner}\boxdot 1)\boxdot(\vartheta^{\ulcorner}\boxdot 1)=\kappa^{\ulcorner}\boxdot\vartheta^{\ulcorner}$ by (G5) and (P17). Similarly, we have $(\kappa\boxplus\vartheta)^{\urcorner}=\kappa^{\urcorner}\boxdot\vartheta^{\urcorner}$.

(G9) For any $\kappa,\vartheta\in S$ with $\kappa\preceq\vartheta$, we have $\tau\boxplus\kappa=(\tau^{\urcorner}\boxdot\kappa^{\urcorner})^{\ulcorner}\preceq(\tau^{\urcorner}\boxdot\vartheta^{\urcorner})^{\ulcorner}=\tau\boxplus\vartheta$ by (P16) and (P6). Similarly, we have $\kappa\boxplus\tau\preceq\vartheta\boxplus\tau$. ■

Below we list the related properties of filters in qpBL algebras.

Definition 3. [6] Let **S** be a qpBL algebra. A non-empty subset T of S is called a *filter* of **S**, if the following conditions hold,

(F1) $\kappa,\vartheta\in T$ implies $\kappa\boxdot\vartheta\in T$;

(F2) $\kappa\in T$ and $\vartheta\in S$ with $\kappa\preceq\vartheta$ imply $\vartheta\in T$.

Definition 4. [16] Let **S** be a qpBL algebra. A non-empty subset T of S is called a *weak filter* of **S**, if the following conditions hold,

(WF1) $\kappa,\vartheta\in T$ implies $\kappa\boxdot\vartheta\in T$;

(WF2) $\kappa\in T$ and $\vartheta\in S$ with $\kappa\preceq\vartheta$ imply $\vartheta\boxdot 1\in T$.

Remark 1. Let **S** be a qpBL algebra and T be a filter of **S**. If $\kappa\in T$ and $\vartheta\in S$ with $\kappa\preceq\vartheta$, then $\kappa\preceq\vartheta\preceq\vartheta\boxdot 1$, it follows that $\vartheta\boxdot 1\in T$, so every filter is a weak filter.

Let **S** be a qpBL algebra. If T is a (weak) filter of **S** and $\kappa\rightharpoondown\vartheta\in T$ iff $\kappa\rightarrowtail\vartheta\in T$ for any $\kappa,\vartheta\in S$, then T is called a *normal* (weak) filter of **S**. If χ is a congruence on **S** and for any $\kappa,\vartheta\in S$, $\langle\kappa\boxdot 1,\vartheta\boxdot 1\rangle\in\chi$ implies $\langle\kappa,\vartheta\rangle\in\chi$, then χ is called a *filter congruence* on **S**. Let χ be a (filter) congruence on **S** and D be a normal (weak) filter of **S**. Define the set $D_\chi=\{\kappa\in S|\langle\kappa,1\rangle\in\chi\}$ and the relation $\chi_D=\left\{\langle\kappa,\vartheta\rangle\in S^2|\kappa\rightharpoondown\vartheta\in D\text{ and }\vartheta\rightharpoondown\kappa\in D\right\}$.

Proposition 3. [6] *Let* **S** *be a qpBL algebra,* χ *be a filter congruence on* **S** *and* D *be a normal filter of* **S**. *Then we have,*

(1) D_χ *is a normal filter of* **S**;

(2) χ_D *is a filter congruence on* $\mathbf{S}$*;*
(3) $\chi = \chi_{D_\chi}$*;*
(4) $D = D_{\chi_D}$.

Proposition 4. [16] *Let* $\mathbf{S}$ *be a qpBL algebra,* χ *be a congruence on* $\mathbf{S}$ *and* D *be a normal weak filter of* $\mathbf{S}$*. Then we have,*
(1) D_χ *is a normal weak filter of* $\mathbf{S}$*;*
(2) χ_D *is a congruence on* $\mathbf{S}$*;*
(3) $\chi \subseteq \chi_{D_\chi}$*;*
(4) $D \subseteq D_{\chi_D}$.

If D is a normal (weak) filter of a qpBL algebra $\mathbf{S}$, we consider the quotient set $S/D = \{\kappa/D | \kappa \in S\}$ where $\kappa/D = \{\vartheta \in S | \kappa \rightharpoondown \vartheta \in D \textit{ and } \vartheta \rightharpoondown \kappa \in D\}$ or equivalently, $\kappa/D = \{\vartheta \in S | \kappa \rightarrowtail \vartheta \in D \textit{ and } \vartheta \rightarrowtail \kappa \in D\}$. On the set S/D, we define $(\kappa/D) \boxdot (\vartheta/D) = (\kappa \boxdot \vartheta)/D$, $(\kappa/D) \Cup (\vartheta/D) = (\kappa \Cup \vartheta)/D$, $(\kappa/D) \Cap (\vartheta/D) = (\kappa \Cap \vartheta)/D$, $(\kappa/D) \rightharpoondown (\vartheta/D) = (\kappa \rightharpoondown \vartheta)/D$ and $(\kappa/D) \rightarrowtail (\vartheta/D) = (\kappa \rightarrowtail \vartheta)/D$. Then $\mathbf{S/D} = \{S/D; \Cup, \Cap, \boxdot, \rightharpoondown, \rightarrowtail, 0/D, 1/D\}$ is a pseudo-BL algebra ([6], [16]). Moreover, we have $(\kappa/D)^{\ulcorner} = \kappa^{\ulcorner}/D$ and $(\kappa/D)^{\urcorner} = \kappa^{\urcorner}/D$ for any $\kappa \in S$.

Let $\mathbf{S}$ be a qpBL algebra and denote $D(S) = \{\kappa \in S | \kappa^{\ulcorner\urcorner} = 1 = \kappa^{\urcorner\ulcorner}\}$ the set of dense elements of S.

Proposition 5. *Let* $\mathbf{S}$ *be a good qpBL algebra. Then* $D(S)$ *is a normal weak filter of* $\mathbf{S}$.

Proof: Clearly $1 \in D(S)$. Let $\kappa, \vartheta \in D(S)$. Then $\kappa^{\ulcorner\urcorner} = 1 = \vartheta^{\ulcorner\urcorner}$, it follows that $(\kappa \boxdot \vartheta)^{\ulcorner\urcorner} = \kappa^{\ulcorner\urcorner} \boxdot \vartheta^{\ulcorner\urcorner} = 1 \boxdot 1 = 1$ by (G5), so $\kappa \boxdot \vartheta \in D(S)$. If $\kappa \in D(S)$ and $\tau \in S$ with $\kappa \preceq \tau$, then $1 = \kappa^{\ulcorner\urcorner} \preceq \tau^{\ulcorner\urcorner}$ by (P16), so $(\tau \boxdot 1)^{\ulcorner\urcorner} = \tau^{\ulcorner\urcorner} \boxdot 1 = 1 \boxdot 1 = 1$ and then $\tau \boxdot 1 \in D(S)$. Therefore $D(S)$ is a weak filter of $\mathbf{S}$. In addition, let $\kappa, \vartheta \in S$ and $\kappa \rightharpoondown \vartheta \in D(S)$. Then $(\kappa \rightharpoondown \vartheta)^{\ulcorner\urcorner} = 1$, it follows that $\kappa^{\ulcorner\urcorner} \rightharpoondown \vartheta^{\ulcorner\urcorner} = 1$ by (G1), so $\kappa^{\ulcorner\urcorner} \preceq \vartheta^{\ulcorner\urcorner}$ by (P9). Since $\mathbf{S}$ is good, we also have $(\kappa \rightarrowtail \vartheta)^{\ulcorner\urcorner} = \kappa^{\ulcorner\urcorner} \rightarrowtail \vartheta^{\ulcorner\urcorner} = 1$ by (G1) and (P9), so $\kappa \rightarrowtail \vartheta \in D(S)$. Similarly, we can prove that $\kappa \rightarrowtail \vartheta \in D(S)$ implies $\kappa \rightharpoondown \vartheta \in D(S)$. Hence $D(S)$ is a normal weak filter of $\mathbf{S}$. ■

Lemma 1. *Let* $\mathbf{S}$ *be a good qpBL algebra. Then for any* $\kappa \in S$*,* $\kappa \rightharpoondown \kappa^{\ulcorner\urcorner} \in D(S)$ *and* $\kappa^{\ulcorner\urcorner} \rightharpoondown \kappa \in D(S)$.

Proof: Since $\kappa \preceq \kappa^{\ulcorner\urcorner}$ and $1 \in D(S)$, we have $\kappa \rightharpoondown \kappa^{\ulcorner\urcorner} = 1 \in D(S)$ by (P9). In addition, we have $(\kappa^{\ulcorner\urcorner} \rightharpoondown \kappa)^{\ulcorner\urcorner} = 1$ by (G3), so $\kappa^{\ulcorner\urcorner} \rightharpoondown \kappa \in D(S)$. ■

Proposition 6. *Let* $\mathbf{S}$ *be a good qpBL algebra. Then* $\mathbf{S}/\mathbf{D}(\mathbf{S})$ *is a pseudo-MV algebra.*

Proof: Since $D(S)$ is a normal weak filter of $\mathbf{S}$ by Proposition 5, we have that $\mathbf{S}/\mathbf{D}(\mathbf{S})$ is a pseudo-BL algebra. By Lemma 1, we have $\kappa/D(S) = \kappa^{\ulcorner\urcorner}/D(S) = (\kappa/D(S))^{\ulcorner\urcorner}$. Moreover, since $\mathbf{S}$ is good, we have $\kappa^{\ulcorner\urcorner} = \kappa^{\urcorner\ulcorner}$, it follows that $\kappa/D(S) = (\kappa/D(S))^{\ulcorner\urcorner} = (\kappa/D(S))^{\urcorner\ulcorner}$. Hence $\mathbf{S}/\mathbf{D}(\mathbf{S})$ is a pseudo-MV algebra. ■

III. Interior Operators on Quasi-pseudo-BL Algebras

In this section, the definitions of multiplicative interior operators, weak mi-operators, wmi-operators and weak wmi-operators on qpBL algebras are given. We study the properties of these operators and discuss the relationship among them.

Definition 5. Let $(S; \Cup, \Cap)$ be a quasi-lattice and $\Gamma : S \longrightarrow S$ be a mapping. Then Γ is called an *interior operator* on $(S; \Cup, \Cap)$, if for any $\kappa, \vartheta \in S$,
(I1) $\Gamma(\kappa) \preceq \kappa$;
(I2) $\Gamma(\Gamma(\kappa)) = \Gamma(\kappa)$;
(I3) $\kappa \preceq \vartheta$ implies $\Gamma(\kappa) \preceq \Gamma(\vartheta)$.

Definition 6. Let $\mathbf{S}$ be a qpBL algebra. A mapping $\Gamma : S \longrightarrow S$ is called a *multiplicative interior operator* (*mi-operator*, for short) on $\mathbf{S}$, if for any $\kappa, \vartheta \in S$,
(MI1) $\Gamma(\kappa \boxdot \vartheta) = \Gamma(\kappa) \boxdot \Gamma(\vartheta)$;
(MI2) $\Gamma(\kappa) \preceq \kappa$;
(MI3) $\Gamma(\Gamma(\kappa)) = \Gamma(\kappa)$;
(MI4) $\Gamma(1) = 1$.

A qpBL algebra $\mathbf{S}$ having an mi-operator Γ is called an *interior qpBL algebra* and denoted by $(\mathbf{S}, \Gamma)$.

Definition 7. Let $\mathbf{S}$ be a qpBL algebra. A mapping $\Gamma : S \longrightarrow S$ is called a *weak multiplicative interior operator* (*weak mi-operator*, for short) on $\mathbf{S}$, if for any $\kappa, \vartheta \in S$,
(WEMI1) $\Gamma(\kappa \boxdot \vartheta) = \Gamma(\kappa) \boxdot \Gamma(\vartheta)$;
(WEMI2) $\Gamma(\kappa) \preceq \kappa$;
(WEMI3) $\Gamma(\Gamma(\kappa)) \boxdot 1 = \Gamma(\kappa) \boxdot 1$;
(WEMI4) $\Gamma(1) = 1$.

Proposition 7. *Let* $(\mathbf{S}, \Gamma)$ *be an interior qpBL algebra. Then* Γ *is a weak mi-operator on* $\mathbf{S}$.

However, if Γ is a weak mi-operator on a qpBL algebra $\mathbf{S}$, then it is not an mi-operator in general.

Example 1. Let $S = \{0, o, \varpi, \upsilon, 1\}$. We define the operations on S as follows:

$\Cup$	0	o	ϖ	υ	1
0	0	0	ϖ	ϖ	1
o	0	0	ϖ	ϖ	1
ϖ	ϖ	ϖ	ϖ	ϖ	1
υ	ϖ	ϖ	ϖ	ϖ	1
1	1	1	1	1	1

$\Cap$	0	o	ϖ	υ	1
0	0	0	0	0	0
o	0	0	0	0	0
ϖ	0	0	ϖ	ϖ	ϖ
υ	0	0	ϖ	ϖ	ϖ
1	0	0	ϖ	ϖ	1

$\boxdot$	0	o	ϖ	υ	1
0	0	0	0	0	0
o	0	0	0	0	0
ϖ	0	0	0	0	ϖ
υ	0	0	0	0	ϖ
1	0	0	ϖ	ϖ	1

$\rightharpoondown$	0	o	ϖ	υ	1
0	1	1	1	1	1
o	1	1	1	1	1
ϖ	ϖ	ϖ	1	1	1
υ	ϖ	ϖ	1	1	1
1	0	0	ϖ	ϖ	1

Then $\mathbf{S} = (S; \Cup, \Cap, \boxdot, \rightharpoondown, 0, 1)$ is a quasi-(pseudo)-BL algebra. Define the mapping $\Gamma : S \longrightarrow S$ by $\Gamma(0) = 0$, $\Gamma(o) = 0$, $\Gamma(\varpi) = 0$, $\Gamma(\upsilon) = o$ and $\Gamma(1) = 1$. Then the mapping Γ is a weak mi-operator on $\mathbf{S}$. However, $\Gamma(\Gamma(\upsilon)) = 0$ and $\Gamma(\upsilon) = o$, we have $\Gamma(\Gamma(\upsilon)) \neq \Gamma(\upsilon)$. Hence Γ is not an mi-operator on $\mathbf{S}$.

Definition 8. Let $\mathbf{S}$ be a qpBL algebra. A mapping $\Gamma : S \longrightarrow S$ is called a *wmi-operator* on $\mathbf{S}$, if for any $\kappa, \vartheta \in S$,
(WMI1) $\Gamma(\kappa \boxdot \vartheta) = \Gamma(\kappa) \boxdot \Gamma(\vartheta)$;
(WMI2) $\Gamma(\kappa) \preceq \kappa^{\ulcorner\urcorner}$ or $\Gamma(\kappa) \preceq \kappa^{\urcorner\ulcorner}$;
(WMI3) $\Gamma(\Gamma(\kappa)) = \Gamma(\kappa)$;
(WMI4) $\Gamma(1) = 1$.

Definition 9. Let $\mathbf{S}$ be a qpBL algebra. A mapping $\Gamma : S \longrightarrow S$ is called a *weak wmi-operator* on $\mathbf{S}$, if for any $\kappa, \vartheta \in S$,
(WWMI1) $\Gamma(\kappa \boxdot \vartheta) = \Gamma(\kappa) \boxdot \Gamma(\vartheta)$;
(WWMI2) $\Gamma(\kappa) \preceq \kappa^{\ulcorner\urcorner}$ or $\Gamma(\kappa) \preceq \kappa^{\urcorner\ulcorner}$;

(WWMI3) $\Gamma(\Gamma(\kappa))\boxdot 1=\Gamma(\kappa)\boxdot 1$;
(WWMI4) $\Gamma(1)=1$.

Remark 2. Let $\mathbf{S}$ be a pseudo-BL algebra. Then wmi-operators and weak wmi-operators are same.

Proposition 8. *Let Γ be a wmi-operator on a qpBL algebra $\mathbf{S}$. Then Γ is a weak wmi-operator on $\mathbf{S}$.*

However, if Γ is a weak wmi-operator on a qpBL algebra $\mathbf{S}$, then it is not a wmi-operator in general.

Example 2. Let $\mathbf{S}=(S;\boxdot,\Cup,\Cap,\rightharpoondown,0,1)$ be a quasi-(pseudo)-BL algebra defined in Example 1. For any element in S, we define the unary operation on $\mathbf{S}$ as follows:

$'$	
0	1
o	1
ϖ	ϖ
υ	υ
1	0

Then Γ defined in Example 1 is a weak wmi-operator on $\mathbf{S}$. According to Example 1, we have $\Gamma(\Gamma(\upsilon))\neq\Gamma(\upsilon)$. Hence Γ is not a wmi-operator on $\mathbf{S}$.

Since $\kappa\preceq\kappa^{\ulcorner\urcorner}$ and $\kappa\preceq\kappa^{\urcorner\ulcorner}$ hold in any qpBL algebra $\mathbf{S}$, we have that any mi-operator is the wmi-operator and any weak mi-operator is the weak wmi-operator on $\mathbf{S}$. The relationship among these operators can be seen in Fig. 1.

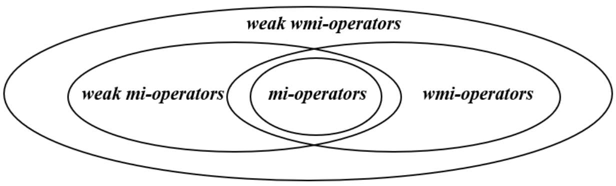

Fig. 1. A Relational Diagram

Lemma 2. *Let Γ be a weak wmi-operator on a qpBL algebra $\mathbf{S}$. Then $\Gamma(\Gamma(\kappa))=\Gamma(\kappa)$ for any $\kappa\in R(S)$.*

Lemma 3. *Let Γ be a weak wmi-operator on a qpBL algebra $\mathbf{S}$. Then Γ is monotone.*

Proof: For any $\kappa,\vartheta\in S$ with $\kappa\preceq\vartheta$, we have $\kappa\Cap\vartheta=\kappa\boxdot 1$ and $\kappa\Cap\vartheta=\vartheta\boxdot(\vartheta\rightarrowtail\kappa)$ by (QPBL3) and (QPBL6), it follows that $\Gamma(\kappa)\preceq\Gamma(\kappa)\boxdot 1=\Gamma(\kappa)\boxdot\Gamma(1)=\Gamma(\kappa\boxdot 1)=\Gamma(\kappa\Cap\vartheta)=\Gamma(\vartheta\boxdot(\vartheta\rightarrowtail\kappa))=\Gamma(\vartheta)\boxdot\Gamma(\vartheta\rightarrowtail\kappa)\preceq\Gamma(\vartheta)\boxdot 1\preceq\Gamma(\vartheta)$ by (P2), (WWMI4), (WWMI1) and (P6), so $\Gamma(\kappa)\preceq\Gamma(\vartheta)$. ■

According to Lemma 3, we have the following results.

Proposition 9. *Let Γ be a weak wmi-operator on a qpBL algebra $\mathbf{S}$. Then $\Gamma(\kappa\Cap\vartheta)\preceq\Gamma(\kappa)\Cap\Gamma(\vartheta)$ and $\Gamma(\kappa)\Cup\Gamma(\vartheta)\preceq\Gamma(\kappa\Cup\vartheta)$ for any $\kappa,\vartheta\in S$.*

Proposition 10. *Let $(\mathbf{S},\Gamma)$ be an interior qpBL algebra. Then Γ is an interior operator on $(S;\Cup,\Cap)$.*

Proposition 11. *Let Γ be a weak wmi-operator on a qpBL algebra $\mathbf{S}$. Then $\Gamma(\kappa\rightharpoondown\vartheta)\preceq\Gamma(\kappa)\rightharpoondown\Gamma(\vartheta)$ and $\Gamma(\kappa\rightarrowtail\vartheta)\preceq\Gamma(\kappa)\rightarrowtail\Gamma(\vartheta)$ for any $\kappa,\vartheta\in S$.*

Proof: For any $\kappa,\vartheta\in S$, since $\kappa\boxdot(\kappa\rightarrowtail\vartheta)\preceq\vartheta$, we have $\Gamma(\kappa)\boxdot\Gamma(\kappa\rightarrowtail\vartheta)\preceq\Gamma(\vartheta)$ by (WWMI1) and Lemma 3, so $\Gamma(\kappa\rightarrowtail\vartheta)\preceq\Gamma(\kappa)\rightarrowtail\Gamma(\vartheta)$ by (QPBL4). Similarly, we have $\Gamma(\kappa\rightharpoondown\vartheta)\preceq\Gamma(\kappa)\rightharpoondown\Gamma(\vartheta)$. ■

Let $\mathbf{S}$ be a qpBL algebra and $\Gamma:S\longrightarrow S$ be a mapping. We define two mappings $\Gamma_{\llcorner}^{\urcorner}:S\longrightarrow S$ by $\Gamma_{\llcorner}^{\urcorner}(\kappa)=\left(\Gamma\left(\kappa^{\ulcorner}\right)\right)^{\urcorner}$ and $\Gamma_{\lrcorner}^{\ulcorner}:S\longrightarrow S$ by $\Gamma_{\lrcorner}^{\ulcorner}(\kappa)=\left(\Gamma\left(\kappa^{\urcorner}\right)\right)^{\ulcorner}$ for any $\kappa\in S$.

Proposition 12. *If $\Gamma:S\longrightarrow S$ is a monotone mapping on a qpBL algebra $\mathbf{S}$, then the mappings $\Gamma_{\llcorner}^{\urcorner}$ and $\Gamma_{\lrcorner}^{\ulcorner}$ are monotone.*

Proof: Let $\kappa,\vartheta\in S$ with $\kappa\preceq\vartheta$. Then we have $\vartheta^{\ulcorner}\boxdot 1=\vartheta\rightharpoondown 0\preceq\kappa\rightharpoondown 0=\kappa^{\ulcorner}\boxdot 1$ by (P7), so $\vartheta^{\ulcorner}\preceq\vartheta^{\ulcorner}\boxdot 1\preceq\kappa^{\ulcorner}\boxdot 1\preceq\kappa^{\ulcorner}$ by (P2) and then $\Gamma(\vartheta^{\ulcorner})\preceq\Gamma\left(\vartheta^{\ulcorner}\boxdot 1\right)\preceq\Gamma\left(\kappa^{\ulcorner}\boxdot 1\right)\preceq\Gamma(\kappa^{\ulcorner})$. Hence $\Gamma_{\llcorner}^{\urcorner}(\kappa)=(\Gamma(\kappa^{\ulcorner}))^{\urcorner}\preceq\left(\Gamma\left(\kappa^{\ulcorner}\boxdot 1\right)\right)^{\urcorner}\preceq\left(\Gamma\left(\vartheta^{\ulcorner}\boxdot 1\right)\right)^{\urcorner}\preceq(\Gamma(\vartheta^{\ulcorner}))^{\urcorner}=\Gamma_{\llcorner}^{\urcorner}(\vartheta)$ by (P16). Analogously for $\Gamma_{\lrcorner}^{\ulcorner}$. ■

Proposition 13. *Let $\Gamma:S\longrightarrow S$ be a weak wmi-operator on a qpBL algebra $\mathbf{S}$. Then for any $\kappa,\vartheta\in S$ we have,*
(1) *$\kappa^{\ulcorner\urcorner}\preceq\Gamma_{\llcorner}^{\urcorner}(\kappa)$ or $\kappa^{\urcorner\ulcorner}\preceq\Gamma_{\lrcorner}^{\ulcorner}(\kappa)$;*
(2) *$\Gamma_{\llcorner}^{\urcorner}(\kappa\Cap\vartheta)\preceq\Gamma_{\llcorner}^{\urcorner}(\kappa)\Cap\Gamma_{\llcorner}^{\urcorner}(\vartheta)$ and $\Gamma_{\lrcorner}^{\ulcorner}(\kappa\Cap\vartheta)\preceq\Gamma_{\lrcorner}^{\ulcorner}(\kappa)\Cap\Gamma_{\lrcorner}^{\ulcorner}(\vartheta)$;*
(3) *$\Gamma_{\llcorner}^{\urcorner}(\kappa)\Cup\Gamma_{\llcorner}^{\urcorner}(\vartheta)\preceq\Gamma_{\llcorner}^{\urcorner}(\kappa\Cup\vartheta)$ and $\Gamma_{\lrcorner}^{\ulcorner}(\kappa)\Cup\Gamma_{\lrcorner}^{\ulcorner}(\vartheta)\preceq\Gamma_{\lrcorner}^{\ulcorner}(\kappa\Cup\vartheta)$;*
(4) *$\Gamma_{\llcorner}^{\urcorner}(0)=0$ and $\Gamma_{\lrcorner}^{\ulcorner}(0)=0$.*

Proof: (1) If $\Gamma(\kappa)\preceq\kappa^{\urcorner\ulcorner}$ for any $\kappa\in S$, we have $\Gamma(\kappa^{\ulcorner})\preceq(\kappa^{-})^{\urcorner\ulcorner}$, so $\kappa^{\ulcorner\urcorner}\preceq\kappa^{\ulcorner\urcorner}\boxdot 1=(\kappa^{\ulcorner}\boxdot 1)^{\urcorner}=(\kappa^{\ulcorner\urcorner\ulcorner}\boxdot 1)^{\urcorner}\preceq\kappa^{-\sim-\sim}=(\kappa^{\ulcorner})^{\urcorner\ulcorner\urcorner}\preceq(\Gamma(\kappa^{\ulcorner}))^{\urcorner}=\Gamma_{\llcorner}^{\urcorner}(\kappa)$ by (P2), (P17) and (P16). If $\Gamma(\kappa)\preceq\kappa^{\ulcorner\urcorner}$, then we can show $\kappa^{\urcorner\ulcorner}\preceq\Gamma_{\lrcorner}^{\ulcorner}(\kappa)$.

(2) and (3) Follow from Proposition 9.

(4) We have $\Gamma_{\llcorner}^{\urcorner}(0)=\left(\Gamma\left(0^{\ulcorner}\right)\right)^{\urcorner}=(\Gamma(1))^{\urcorner}=1^{\urcorner}=0$ by (P13). Analogously for $\Gamma_{\lrcorner}^{\ulcorner}(0)=0$. ■

Below we use the notion $(\mathbf{S},\Gamma)$ to represent a qpBL algebra $\mathbf{S}$ with a weak wmi-operator Γ. If T is a (weak) filter of $\mathbf{S}$ and $\kappa\in T$ implies $\Gamma(\kappa)\in T$, then T is called a (*weak*) *Γ-filter* of $(\mathbf{S},\Gamma)$. If χ is a congruence on $\mathbf{S}$ and $\langle\kappa,\vartheta\rangle\in\chi$ implies $\langle\Gamma(\kappa),\Gamma(\vartheta)\rangle\in\chi$, then χ is called a *congruence* on $(\mathbf{S},\Gamma)$. According to Proposition 3 and Proposition 4, we have the following results.

Proposition 14. *Let $(\mathbf{S},\Gamma)$ be a qpBL algebra $\mathbf{S}$ with a weak wmi-operator Γ, χ be a filter congruence on $(\mathbf{S},\Gamma)$ and D be a normal Γ-filter of $(\mathbf{S},\Gamma)$. Then we have,*
(1) *D_{χ} is a normal Γ-filter of $(\mathbf{S},\Gamma)$;*
(2) *χ_D is a filter congruence on $(\mathbf{S},\Gamma)$;*
(3) *$\chi=\chi_{D_{\chi}}$;*
(4) *$D=D_{\chi_D}$.*

Proposition 15. *Let $(\mathbf{S},\Gamma)$ be a qpBL algebra $\mathbf{S}$ with a weak wmi-operator Γ, χ be a congruence on $(\mathbf{S},\Gamma)$ and D be a normal weak Γ-filter of $(\mathbf{S},\Gamma)$. Then we have,*
(1) *D_{χ} is a normal weak Γ-filter of $(\mathbf{S},\Gamma)$;*
(2) *χ_D is a congruence on $(\mathbf{S},\Gamma)$;*
(3) *$\chi\subseteq\chi_{D_{\chi}}$;*
(4) *$D\subseteq D_{\chi_D}$.*

Let $(\mathbf{S},\Gamma)$ be a qpBL algebra $\mathbf{S}$ with a weak wmi-operator Γ and D be a normal weak Γ-filter of $(\mathbf{S},\Gamma)$. Then $\mathbf{S}/\mathbf{D}$ is a pseudo-BL algebra. Define $\tilde{\Gamma}:S/D\longrightarrow S/D$ by $\tilde{\Gamma}(\kappa/D)=\Gamma(\kappa)/D$ for any $\kappa\in S$. We can show the following results.

Theorem 1. *Let $(\mathbf{S},\Gamma)$ be a qpBL algebra $\mathbf{S}$ with a weak wmi-operator Γ and D be a normal weak Γ-filter of $(\mathbf{S},\Gamma)$.*

Then $\tilde{\Gamma}$ is a wmi-operator on $\mathbf{S}/\mathbf{D}$ and $(\mathbf{S}/\mathbf{D},\tilde{\Gamma})$ is a pseudo-BL algebra with a wmi-operator $\tilde{\Gamma}$.

Proof: We check the conditions one by one. For any $\kappa,\vartheta\in S$:

(WMI1) We have $\tilde{\Gamma}((\kappa/D)\boxdot(\vartheta/D))=\tilde{\Gamma}((\kappa\boxdot\vartheta)/D)=\Gamma(\kappa\boxdot\vartheta)/D=(\Gamma(\kappa)\boxdot\Gamma(\vartheta))/D=(\Gamma(\kappa)/D)\boxdot(\Gamma(\vartheta)/D)=\tilde{\Gamma}(\kappa/D)\boxdot\tilde{\Gamma}(\vartheta/D)$.

(WMI2) We have $\tilde{\Gamma}(\kappa/D)=\Gamma(\kappa)/D\preceq\kappa^{\ulcorner\urcorner}/D=(\kappa/D)^{\ulcorner\urcorner}$ or $\tilde{\Gamma}(\kappa/D)=\Gamma(\kappa)/D\preceq\kappa^{\urcorner\ulcorner}/D=(\kappa/D)^{\urcorner\ulcorner}$.

(WMI3) We have $\tilde{\Gamma}(\tilde{\Gamma}(\kappa/D))=\tilde{\Gamma}(\Gamma(\kappa)/D)=\Gamma(\Gamma(\kappa))/D=(\Gamma(\Gamma(\kappa))/D)\boxdot(1/D)=(\Gamma(\Gamma(\kappa))\boxdot 1)/D=(\Gamma(\kappa)\boxdot 1)/D=(\Gamma(\kappa)/D)\boxdot(1/D)=\Gamma(\kappa)/D=\tilde{\Gamma}(\kappa/D)$.

(WMI4) We have $\tilde{\Gamma}(1/D)=\Gamma(1)/D=1/D$.

Thus $\tilde{\Gamma}$ is a wmi-operator on $\mathbf{S}/\mathbf{D}$ and $(\mathbf{S}/\mathbf{D},\tilde{\Gamma})$ is a pseudo-BL algebra with a wmi-operator $\tilde{\Gamma}$. ■

Corollary 1. *Let $(\mathbf{S},\Gamma)$ be an interior qpBL algebra and D be a normal weak Γ-filter of $(\mathbf{S},\Gamma)$. Then $\tilde{\Gamma}$ is an mi-operator on $\mathbf{S}/\mathbf{D}$ and $(\mathbf{S}/\mathbf{D},\tilde{\Gamma})$ is an interior pseudo-BL algebra.*

IV. Closure Operators on Good Quasi-pseudo-BL Algebras

In this section, the definitions of additive closure operators, weak ac-operators, sac-operators and weak sac-operators on good qpBL algebras are given. We study their related properties and discuss the relationship among them. Moreover, we present the relations between (weak) wmi-operators and (weak) ac-operators on good qpBL algebras.

Definition 10. Let $\mathbf{S}$ be a good qpBL algebra. A mapping $\Upsilon: S\longrightarrow S$ is called an *additive closure operator* (*ac-operator*, for short) on $\mathbf{S}$, if for any $\kappa,\vartheta\in S$,

(AC1) $\Upsilon(\kappa\boxplus\vartheta)=\Upsilon(\kappa)\boxplus\Upsilon(\vartheta)$;

(AC2) $\kappa\preceq\Upsilon(\kappa)$;

(AC3) $\Upsilon(\Upsilon(\kappa))=\Upsilon(\kappa)$;

(AC4) $\Upsilon(0)=0$.

A good qpBL algebra $\mathbf{S}$ having an ac-operator Υ is called a *closure qpBL algebra* and denoted by $(\mathbf{S},\Upsilon)$.

Definition 11. Let $\mathbf{S}$ be a good qpBL algebra. A mapping $\Upsilon: S\longrightarrow S$ is called a *weak additive closure operator* (*weak ac-operator*, for short) on $\mathbf{S}$, if for any $\kappa,\vartheta\in S$,

(WAC1) $\Upsilon(\kappa\boxplus\vartheta)=\Upsilon(\kappa)\boxplus\Upsilon(\vartheta)$;

(WAC2) $\kappa\preceq\Upsilon(\kappa)$;

(WAC3) $\Upsilon(\Upsilon(\kappa))\boxplus 0=\Upsilon(\kappa)\boxplus 0$;

(WAC4) $\Upsilon(0)=0$.

Proposition 16. *Let $(\mathbf{S},\Upsilon)$ be a closure qpBL algebra. Then Υ is a weak ac-operator on $\mathbf{S}$.*

However, if Υ is a weak ac-operator on a qpBL algebra $\mathbf{S}$, then it is not an ac-operator in general.

Example 3. Let $S=\{0,o,\varpi,\upsilon,1\}$. We define the operations on S as follows:

$\Cup$	0	o	ϖ	υ	1
0	0	ϖ	ϖ	ϖ	1
o	ϖ	ϖ	ϖ	ϖ	1
ϖ	ϖ	ϖ	ϖ	ϖ	1
υ	ϖ	ϖ	ϖ	ϖ	1
1	1	1	1	1	1

$\Cap$	0	o	ϖ	υ	1
0	0	0	0	0	0
o	0	ϖ	ϖ	ϖ	ϖ
ϖ	0	ϖ	ϖ	ϖ	ϖ
υ	0	ϖ	ϖ	ϖ	ϖ
1	0	ϖ	ϖ	ϖ	1

$\boxdot$	0	o	ϖ	υ	1
0	0	0	0	0	0
o	0	0	0	0	ϖ
ϖ	0	0	0	0	ϖ
υ	0	0	0	0	ϖ
1	0	ϖ	ϖ	ϖ	1

$\rightharpoondown$	0	o	ϖ	υ	1
0	1	1	1	1	1
o	ϖ	1	1	1	1
ϖ	ϖ	1	1	1	1
υ	ϖ	1	1	1	1
1	0	ϖ	ϖ	ϖ	1

Then $\mathbf{S}=(S;\Cup,\Cap,\boxdot,\rightharpoondown,0,1)$ is a quasi-(pseudo)-BL algebra. For any element in S, we define the unary operation on $\mathbf{S}$ as follows:

$'$	
0	1
o	υ
ϖ	ϖ
υ	o
1	0

Then $\mathbf{S}$ is a good quasi-(pseudo)-BL algebra. We define $\kappa\boxplus\vartheta=(\kappa'\boxdot\vartheta')'$ for any $\kappa,\vartheta\in S$ and define $\Upsilon: S\longrightarrow S$ by $\Upsilon(0)=0$, $\Upsilon(o)=\upsilon$, $\Upsilon(\varpi)=\varpi$, $\Upsilon(\upsilon)=o$ and $\Upsilon(1)=1$. Then the mapping Υ is a weak ac-operator on $\mathbf{S}$. However, $\Upsilon(\Upsilon(\upsilon))=\upsilon$ and $\Upsilon(\upsilon)=o$, we have $\Upsilon(\Upsilon(\upsilon))\neq\Upsilon(\upsilon)$. Hence Υ is not an ac-operator on $\mathbf{S}$.

Definition 12. Let $\mathbf{S}$ be a good qpBL algebra. A mapping $\Upsilon: S\longrightarrow S$ is called an *sac-operator* on $\mathbf{S}$, if for any $\kappa,\vartheta\in S$,

(SAC1) $\Upsilon(\kappa\boxplus\vartheta)=\Upsilon(\kappa)\boxplus\Upsilon(\vartheta)$;

(SAC2) $\kappa^{\ulcorner\urcorner}\preceq\Upsilon(\kappa)$;

(SAC3) $\Upsilon(\Upsilon(\kappa))=\Upsilon(\kappa)$;

(SAC4) $\Upsilon(0)=0$.

Definition 13. Let $\mathbf{S}$ be a good qpBL algebra. A mapping $\Upsilon: S\longrightarrow S$ is called a *weak sac-operator* on $\mathbf{S}$, if for any $\kappa,\vartheta\in S$,

(WSAC1) $\Upsilon(\kappa\boxplus\vartheta)=\Upsilon(\kappa)\boxplus\Upsilon(\vartheta)$;

(WSAC2) $\kappa^{\ulcorner\urcorner}\preceq\Upsilon(\kappa)$;

(WSAC3) $\Upsilon(\Upsilon(\kappa))\boxplus 0=\Upsilon(\kappa)\boxplus 0$;

(WSAC4) $\Upsilon(0)=0$.

Remark 3. If $\mathbf{S}$ is a good qpBL algebra, then $\kappa^{\urcorner\ulcorner}=\kappa^{\ulcorner\urcorner}$ for any $\kappa\in S$, so $\kappa^{\ulcorner\urcorner}\preceq\Upsilon(\kappa)$ is equivalent to $\kappa^{\urcorner\ulcorner}\preceq\Upsilon(\kappa)$.

We have $\Upsilon(1)=\Upsilon(1\boxplus 0)=\Upsilon(1)\boxplus\Upsilon(0)=\Upsilon(1)\boxplus 0$, so $\Upsilon(1)$ is regular. Moreover, we have $1=1^{\ulcorner\urcorner}\preceq\Upsilon(1)$ and $\Upsilon(1)\preceq 1$, so $\Upsilon(1)=1$.

Proposition 17. *Let Υ be an sac-operator on a qpBL algebra $\mathbf{S}$. Then Υ is a weak sac-operator on $\mathbf{S}$.*

However, if Υ is a weak sac-operator on qpBL algebra $\mathbf{S}$, then it is not an sac-operator in general.

Example 4. Let $(\mathbf{S},\Upsilon)=(S;\boxdot,\Cup,\Cap,\rightharpoondown,0,1)$ be a good quasi-(pseudo)-BL algebra defined in Example 3. Then Υ defined in Example 3 is a weak sac-operator on $\mathbf{S}$. According to Example 3, we have $\Upsilon(\Upsilon(\upsilon))\neq\Upsilon(\upsilon)$. Hence Υ is not an sac-operator on $\mathbf{S}$.

Since $\kappa\preceq\kappa^{\ulcorner\urcorner}=\kappa^{\urcorner\ulcorner}$ holds in any good qpBL algebra $\mathbf{S}$, we have that any sac-operator is the ac-operator and any weak sac-operator is the weak ac-operator on $\mathbf{S}$. The relationship among these operators can be seen in Fig. 2.

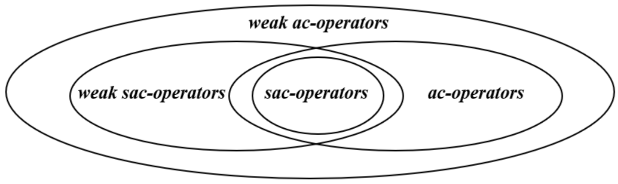

Fig. 2. A Relational Diagram

Below we will see the relationship between (weak) wmi-operators and (weak) ac-operators.

Theorem 2. *Let* **S** *be a good qpBL algebra. If* Γ *is a weak wmi-operator on* **S**, *then the mappings* $\Gamma_{\ulcorner}^{\urcorner}$ *and* $\Gamma_{\urcorner}^{\ulcorner}$ *are weak ac-operators on* **S**.

Proof: We check the conditions one by one. For any $\kappa, \vartheta \in S$:

(WAC1) We have $\Gamma_{\ulcorner}^{\urcorner}(\kappa \boxplus \vartheta) = (\Gamma((\kappa \boxplus \vartheta)^{\ulcorner}))^{\urcorner} = (\Gamma((\kappa^{\ulcorner} \boxdot \vartheta^{\ulcorner})^{\urcorner\ulcorner}))^{\urcorner} = (\Gamma(\kappa^{\ulcorner\urcorner\urcorner\ulcorner} \boxdot \vartheta^{\ulcorner\urcorner\urcorner\ulcorner}))^{\urcorner} = (\Gamma((\kappa^{\ulcorner\urcorner\urcorner\ulcorner} \boxdot 1) \boxdot (\vartheta^{\ulcorner\urcorner\urcorner\ulcorner} \boxdot 1)))^{\urcorner} = (\Gamma((\kappa^{\ulcorner} \boxdot 1) \boxdot (\vartheta^{\ulcorner} \boxdot 1)))^{\urcorner} = (\Gamma(\kappa^{\ulcorner} \boxdot 1) \boxdot \Gamma(\vartheta^{\ulcorner} \boxdot 1))^{\urcorner} = ((\Gamma(\kappa^{\ulcorner}) \boxdot \Gamma(1)) \boxdot (\Gamma(\vartheta^{\ulcorner}) \boxdot \Gamma(1)))^{\urcorner} = (\Gamma(\kappa^{\ulcorner}) \boxdot 1 \boxdot \Gamma(\vartheta^{\ulcorner}) \boxdot 1)^{\urcorner} = (\Gamma(\kappa^{\ulcorner}) \boxdot \Gamma(\vartheta^{\ulcorner}))^{\urcorner} = (\Gamma(\kappa^{\ulcorner}))^{\urcorner} \boxplus (\Gamma(\vartheta^{\ulcorner}))^{\urcorner} = \Gamma_{\ulcorner}^{\urcorner}(\kappa) \boxplus \Gamma_{\ulcorner}^{\urcorner}(\vartheta)$ by (G5), (P17) and (G7).

(WAC2) Since $\kappa \preceq \kappa^{\ulcorner\urcorner} = \kappa^{\urcorner\ulcorner}$, we have $\kappa \preceq \Gamma_{\ulcorner}^{\urcorner}(\kappa)$ by Proposition 13(1).

(WAC3) We have $\Gamma(\kappa^{\ulcorner}) \preceq \Gamma(\kappa^{\ulcorner}) \boxdot 1 = \Gamma(\Gamma(\kappa^{\ulcorner})) \boxdot 1 \preceq \Gamma(\Gamma(\kappa^{\ulcorner}))$ by (P2) and (WWMI3), so $\Gamma_{\ulcorner}^{\urcorner}\left(\Gamma_{\ulcorner}^{\urcorner}(\kappa)\right) \boxplus 0 = \Gamma_{\ulcorner}^{\urcorner}((\Gamma(\kappa^{\ulcorner}))^{\urcorner}) \boxplus 0 = (\Gamma((\Gamma(\kappa^{\ulcorner}))^{\urcorner\ulcorner}))^{\urcorner} \boxplus 0 \preceq (\Gamma(\Gamma(\kappa^{\ulcorner})))^{\urcorner} \boxplus 0 \preceq (\Gamma(\kappa^{\ulcorner}))^{\urcorner} \boxplus 0 = \Gamma_{\ulcorner}^{\urcorner}(\kappa) \boxplus 0$ by (P15), (P16) and (G9). Moreover, we have $\Gamma_{\ulcorner}^{\urcorner}(\kappa) \boxplus 0 \preceq (\Gamma_{\ulcorner}^{\urcorner}(\kappa))^{\ulcorner\urcorner} \boxplus 0 \preceq \Gamma_{\ulcorner}^{\urcorner}\left(\Gamma_{\ulcorner}^{\urcorner}(\kappa)\right) \boxplus 0$ by (P15), (G9) and Proposition 13(1). Since $\Gamma_{\ulcorner}^{\urcorner}\left(\Gamma_{\ulcorner}^{\urcorner}(\kappa)\right) \boxplus 0$ and $\Gamma_{\ulcorner}^{\urcorner}(\kappa) \boxplus 0$ are regular, we have $\Gamma_{\ulcorner}^{\urcorner}\left(\Gamma_{\ulcorner}^{\urcorner}(\kappa)\right) \boxplus 0 = \Gamma_{\ulcorner}^{\urcorner}(\kappa) \boxplus 0$.

(WAC4) Follows from Proposition 13(4).

Hence $\Gamma_{\ulcorner}^{\urcorner}$ is a weak ac-operator on **S**. Analogously for $\Gamma_{\urcorner}^{\ulcorner}$. ∎

Let **S** be a good qpBL algebra and $\Upsilon : S \longrightarrow S$ be a mapping. We consider two mappings $\Upsilon_{\ulcorner}^{\urcorner} : S \longrightarrow S$ by $\Upsilon_{\ulcorner}^{\urcorner}(\kappa) = \left(\Upsilon\left(\kappa^{\ulcorner}\right)\right)^{\urcorner}$ and $\Upsilon_{\urcorner}^{\ulcorner} : S \longrightarrow S$ by $\Upsilon_{\urcorner}^{\ulcorner}(\kappa) = \left(\Upsilon\left(\kappa^{\urcorner}\right)\right)^{\ulcorner}$ for any $\kappa \in S$.

Proposition 18. *Let* **S** *be a good qpBL algebra and* Υ *be a weak ac-operator on* **S**. *If* Υ *is monotone, then the mappings* $\Upsilon_{\ulcorner}^{\urcorner}$ *and* $\Upsilon_{\urcorner}^{\ulcorner}$ *are monotone.*

Proof: Let Υ be monotone and $\kappa \preceq \vartheta$. Then $\vartheta^{\ulcorner} \preceq \kappa^{\ulcorner}$ by (P16), it follows that $\Upsilon(\vartheta^{\ulcorner}) \preceq \Upsilon(\kappa^{\ulcorner})$, so $\Upsilon_{\ulcorner}^{\urcorner}(\kappa) = (\Upsilon(\kappa^{\ulcorner}))^{\urcorner} \preceq (\Upsilon(\vartheta^{\ulcorner}))^{\urcorner} = \Upsilon_{\ulcorner}^{\urcorner}(\vartheta)$. Analogously for $\Upsilon_{\urcorner}^{\ulcorner}$. ∎

Theorem 3. *Let* **S** *be a good qpBL algebra. If* Υ *is a weak ac-operator on* **S**, *then the mappings* $\Upsilon_{\ulcorner}^{\urcorner}$ *and* $\Upsilon_{\urcorner}^{\ulcorner}$ *are weak wmi-operators on* **S**.

Proof: We check the conditions one by one. For any $\kappa, \vartheta \in S$:

(WWMI1) We have $\Upsilon_{\ulcorner}^{\urcorner}(\kappa \boxdot \vartheta) = (\Upsilon((\kappa \boxdot \vartheta)^{\ulcorner}))^{\urcorner} = (\Upsilon(\kappa^{\ulcorner} \boxplus \vartheta^{\ulcorner}))^{\urcorner} = (\Upsilon(\kappa^{\ulcorner}) \boxplus \Upsilon(\vartheta^{\ulcorner}))^{\urcorner} = (\Upsilon(\kappa^{\ulcorner}))^{\urcorner} \boxdot (\Upsilon(\vartheta^{\ulcorner}))^{\urcorner} = \Upsilon_{\ulcorner}^{\urcorner}(\kappa) \boxdot \Upsilon_{\ulcorner}^{\urcorner}(\vartheta)$ by (G7) and (G8).

(WWMI2) Since $\kappa^{\ulcorner} \preceq \Upsilon(\kappa^{\ulcorner})$, we have $\Upsilon_{\ulcorner}^{\urcorner}(\kappa) = (\Upsilon(\kappa^{\ulcorner}))^{\urcorner} \preceq \kappa^{\ulcorner\urcorner}$ by (P16). Note that **S** is good, we have $\Upsilon_{\ulcorner}^{\urcorner}(\kappa) \preceq \kappa^{\urcorner\ulcorner}$.

(WWMI3) We have $\Upsilon_{\ulcorner}^{\urcorner}(\Upsilon_{\ulcorner}^{\urcorner}(\kappa)) \boxdot 1 = (\Upsilon((\Upsilon(\kappa^{\ulcorner}))^{\urcorner\ulcorner}))^{\urcorner} \boxdot 1 = (\Upsilon((\Upsilon(\kappa^{\ulcorner}))^{\urcorner\ulcorner}) \boxplus 0)^{\urcorner} = (\Upsilon((\Upsilon(\kappa^{\ulcorner}))^{\ulcorner\urcorner} \boxplus 0))^{\urcorner} = (\Upsilon(\Upsilon(\kappa^{\ulcorner}) \boxplus 0))^{\urcorner} = (\Upsilon(\Upsilon(\kappa^{\ulcorner})) \boxplus \Upsilon(0))^{\urcorner} = (\Upsilon(\Upsilon(\kappa^{\ulcorner})) \boxplus 0)^{\urcorner} = (\Upsilon(\kappa^{\ulcorner}) \boxplus 0)^{\urcorner} = (\Upsilon(\kappa^{\ulcorner}))^{\urcorner} \boxdot 1 = \Upsilon_{\ulcorner}^{\urcorner}(\kappa) \boxdot 1$ by (P13), (G8), (G6) and (WAC3).

(WWMI4) We have $\Upsilon_{\ulcorner}^{\urcorner}(1) = \left(\Upsilon\left(1^{\ulcorner}\right)\right)^{\urcorner} = (\Upsilon(0))^{\urcorner} = 0^{\urcorner} = 1$ by (P13).

Hence $\Upsilon_{\ulcorner}^{\urcorner}$ is a weak wmi-operator on **S**. Analogously for $\Upsilon_{\urcorner}^{\ulcorner}$. ∎

Let **S** be a good qpBL algebra. We denote $\Im(S)$ the set of weak wmi-operators on **S** and $\Re(S)$ the set of monotone weak ac-operators on **S**. Suppose that $\Im(S)$ and $\Re(S)$ are pointwise ordered. Define the mappings $\phi,\ \Phi : \Im(S) \longrightarrow \Re(S)$ such that $\phi(\Gamma) = \Gamma_{\ulcorner}^{\urcorner}$ and $\Phi(\Gamma) = \Gamma_{\urcorner}^{\ulcorner}$ for any $\Gamma \in \Im(S)$. Also define the mappings $\psi,\ \Psi : \Re(S) \longrightarrow \Im(S)$ such that $\psi(\Upsilon) = \Upsilon_{\ulcorner}^{\urcorner}$ and $\Psi(\Upsilon) = \Upsilon_{\urcorner}^{\ulcorner}$ for any $\Upsilon \in \Re(S)$.

Theorem 4. *Let* **S** *be a good qpBL algebra. Then we have,*

(1) ϕ *and* Ψ *have an antitone Galois connection, i.e.,* $\Gamma \preceq \Psi(\Upsilon)$ *iff* $\Upsilon \preceq \phi(\Gamma)$, *for any* $\Gamma \in \Im(S)$ *and* $\Upsilon \in \Re(S)$.

(2) Φ *and* ψ *have an antitone Galois connection, i.e.,* $\Gamma \preceq \psi(\Upsilon)$ *iff* $\Upsilon \preceq \Phi(\Gamma)$, *for any* $\Gamma \in \Im(S)$ *and* $\Upsilon \in \Re(S)$.

Proof: (1) For $\Gamma \in \Im(S)$ and $\Upsilon \in \Re(S)$, if $\Gamma \preceq \Psi(\Upsilon) = \Upsilon_{\urcorner}^{\ulcorner}$, then for any $\kappa \in S$, $\Gamma(\kappa) \preceq \Upsilon_{\urcorner}^{\ulcorner}(\kappa) = (\Upsilon(\kappa^{\urcorner}))^{\ulcorner}$ holds, it turns out that $(\Upsilon(\kappa^{\urcorner}))^{\ulcorner\urcorner} \preceq (\Gamma(\kappa))^{\urcorner}$ by (P16), so $\Upsilon(\kappa) \preceq \Upsilon(\kappa^{\ulcorner\urcorner}) \preceq (\Upsilon(\kappa^{\ulcorner\urcorner}))^{\ulcorner\urcorner} \preceq (\Gamma(\kappa^{\ulcorner}))^{\urcorner} = \Gamma_{\ulcorner}^{\urcorner}(\kappa)$ by (P15) and then $\Upsilon(\kappa) \preceq \phi(\Gamma)(\kappa)$ for any $\kappa \in S$. Hence $\Upsilon \preceq \phi(\Gamma)$. Conversely, let $\Upsilon \preceq \phi(\Gamma) = \Gamma_{\ulcorner}^{\urcorner}$. Then for any $\kappa \in S$, we have $\Upsilon(\kappa) \preceq \Gamma_{\ulcorner}^{\urcorner}(\kappa) = (\Gamma(\kappa^{\ulcorner}))^{\urcorner}$, it follows that $(\Gamma(\kappa^{\ulcorner}))^{\urcorner\ulcorner} \preceq (\Upsilon(\kappa))^{\ulcorner}$, so $\Gamma(\kappa) \preceq \Gamma(\kappa^{\urcorner\ulcorner}) \preceq (\Gamma(\kappa^{\urcorner\ulcorner}))^{\urcorner\ulcorner} \preceq (\Upsilon(\kappa^{\urcorner}))^{\ulcorner} = \Upsilon_{\urcorner}^{\ulcorner}(\kappa)$ by (P15) and then $\Gamma(\kappa) \preceq \Psi(\Upsilon)(\kappa)$ for any $\kappa \in S$. Hence $\Gamma \preceq \Psi(\Upsilon)$.

(2) The proof is similar to (1). ∎

Theorem 5. *Let* **S** *be a good qpBL algebra.*

(1) *If* Γ *is a weak wmi-operator on* **S** *and* $h = (\Gamma_{\urcorner}^{\ulcorner})_{\ulcorner}^{\urcorner} = (\Gamma_{\ulcorner}^{\urcorner})_{\urcorner}^{\ulcorner}$, *then* $\Gamma_{\ulcorner}^{\urcorner}|_{R(S)} = h_{\ulcorner}^{\urcorner}|_{R(S)}$ *and* $\Gamma_{\urcorner}^{\ulcorner}|_{R(S)} = h_{\urcorner}^{\ulcorner}|_{R(S)}$.

(2)*If* Υ *is a monotone weak ac-operator on* **S** *and* $k = (\Upsilon_{\urcorner}^{\ulcorner})_{\ulcorner}^{\urcorner} = (\Upsilon_{\ulcorner}^{\urcorner})_{\urcorner}^{\ulcorner}$, *then* $\Upsilon_{\ulcorner}^{\urcorner}|_{R(S)} = k_{\ulcorner}^{\urcorner}|_{R(S)}$ *and* $\Upsilon_{\urcorner}^{\ulcorner}|_{R(S)} = k_{\urcorner}^{\ulcorner}|_{R(S)}$.

Proof: (1) Let Γ be a weak wmi-operator on **S**. Since $h = \Psi(\Gamma_{\ulcorner}^{\urcorner}) \in \Im(S)$, we have $\Gamma_{\ulcorner}^{\urcorner} \preceq \phi(h)$ by Theorem 4, so $\Gamma_{\ulcorner}^{\urcorner} \preceq h_{\ulcorner}^{\urcorner}$. Meanwhile, since $\phi(\Gamma) = \phi(\Gamma)$, we have $\Gamma \preceq \Psi\phi(\Gamma)$ using Theorem 4 again, so $\Gamma \preceq \Psi\phi(\Gamma) = (\Gamma_{\ulcorner}^{\urcorner})_{\urcorner}^{\ulcorner} = h$. It turns out that for any $\kappa \in S$, $\Gamma(\kappa^{\ulcorner}) \preceq h(\kappa^{\ulcorner})$ and then $h_{\ulcorner}^{\urcorner}(\kappa) = (h(\kappa^{\ulcorner}))^{\urcorner} \preceq (\Gamma(\kappa^{\ulcorner}))^{\urcorner} = \Gamma_{\ulcorner}^{\urcorner}(\kappa)$ which means that $h_{\ulcorner}^{\urcorner} \preceq \Gamma_{\ulcorner}^{\urcorner}$. Hence we have $\Gamma_{\ulcorner}^{\urcorner}(\kappa \boxdot 1) = h_{\ulcorner}^{\urcorner}(\kappa \boxdot 1)$ and then $\Gamma_{\ulcorner}^{\urcorner}|_{R(S)} = h_{\ulcorner}^{\urcorner}|_{R(S)}$. Analogously for $\Gamma_{\urcorner}^{\ulcorner}|_{R(S)} = h_{\urcorner}^{\ulcorner}|_{R(S)}$.

(2) The proof is similar to (1). ∎

Below the notion $(\mathbf{S}, \Upsilon)$ also denotes a good qpBL algebra **S** with a weak ac-operator Υ. According to Proposition 5, we have that $D(S)$ is a normal weak filter of **S**.

Proposition 19. *Let* $(\mathbf{S}, \Upsilon)$ *be a good qpBL algebra* **S** *with a weak ac-operator* Υ. *If* $\kappa \in D(S)$, *then* $\Upsilon(\kappa \boxplus 0) \in D(S)$.

Proof: For any $\kappa \in D(S)$, we have $\kappa \preceq \Upsilon(\kappa) \preceq \Upsilon(\kappa) \boxdot 1 \preceq (\Upsilon(\kappa) \boxdot 1)^{\urcorner\ulcorner} = ((\Upsilon(\kappa))^{\urcorner} \boxdot 1)^{\ulcorner} = ((\Upsilon(\kappa))^{\urcorner} \boxdot 0^{\urcorner})^{\ulcorner} = \Upsilon(\kappa) \boxplus 0 = \Upsilon(\kappa \boxplus 0) = \Upsilon(\kappa \boxplus 0) \boxdot 1$ by (WAC2), (P2), (P15) and (P13). Since $D(S)$ is a weak filter of **S**, we have $\Upsilon(\kappa \boxplus 0) \in D(S)$. ∎

Theorem 6. *Let* $(\mathbf{S}, \Upsilon)$ *be a good qpBL algebra* **S** *with a weak ac-operator* Υ. *Define* $\tilde{\Upsilon} : S/D(S) \longrightarrow S/D(S)$ *by* $\tilde{\Upsilon}(\kappa/D(S)) = \Upsilon(\kappa \boxplus 0)/D(S)$ *for any* $\kappa \in S$. *Then* $\tilde{\Upsilon}$ *is an ac-operator on* $\mathbf{S}/\mathbf{D}(\mathbf{S})$ *and* $(\mathbf{S}/\mathbf{D}(\mathbf{S}), \tilde{\Upsilon})$ *is a pseudo-MV algebra with an ac-operator* $\tilde{\Upsilon}$.

Proof: First, we show that the mapping $\tilde{\Upsilon}$ is well-defined. For any $\kappa/D(S) = \vartheta/D(S) \in S/D(S)$, we have $\kappa \in \vartheta/D(S)$ and $\kappa^{\ulcorner\urcorner} \to \vartheta^{\ulcorner\urcorner} = 1 = \vartheta^{\ulcorner\urcorner} \to \kappa^{\ulcorner\urcorner}$, it follows that $\kappa^{\ulcorner\urcorner} \preceq \vartheta^{\ulcorner\urcorner}$ and $\vartheta^{\ulcorner\urcorner} \preceq \kappa^{\ulcorner\urcorner}$ by (P9), so $\vartheta^{\ulcorner\urcorner} \boxdot 1 = \kappa^{\ulcorner\urcorner} \boxdot 1$ by (P1). Since $\kappa \boxplus 0 = (\kappa^{\ulcorner} \boxdot 0^{\ulcorner})^{\urcorner} = (\kappa^{\ulcorner} \boxdot 1)^{\urcorner} = \kappa^{\ulcorner\urcorner} \boxdot 1$ and $\vartheta \boxplus 0 = \vartheta^{\ulcorner\urcorner} \boxdot 1$, we have $\kappa \boxplus 0 = \vartheta \boxplus 0$, so $\Upsilon(\kappa \boxplus 0) = \Upsilon(\vartheta \boxplus 0)$ and then $\Upsilon(\kappa \boxplus 0)/D(S) = \Upsilon(\vartheta \boxplus 0)/D(S)$, it follows that $\tilde{\Upsilon}(\kappa) = \tilde{\Upsilon}(\vartheta)$.

Now, we check the conditions of Definition 10 one by one.
(1) We have $\tilde{\Upsilon}((\kappa/D(S)) \boxplus (\vartheta/D(S))) = \tilde{\Upsilon}((\kappa \boxplus \vartheta)/D(S)) = \Upsilon(\kappa \boxplus \vartheta \boxplus 0)/D(S) = (\Upsilon(\kappa) \boxplus \Upsilon(\vartheta) \boxplus \Upsilon(0))/D(S) = (\Upsilon(\kappa) \boxplus \Upsilon(\vartheta) \boxplus \Upsilon(0) \boxplus \Upsilon(0))/D(S) = ((\Upsilon(\kappa) \boxplus \Upsilon(0)) \boxplus (\Upsilon(\vartheta) \boxplus \Upsilon(0)))/D(S) = (\Upsilon(\kappa \boxplus 0) \boxplus \Upsilon(\vartheta \boxplus 0))/D(S) = ((\Upsilon(\kappa \boxplus 0))/D(S)) \boxplus ((\Upsilon(\vartheta \boxplus 0))/D(S)) = \tilde{\Upsilon}(\kappa/D(S)) \boxplus \tilde{\Upsilon}(\vartheta/D(S))$ by (WAC4) and (WAC1).
(2) By Proposition 6, we have that $\mathbf{S/D(S)}$ is a pseudo-MV algebra, so $\kappa/D(S) = (\kappa/D(S))^{\ulcorner\urcorner} = (\kappa/D(S))^{\ulcorner\urcorner} \boxdot (1/D(S)) = (\kappa/D(S)) \boxplus (0/D(S)) = (\kappa \boxplus 0)/D(S) \preceq \Upsilon(\kappa \boxplus 0)/D(S) = \tilde{\Upsilon}(\kappa/D(S))$.
(3) We have $\tilde{\Upsilon}(\tilde{\Upsilon}(\kappa/D(S))) = \tilde{\Upsilon}(\Upsilon(\kappa \boxplus 0)/D(S)) = \Upsilon(\Upsilon(\kappa \boxplus 0) \boxplus 0)/D(S) = \Upsilon(\Upsilon(\kappa) \boxplus \Upsilon(0) \boxplus \Upsilon(0))/D(S) = \Upsilon(\Upsilon(\kappa) \boxplus 0)/D(S) = (\Upsilon(\Upsilon(\kappa)) \boxplus \Upsilon(0))/D(S) = (\Upsilon(\Upsilon(\kappa)) \boxplus 0)/D(S) = (\Upsilon(\kappa) \boxplus 0)/D(S) = \Upsilon(\kappa \boxplus 0)/D(S) = \tilde{\Upsilon}(\kappa/D(S))$.
(4) We have $\tilde{\Upsilon}(0/D(S)) = \Upsilon(0 \boxplus 0)/D(S) = 0/D(S)$.

Hence $\tilde{\Upsilon}$ is an ac-operator on $\mathbf{S/D(S)}$ and $(\mathbf{S/D(S)}, \tilde{\Upsilon})$ is a pseudo-MV algebra with an ac-operator $\tilde{\Upsilon}$. ■

V. Operators on Quasi-pseudo-MV Algebra K(S)

Let $\mathbf{S}$ be a good qpBL algebra. We denote $K(S) = \{\kappa \in S | \kappa = \kappa^{\ulcorner\urcorner} = \kappa^{\urcorner\ulcorner}\}$. Then we have the following results.

Lemma 4. [6] *Let* $\mathbf{S}$ *be a good qpBL algebra. Then for any* $\kappa, \vartheta \in S$ *we have,*

(M1) $0, 1 \in K(S)$*;*

(M2) $\kappa^{\ulcorner} \boxdot 1$, $\kappa^{\urcorner} \boxdot 1 \in K(S)$. *Especially, if* $\kappa \in K(S)$*, then* $\kappa^{\ulcorner}$, $\kappa^{\urcorner} \in K(S)$*;*

(M3) $\kappa \boxplus \vartheta \in K(S)$*;*

For any $\kappa, \vartheta \in K(S)$ *we have,*

(M4) $\kappa \boxdot \vartheta \in K(S)$*;*

(M5) $\kappa \rightharpoondown \vartheta = \vartheta^{\ulcorner} \rightarrowtail \kappa^{\ulcorner}$ *and* $\kappa \rightarrowtail \vartheta = \vartheta^{\urcorner} \rightharpoondown \kappa^{\urcorner}$*;*

(M6) $\kappa \boxplus \vartheta = (\kappa^{\ulcorner} \boxdot \vartheta^{\ulcorner})^{\urcorner} = (\kappa^{\urcorner} \boxdot \vartheta^{\urcorner})^{\ulcorner} = \kappa^{\urcorner} \rightharpoondown \vartheta = \vartheta^{\ulcorner} \rightarrowtail \kappa$.

Following from [6], we have that $\mathbf{K(S)} = (K(S); \boxplus, ^{\ulcorner}, ^{\urcorner}, 0, 1)$ is a quasi-pseudo-MV subalgebra of $\mathbf{S}$.

Theorem 7. *Let* $(\mathbf{S}, \Gamma)$ *be a good qpBL algebra* $\mathbf{S}$ *with a weak wmi-operator* Γ*. If a mapping* $\Gamma^* : K(S) \longrightarrow K(S)$ *is defined by* $\Gamma^*(\kappa) = (\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1$ *for any* $\kappa \in K(S)$*, then* Γ^* *is an mi-operator on* $\mathbf{K(S)}$.

Proof: For any $\kappa, \vartheta \in K(S)$, then $\kappa \boxdot \vartheta \in K(S)$ by (M4). So,

(MI1) We have $\Gamma^*(\kappa \boxdot \vartheta) = (\Gamma(\kappa \boxdot \vartheta))^{\ulcorner\urcorner} \boxdot 1 = (\Gamma(\kappa) \boxdot \Gamma(\vartheta))^{\ulcorner\urcorner} \boxdot 1 = (\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot (\Gamma(\vartheta))^{\ulcorner\urcorner} \boxdot 1 = ((\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1) \boxdot ((\Gamma(\vartheta))^{\ulcorner\urcorner} \boxdot 1) = \Gamma^*(\kappa) \boxdot \Gamma^*(\vartheta)$ by (G5).

(MI2) Since $\mathbf{S}$ is good, we have $\Gamma^*(\kappa) = (\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1 \preceq (\Gamma(\kappa))^{\ulcorner\urcorner} \preceq (\kappa^{\ulcorner\urcorner})^{\ulcorner\urcorner} = (\kappa^{\urcorner\ulcorner})^{\ulcorner\urcorner} = \kappa$ by (P2).

(MI3) We have $\Gamma^*(\kappa) = (\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1 = (\Gamma(\kappa) \boxdot 1)^{\ulcorner\urcorner} = (\Gamma(\Gamma(\kappa)) \boxdot 1)^{\ulcorner\urcorner} = (\Gamma(\Gamma(\kappa)))^{\ulcorner\urcorner} \boxdot 1 \preceq (\Gamma((\Gamma(\kappa))^{\ulcorner\urcorner}))^{\ulcorner\urcorner} \boxdot 1 = (\Gamma((\Gamma(\kappa))^{\ulcorner\urcorner}))^{\ulcorner\urcorner} \boxdot 1 \boxdot 1 = (\Gamma((\Gamma(\kappa))^{\ulcorner\urcorner}) \boxdot 1)^{\ulcorner\urcorner} \boxdot 1 = (\Gamma((\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1))^{\ulcorner\urcorner} \boxdot 1 = (\Gamma(\Gamma^*(\kappa))^{\ulcorner\urcorner} \boxdot 1 = \Gamma^*(\Gamma^*(\kappa))$. Conversely, $\Gamma^*(\Gamma^*(\kappa)) = \Gamma^*((\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1) \preceq (\Gamma(\kappa))^{\ulcorner\urcorner} \boxdot 1 = \Gamma^*(\kappa)$ by (MI2). Thus we have $\Gamma^*(\Gamma^*(\kappa)) = (\Gamma(\Gamma^*(\kappa)))^{\ulcorner\urcorner} \boxdot 1 = (\Gamma(\Gamma^*(\kappa)))^{\ulcorner\urcorner} \boxdot 1 \boxdot 1 = \Gamma^*(\Gamma^*(\kappa)) \boxdot 1 = \Gamma^*(\kappa) \boxdot 1 = \Gamma^*(\kappa)$ by (P1).

(MI4) We have $\Gamma^*(1) = (\Gamma(1))^{\ulcorner\urcorner} \boxdot 1 = 1^{\ulcorner\urcorner} \boxdot 1 = 1 \boxdot 1 = 1$ by (P13).

Thus Γ^* is an mi-operator on $\mathbf{K(S)}$. ■

Theorem 8. *Let* $\mathbf{S}$ *be a good qpBL algebra and* Γ *be a weak wmi-operator on* $\mathbf{K(S)}$*. If a mapping* $\Gamma^+ : S \longrightarrow S$ *is defined by* $\Gamma^+(\kappa) = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1)$ *for any* $\kappa \in S$*, then* Γ^+ *is a weak wmi-operator on* $\mathbf{S}$.

Proof: Let Γ be a weak wmi-operator on $\mathbf{K(S)}$. For any $\kappa, \vartheta \in S$, we check the conditions of Definition 9 one by one.

(WWMI1) We have $\Gamma^+(\kappa \boxdot \vartheta) = \Gamma((\kappa \boxdot \vartheta)^{\ulcorner\urcorner} \boxdot 1) = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot \vartheta^{\ulcorner\urcorner} \boxdot 1) = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot \vartheta^{\ulcorner\urcorner} \boxdot 1 \boxdot 1) = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1 \boxdot \vartheta^{\ulcorner\urcorner} \boxdot 1) = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1) \boxdot \Gamma(\vartheta^{\ulcorner\urcorner} \boxdot 1) = \Gamma^+(\kappa) \boxdot \Gamma^+(\vartheta)$ by (G5).

(WWMI2) Note that $\mathbf{S}$ is good, we have $\Gamma^+(\kappa) = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1) \preceq (\kappa^{\ulcorner\urcorner} \boxdot 1)^{\ulcorner\urcorner} = \kappa^{\ulcorner\urcorner} \boxdot 1 \preceq \kappa^{\ulcorner\urcorner} = \kappa^{\urcorner\ulcorner}$ by (M2) and (P2).

(WWMI3) We have $\Gamma^+(\Gamma^+(\kappa)) \boxdot 1 = \Gamma((\Gamma^+(\kappa))^{\ulcorner\urcorner} \boxdot 1) \boxdot 1 = \Gamma((\Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1))^{\ulcorner\urcorner} \boxdot 1) \boxdot 1 = \Gamma(\Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1) \boxdot 1) \boxdot 1 = \Gamma(\Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1)) \boxdot 1 = \Gamma(\kappa^{\ulcorner\urcorner} \boxdot 1) \boxdot 1 = \Gamma^+(\kappa) \boxdot 1$.

(WWMI4) Since $1 \in K(S)$, we have $\Gamma^+(1) = \Gamma(1^{\ulcorner\urcorner} \boxdot 1) = \Gamma(1) = 1$ by (P13).

Thus Γ^+ is a weak wmi-operator on $\mathbf{S}$. ■

Theorem 9. *Let* $(\mathbf{S}, \Upsilon)$ *be a good qpBL algebra* $\mathbf{S}$ *with a weak ac-operator* Υ*. If a mapping* $\Upsilon^* : K(S) \longrightarrow K(S)$ *is defined by* $\Upsilon^*(\kappa) = (\Upsilon(\kappa))^{\ulcorner\urcorner} \boxplus 0$ *for any* $\kappa \in K(S)$*, then* Υ^* *is an ac-operator on* $\mathbf{K(S)}$.

Proof: Let Υ be a weak ac-operator on $\mathbf{S}$. For any $\kappa, \vartheta \in K(S)$,

(AC1) We have $\Upsilon^*(\kappa \boxplus \vartheta) = (\Upsilon(\kappa \boxplus \vartheta))^{\ulcorner\urcorner} \boxplus 0 = (\Upsilon(\kappa) \boxplus \Upsilon(\vartheta))^{\ulcorner\urcorner} \boxplus 0 = \Upsilon(\kappa) \boxplus \Upsilon(\vartheta) \boxplus 0 = (\Upsilon(\kappa))^{\ulcorner\urcorner} \boxplus (\Upsilon(\vartheta))^{\ulcorner\urcorner} \boxplus 0 = ((\Upsilon(\kappa))^{\ulcorner\urcorner} \boxplus 0) \boxplus ((\Upsilon(\vartheta))^{\ulcorner\urcorner} \boxplus 0) = \Upsilon^*(\kappa) \boxplus \Upsilon^*(\vartheta)$ by (M3) and (G6).

(AC2) We have $\kappa \preceq \Upsilon(\kappa)$, then $\kappa = \kappa^{\ulcorner\urcorner} \preceq (\Upsilon(\kappa))^{\ulcorner\urcorner} \preceq (\Upsilon(\kappa))^{\ulcorner\urcorner} \boxdot 1 = ((\Upsilon(\kappa))^{\ulcorner} \boxdot 1^{\ulcorner})^{\urcorner} = \Upsilon(\kappa) \boxplus 0 = (\Upsilon(\kappa))^{\ulcorner\urcorner} \boxplus 0 = \Upsilon^*(\kappa)$ by (P2) and (P13).

(AC3) We have $\Upsilon^*(\Upsilon^*(\kappa)) = (\Upsilon((\Upsilon(\kappa))^{\ulcorner\urcorner} \boxplus 0))^{\ulcorner\urcorner} \boxplus 0 = (\Upsilon(\Upsilon(\kappa) \boxplus 0))^{\ulcorner\urcorner} \boxplus 0 = \Upsilon(\Upsilon(\kappa) \boxplus 0) \boxplus 0 = \Upsilon(\Upsilon(\kappa)) \boxplus 0 \boxplus 0 = \Upsilon(\Upsilon(\kappa)) \boxplus 0 = \Upsilon(\kappa) \boxplus 0 = (\Upsilon(\kappa))^{\ulcorner\urcorner} \boxplus 0 = \Upsilon^*(\kappa)$ by (G6) and (P13).

(AC4) We have $\Upsilon^*(0) = (\Upsilon(0))^{\ulcorner\urcorner} \boxplus 0 = 0^{\ulcorner\urcorner} \boxplus 0 = 0 \boxplus 0 = 0$ by (P13).

Thus Υ^* is an ac-operator on $\mathbf{K(S)}$. ■

Theorem 10. *Let* $\mathbf{S}$ *be a good qpBL algebra and* Υ *be a weak ac-operator on* $\mathbf{K(S)}$*. If a mapping* $\Upsilon^+ : S \longrightarrow S$ *is defined by* $\Upsilon^+(\kappa) = \Upsilon(\kappa^{\ulcorner\urcorner} \boxplus 0)$ *for any* $\kappa \in S$*, then* Υ^+ *is a weak sac-operator on* $\mathbf{S}$.

Proof: Let Υ be a weak ac-operator on $\mathbf{K(S)}$. For any $\kappa, \vartheta \in S$, we check the conditions of Definition 13 one by one.

(WSAC1) We have $\Upsilon^+(\kappa \boxplus \vartheta) = \Upsilon((\kappa \boxplus \vartheta)^{\ulcorner\urcorner} \boxplus 0) = \Upsilon(\kappa \boxplus \vartheta \boxplus 0) = \Upsilon(\kappa^{\ulcorner\urcorner} \boxplus \vartheta^{\ulcorner\urcorner} \boxplus 0) = \Upsilon(\kappa^{\ulcorner\urcorner} \boxplus \vartheta^{\ulcorner\urcorner} \boxplus 0 \boxplus 0) = \Upsilon(\kappa^{\ulcorner\urcorner} \boxplus 0 \boxplus \vartheta^{\ulcorner\urcorner} \boxplus 0) = \Upsilon(\kappa^{\ulcorner\urcorner} \boxplus 0) \boxplus \Upsilon(\vartheta^{\ulcorner\urcorner} \boxplus 0) = \Upsilon^+(\kappa) \boxplus \Upsilon^+(\vartheta)$ by (M3) and (G6).

(WSAC2) We have $\kappa^{\ulcorner\urcorner} \preceq \kappa^{\ulcorner\urcorner} \boxdot 1 = \kappa^{\ulcorner\urcorner} \boxplus 0 \preceq \Upsilon(\kappa^{\ulcorner\urcorner} \boxplus 0) = \Upsilon^+(\kappa)$ by (P2).

(WSAC3) We have $\Upsilon^{+}(\Upsilon^{+}(\kappa))\boxplus 0=\Upsilon((\Upsilon^{+}(\kappa))^{\ulcorner\urcorner}\boxplus 0)\boxplus 0=\Upsilon((\Upsilon(\kappa^{\ulcorner\urcorner}\boxplus 0))^{\ulcorner\urcorner}\boxplus 0)\boxplus 0=\Upsilon(\Upsilon(\kappa^{\ulcorner\urcorner}\boxplus 0)\boxplus 0)\boxplus 0=\Upsilon(\Upsilon(\kappa^{\ulcorner\urcorner}\boxplus 0))\boxplus 0=\Upsilon(\kappa^{\ulcorner\urcorner}\boxplus 0)\boxplus 0=\Upsilon^{+}(\kappa)\boxplus 0$ by (G6) and (M3).

(WSAC4) Since $0\in K(S)$, we have $\Upsilon^{+}(0)=\Upsilon(0^{\ulcorner\urcorner}\boxplus 0)=\Upsilon(0\boxplus 0)=\Upsilon(0)=0$ by (P13).

Thus Υ^{+} is a weak sac-operator on **S**. ■

VI. Conclusion

In this paper, we introduce and investigate the multiplicative interior operators, additive closure operators and the relations between them on qpBL algebras. This is the study of qpBL algebras in operator theory. In the future, we will further study qpBL algebras from the perspective of algebraic structure.

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