An Analysis of Three Servers Markovian Multiple Vacation Queueing System with Servers Breakdown

V. Karthick and V. Suvitha*, *Member, IAENG*

Abstract—**An $M/M/3$ repairable Markovian queueing system with servers on vacation has been taken into consideration. The service rates of the servers in this system differ. Each server is allowed to go on its own vacation when there are no customers in system. At the end of the vacation period, if the system is empty, then servers take another vacation; otherwise, they turn on to the busy state. Besides, a breakdown may occur in a Poisson manner in a particular state and the repair process will start immediately. The steady state probability vector of the number of customers in the system was evaluated as a quasi-birth and death (QBD) process, and the matrix geometric technique was used to derive the stationary condition. Some system performance measures are obtained. The effects of arrival rates, breakdown rates and repair rates on the mean number of customers and the influence of arrival rate on steady probabilities are shown through graphs.**

Index Terms—**Heterogenous servers, Matrix geometric method, Multiple vacation, Quasi birth and death process, Server breakdown, Three servers.**

I. Introduction

A server in a queueing system is said to be on vacation if it becomes unavailable to its customers for a random period of time when the system becomes empty. After completion of vacation, if the servers are found empty, again the servers go on vacation. This type of vacation is called a multiple vacation. Over the last two decades, queuing system with vacations have been made use of in industrial and computer communication systems. Doshi [1], Takagi [2], Tian and Zang [3], Ke et al. [4] and Upadhyaya [5] conducted several many surveys on vacation models.

A number of academics have looked into multi-server queues with vacations. Initially, Levy and Yechiali [6] and Vinod [7] studied the $M/M/c$ exponential queue with server vacation. Early research did not provide precise information on the stationary distributions of variables like waiting time and queue length. The $M/M/c$ queueing system with vacation was thoroughly described by Tian et al.[8], who also provided conditional stochastic decomposition estimates for waiting time and queue length. Numerous scholars, including Kao and Narayanan [9], Igaki [10], Chao and Zhang [11], Zhang and Tian [12], [13], and Houalef et al. [14], have evaluated multi-server queueing systems with vacations. In addition Li et al. [15] analysed about a single server retrial queue with the working vacation.

The studies on multi-server queueing systems described above presume that the servers are all homogeneous, i.e., the service rates of all the servers are equal for the whole system. The above assumption is difficult to realise in a queueing system with human servers. Only a few studies have been carried out on a multi-server queueing system with server vacation and different service rates for different servers. Madan et al. [16] carried out an analysis on a $M/M/2$ queue with Bernoulli schedules and a single vacation policy, where two servers bring heterogeneous exponential service to customers. They obtained steady-state probability generating functions for the system size for various server states. The two server Markovian queues with balking is discussed by Singh [17], and comparison between heterogeneous and homogeneous servers is also given. Lin and Ke [18] analysed the multi-server queueing system with single working vacation.

In recent decades, interest in queueing systems with server breakdown has increased. This is a result of their use in computer systems, telecommunications and production systems. For instance, in a facility that processes machines, equipment malfunctions could be caused by things like power outages, a lack of preventive maintenance, or the use of poor raw materials. Recently, Kalidass et al.[19], Shengli et al.[20], Shengli [21], Jing and Tao [22], Chakravarthy et al.[23], Seenivasan and Abinaya [24], Dasa et al. [25], Atencia [26], Gupur [27] and Jain and Jain [28], Tsai et al. [29] have analysed the single server queueing system with the server breakdown. Recently, Yang et al. [30] have studied a two server multiple vacation queueing system with server breakdown.

In this paper, we present a model that has several practical applications. For instance, there is a University with three servers to check the exam results for students. If there are no students checking their results, then all the three servers will be at rest. If at least one student is willing to check his results, then any one server can give the response to the student. At that point, the other two servers are at rest. If at least two students checks their results at the time, any two servers will give their response, while the third server is at rest. If at least three students to check their results at the same time, all the three servers will give their response. When all the three servers are used simultaneously, there is a possibility of system breakdown.

In section II, we describe the model and describe a quasi-birth-death process. In Section III, we present the steady state solution via the matrix geometric technique. We provided

Manuscript received September 26, 2022; revised April 08, 2023.

V. Karthick is Research Scholar in Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603 203, Tamil Nadu, INDIA e-mail: kv0994@srmist.edu.in

*Corresponding Author - V. Suvitha is Assistant Professor in Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603 203, Tamil Nadu, INDIA e-mail: suvithav@srmist.edu.in

some system performance measures and a numerical analysis in sections IV and V. A conclusion has been offered.

II. The Model Description

Here we are dealing with a three-servers multiple working vacation queueing system with servers breakdown, and the service rates of the servers are considered to be different. The following is a description of the model's assumptions:

1) Customers arrive to the system in a Poisson manner with rate λ. In the order of arrival, customers form a waiting line. It is assumed that both the system capacity and the total number of possible consumers are unlimited.
2) The three servers provides heterogeneous exponential service to clients on a First-Come First-Service (FCFS) basis with rates μ_1, μ_2 and μ_3 ($\mu_1 > \mu_2 > \mu_3$).
3) When there are no customers in the system, each server is permitted to take a vacation on its own. If a customer is present in the system at the end of a vacation time, service will begin. Otherwise, the server immediately takes another vacation and continues in this fashion until the server returns from vacation and finds at least one customer waiting in the system.
4) The vacation periods of the servers follow heterogenous exponential distribution with vacation rates $\theta_j, j = 1, 2, 3$.
5) In addition servers may breakdown and then repair process starts instantly.
6) Here both breakdown and repair process follow exponential distribution with the rates α and β respectively.

A. The Quasi-Birth-and-Death (QBD) process

At the time t, the number of customers in the systems is consider $H(t)$ and let $I(t)$ be the servers state. Then

$$I(t) = \begin{cases} 0, & \text{all the servers are on vacation} \\ 1, & \text{the server 1 is alone busy} \\ 2, & \text{the server 2 alone is busy} \\ 3, & \text{the server 3 alone is busy} \\ 4, & \text{the servers 1 and 2 are busy} \\ 5, & \text{the servers 2 and 3 are busy} \\ 6, & \text{the servers 1 and 3 are busy} \\ 7, & \text{all the servers are busy} \\ 8, & \text{all the servers are breakdown} \end{cases}$$

Then $X(t) = \{H(t); I(t)\}$, is a QBD process with a state space denoted by ω as follows:
$\omega = \{(0,0)\} \cup \{(1,l), l = 0,1,2,3\} \cup \{(2,l), l = 0,1,2,3,4,5,6\} \cup \{(i,l), i \geq 3, l = 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
Using lexicographical sequence for the states, Q is the infinitesimal generator of the Markov chain and is given by

$$Q = \begin{bmatrix} B_{00} & B_{01} & \cdots & & & & & \\ B_{10} & B_{11} & B_{12} & \cdots & & & & \\ \vdots & B_{21} & B_{22} & B_{23} & \cdots & & & \\ & \vdots & B_{32} & A_1 & A_0 & \cdots & & \\ & & \vdots & A_2 & A_1 & A_0 & \cdots & \\ & & & \vdots & A_2 & A_1 & A_0 & \cdots \\ & & & & \vdots & \vdots & \vdots & \end{bmatrix}$$

where
$B_{00} = \begin{bmatrix} -\lambda \end{bmatrix}$
$B_{01} = \begin{bmatrix} \lambda & 0 & 0 & 0 \end{bmatrix}$
$B_{10} = \begin{bmatrix} 0 & \mu_1 & \mu_2 & \mu_3 \end{bmatrix}^T$

$$B_{11} = \begin{bmatrix} V_0 & \theta_1 & \theta_2 & \theta_3 \\ 0 & -(\lambda+\mu_1) & 0 & 0 \\ 0 & 0 & -(\lambda+\mu_2) & 0 \\ 0 & 0 & 0 & -(\lambda+\mu_3) \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_3 \\ 0 & \mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 & \mu_2 \\ 0 & \mu_3 & 0 & \mu_1 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} V_0 & \theta_1 & \theta_2 & \theta_3 & 0 & 0 & 0 \\ 0 & V_1 & 0 & 0 & \theta_2 & 0 & \theta_3 \\ 0 & 0 & V_2 & 0 & \theta_1 & \theta_3 & 0 \\ 0 & 0 & 0 & V_3 & 0 & \theta_2 & \theta_1 \\ 0 & 0 & 0 & 0 & V_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_6 \end{bmatrix}$$

$$B_{23} = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \end{bmatrix}$$

$$B_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_9 \\ 0 & 0 & 0 & 0 & \mu_3 & \mu_1 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} V_0 & \theta_1 & \theta_2 & \theta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & V_1 & 0 & 0 & \theta_2 & 0 & \theta_3 & 0 & 0 \\ 0 & 0 & V_2 & 0 & \theta_1 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_3 & 0 & \theta_2 & \theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{10} & 0 & 0 & \theta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{11} & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{12} & \theta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{13} & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & V_{14} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $A_0 = \lambda I_9$
$V_0 = -(\lambda + \theta_1 + \theta_2 + \theta_3)$
$V_1 = -(\lambda + \mu_1 + \theta_2 + \theta_3)$
$V_2 = -(\lambda + \mu_2 + \theta_1 + \theta_3)$
$V_3 = -(\lambda + \mu_3 + \theta_1 + \theta_2)$
$V_4 = -(\lambda + \mu_1 + \mu_2)$
$V_5 = -(\lambda + \mu_2 + \mu_3)$
$V_6 = -(\lambda + \mu_1 + \mu_3)$
$V_7 = \mu_1 + \mu_2$
$V_8 = \mu_2 + \mu_3$
$V_9 = \mu_1 + \mu_3$
$V_{10} = -(\lambda + \mu_1 + \mu_2 + \theta_3)$
$V_{11} = -(\lambda + \mu_2 + \mu_3 + \theta_1)$
$V_{12} = -(\lambda + \mu_1 + \mu_3 + \theta_2)$
$V_{13} = -(\lambda + \mu_1 + \mu_2 + \mu_3 + \alpha)$
$V_{14} = -(\lambda + \beta)$
$V_{15} = \mu_1 + \mu_2 + \mu_3$

III. Steady-state Analysis

In this section we find the condition for the system to reach a steady state. The system's steady-state probability vectors are calculated using a method of matrix-geometric. Also, the rate matrix and the boundary probability vectors are found. Finally, using the steady-state probability vectors, we give several system performance measures.

A. Stationary Condition

To obtain the stationary condition, we first define the matrix $A = A_0 + A_1 + A_2$. Then the matrix A can be written as

$$A = \begin{bmatrix} X_1 & \theta_1 & \theta_2 & \theta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 & \theta_2 & 0 & \theta_3 & 0 & 0 \\ 0 & 0 & X_3 & 0 & \theta_1 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_4 & 0 & \theta_2 & \theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_3 & 0 & 0 & \theta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\theta_1 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\theta_2 & \theta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & -\beta \end{bmatrix}$$

where
$X_1 = -(\theta_1 + \theta_2 + \theta_3)$
$X_2 = -(\theta_2 + \theta_3)$
$X_3 = -(\theta_1 + \theta_3)$
$X_4 = -(\theta_1 + \theta_2)$

It is obvious that A is an irreducible Markov process generator.
Let $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8)$ be a stationary probability vector of this Markov process. Then, π satisfies the linear equations:
$\pi A = 0$, $\pi e = 1$
Following Neuts [31], the system is stable if and only if

$$\pi A_0 e < \pi A_2 e.$$

That is the system is stable if and only if $\rho < 1$ where

$$\rho = \frac{\lambda(\alpha + \beta)}{(\mu_1 + \mu_2 + \mu_3)\beta}$$

B. Matrix geometric Solution

Let H and I be the stationary random variables for the number of customers in the system and the states of servers. We denote the stationary probability by
$P_{n,i} = \lim_{t\to\infty} P\{H(t) = i, I(t) = l\}$, where $(i, l) \in \omega$
Under the stationary condition $\rho < 1$, the stationary probability vector P of the generator Q exists. This stationary probability vector P is partitioned as $P = (p_0, p_1, p_2, ...)$ where $p_0 = p_{00}$
$p_1 = (p_{10}, p_{11}, p_{12}, p_{13})$
$p_2 = (p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16})$
$p_i = (p_{i0}, p_{i1}, p_{i2}, p_{i3}, p_{i4}, p_{i4}, p_{i5}, p_{i6}, p_{i7}, p_{i8})$ for $i \geq 3$.
The stationary probability vector P is generated by using Netus [31] matrix-geometric solution approach, which is given by

$$p_0 B_{00} + p_1 B_{10} = 0 \tag{1}$$

$$p_0 B_{01} + p_1 B_{11} + p_2 B_{21} = 0 \tag{2}$$

$$p_1 B_{12} + p_2 B_{22} + p_3 B_{32} = 0 \tag{3}$$

$$p_2 B_{23} + p_3 A_1 + p_4 A_2 = 0 \tag{4}$$

$$p_i A_0 + p_{i+1} A_1 + p_{i+2} A_2 = 0 \text{ for } i \geq 3$$

$$p_i = p_3 R^{(i-3)} \text{ for } i \geq 3 \tag{5}$$

and the normalizing condition is

$$p_0 + p_1 e_1 + +p_2 e_2 + p_3 (I - R)^{-1} e_3 = 1 \tag{6}$$

where e_1, e_2, and e_3 are column vectors with elements that are all one and in the proper order, and R is called the Rate matrix, which has the minimal non-negative solution of the matrix quadratic equation as follows:

$$R^2 A_2 + R A_1 + A_0 = 0 \tag{7}$$

Theorem 3.1: If $\rho < 1$ then (7) has the minimal non-negative solution $R = -[A_0 + R^2 A_2] A_1^{-1}$.

Proof: Since A is reducible. The analysis present in Netus [31] is not applicable. According to Lucantoni [32], similar reducible matrix is treated for the case when the elements are probabilities. Equation (7), can be written as,
$A_0 A_1^{-1} + R A_1 A_1^{-1} + R^2 A_2 A_1^{-1} = 0 A_1^{-1}$
Since A_1 is non-singular, A_1^{-1} exists. Therefore

$$R = -[A_0 + R^2 A_2] A_1^{-1} \tag{8}$$

Using Netus and Lucantoni [33] the matrix R is numerically computed by using the recurrence relation with $R(0) = 0$ in equation (8). ■

C. Boundary Probability Vectors

We should solve equations (1)-(4) and (6) to obtain the stationary boundary probability vectors p_0, p_1, p_2 and p_3. Additionally, we can define matrices as follows:
$D_1 = B_{11} - B_{10}B_{00}^{-1}B_{01}$, $D_2 = B_{22} - B_{21}D_1^{-1}B_{12}$
$D_3 = B_{32}D_2^{-1}B_{21}D_1^{-1}$
It is easy to show that matrices D_1, D_2, D_3 are invertible. The following theorem gives the boundary probability vectors p_0, p_1, p_2 and p_3.

Theorem 3.2: The boundary probability vectors are given by
$p_0 = -p_3D_3B_{10}B_{00}^{-1}$
$p_1 = p_3D_3$
$p_2 = -p_3B_{32}D_2^{-1}$ and the following equations determine p_3:

$$\begin{cases} p_3(-B_{32}D_2^{-1}B_{23} + A_1 + RA_2) = 0 \\ p_3(-D_3B_{10}B_{00}^{-1} + D_3e_1 - B_{22}D_2^{-1}e_2 + (I-R)^{-1}e_3) = 1 \end{cases} \tag{9}$$

Proof: We know B_{00} is invertible and from (1)-(3) we get

$$p_0 = -p_1B_{10}B_{00}^{-1} \tag{10}$$

$$p_1 = -p_2B_{21}D_1^{-1} \tag{11}$$

$$p_2 = -p_3B_{32}D_2^{-1} \tag{12}$$

using equations (10)-(12) we get

$$p_0 = -p_3D_3B_{10}B_{00}^{-1}$$

$$p_1 = p_3D_3$$

Substituting equation (5) in equation (4) we have

$$p_2B_{23} + p_3(A_1 + RA_2) = 0 \tag{13}$$

using equations (6) and (13) we get (9)
This gives the required proof. ■

D. Remark

- If $\mu_3 = \theta_3 = 0$, the current model reduces to $M/M/2$ multiple vacation queueing system with server breakdown.
- If $\mu_2 = \mu_3 = \theta_2 = \theta_3 = 0$, our proposed model reduces to $M/M/1$ multiple vacation queueing system with server breakdown.
- If $\alpha = \mu_3 = \theta_3 = 0$ and β=1, the current model reduces to $M/M/2$ multiple vacation queueing system, which is studied by Kumar and Madheshwari [34].
- If $\alpha = \mu_2 = \mu_3 = \theta_2 = \theta_3 = 0$, $\beta = 1$, our proposed model reduces to $M/M/1$ multiple vacation queueing system.

IV. Performance Measures

By using the normal calculations, performance measures are calculated as follows:

1) Mean number of customers in the system
$E(L) = p_1e_1 + 2p_2e_2 + p_3r^{-2}((I-R)^{-2} - (1+2R))e_3$
2) The probability that
   - all the servers are on vacation $(P_v) = \sum_{i=0}^{\infty} p_{i0}$
   - only the server 1 is busy $(P_{1b}) = \sum_{i=1}^{\infty} p_{i1}$
   - only the server 2 is busy $(P_{2b}) = \sum_{i=1}^{\infty} p_{i2}$
   - only the server 3 is busy $(P_{3b}) = \sum_{i=1}^{\infty} p_{i3}$
   - the servers 1 and 2 are busy $(P_{(1,2)b}) = \sum_{i=2}^{\infty} p_{i4}$
   - the servers 2 and 3 are busy $(P_{(2,3)b}) = \sum_{i=2}^{\infty} p_{i5}$
   - the servers 1 and 3 are busy $(P_{(1,3)b}) = \sum_{i=2}^{\infty} p_{i6}$
   - all the servers are busy $(P_b) = \sum_{i=3}^{\infty} p_{i7}$
   - all the servers have breakdown $(P_{br}) = \sum_{i=3}^{\infty} p_{i8}$

V. Numerical Results

In this section, we discuss the numerical results we have obtained by the model described in this paper. To gain an understanding of the performance of this system, we study the effects of the parameters on the specific probabilistic descriptions, which means the number of customers in the system. Figures (1)-(4) show the effect of the arrival rate on steady state probabilities. Other values are as follows: μ_1=15, μ_2=12, μ_3=9, θ_1=7, θ_2=5, θ_3=3, α=4, β=6. Figure 1, tells us that if the arrival rate increases, the probability that all the servers will be on vacation state will decrease. As the arrival rate increases, the system will progress to the next state. That is, the system moves on to servers busy states. From Figure 2, we notice that if the arrival rate increases, the probability of a single server being in a busy state (any one server being busy while others are on vacation) decreases. When the arrival rate increases, the system moves on to the state where any two servers out of the three are busy or all the servers busy. From figure 3, we observe that if the arrival rate increases, the probability that any two servers busy states (any two servers are busy, another one is on vacation) increases. And also from Figure 4, we notice that if the arrival rate increases, the probability that all the servers are busy and the servers are in breakdown states increases. If the arrival rates increase, the system will most likely switch to any two or all three busy servers.

Figures (5)-(7) show the effect of the arrival rate on the mean number of customers in the system for different values of service rate. We fix the other parameters as θ_1=7, θ_2=5, θ_3=3, α=4, β=6. In figure 5, we fix μ_1 and μ_2 values as 15 and 12 respectively and draw the graphs for different values (7 ,9, 11) of μ_3. From figure 5, we observe that if the arrival rate increases, the mean number of customers in the system also increases. Also, we notice that if the service rate of server 3 increases, the mean number of customers in the system decreases. In figure 6, we fix μ_1 and μ_3 values as 15 and 9 respectively and draw the graphs for different values (12, 14, 16) of μ_2. From figure 6, we observe that if the arrival rate increases, the mean number of customers in the system also increases. Also, we notice that if the service rate of server 2 increases, the mean number of customers in the system decreases. In figure 7, we fix μ_2 and μ_3 values as 12 and 9 respectively and draw the graphs for different values (13, 15, 17) of μ_1. From figure 7, we observe that if the arrival rate increases, the mean number of customers in the system also increases. Also, we notice that if the service rate of server 1 increases, the mean number of customers in the system decreases.

Figures (8)-(10) show the effect of the breakdown rate on the mean number of customers in the system for different values of service rates. We fix the other parameters as λ=13, θ_1=7, θ_2=5, θ_3=3, β=6. In figure 8, we fix μ_1 and μ_2 values as 15 and 12 respectively and draw the graphs for different values (7, 9, 11) of μ_3. From figure 8, we observe that if the breakdown rate increases, the mean number of customers in the system also increases. Also, we notice that if the service rate of server 3 increases, the mean number of customers in the system decreases. In figure 9 we fix μ_1 and μ_3 values as 15 and 9 respectively and draw the graphs for different values (10, 12, 14) of μ_2. From figure 9 we observe that if the breakdown rate increases, the mean number of customers in the system also increases. Also, we notice that if the service rate of server 2 increases, the mean number of customers in the system decreases. In figure 10, we fix μ_2 and μ_3 values as 12 and 9 respectively and draw the graphs for different values (13, 15, 17) of μ_1. From figure 10, we observe that if the breakdown rate increases, the mean number of customers in the system also increases. Also, we notice that if the service rate of server 1 increases, the mean number of customers in the system decreases.

Figures (11)-(13) show the effect of the repair rate on the mean number of customers in the system for different values of service rate. We fix the other parameters as λ=13, θ_1=7, θ_2=5, θ_3=3, α=4. In figure 11 we fix μ_1 and μ_2 values as 15 and 12 respectively and draw the graphs for different values (7, 9, 11) of μ_3. From figure 11, we observe that if the repair rate increases, the mean number of customers in the system decreases. Also, we notice that if the service rate of server 3 increases, the mean number of customers in the system decreases. In figure 12 we fix μ_1 and μ_3 values as 15 and 9 respectively and draw the graphs for different values (10, 12, 14) of μ_2. From figure 12, we observe that if the repair rate increases, then the mean number of customers in the system decreases. Also, we notice that if the service rate of server 2 increases, the mean number of customers in the system decreases. In figure 13, we fix μ_2 and μ_3 values as 12 and 9 respectively and draw the graphs for diffent values (13,15,17) of μ_1. From figure 13, we observe that if the repair rate increases, the mean number of customers in the system decreases. Also, we notice that if the service rate of server 1 increases, the mean number of customers in the system decreases.

A. *Comparison of Heterogeneous and Homogeneous service rates*

In this section we give comparison between the heterogenous and homogeneous service rates. In tables I-III, for the case heterogenous service rates we take $\mu_1 = 0.8\mu$, $\mu_2 = 1.1\mu$, $\mu_3 = 1.4\mu$ and for homogenous service rates we take $\mu_1 = \mu_2 = \mu_3 = \mu$ also we are varying the values of μ as 11, 11.5 and 12.

For table I we take the values of $\alpha = 4$ and $\beta = 6$ and varying the values of λ as 13, 14, 15, 16, 17. From that table we observe that if the arrival rate increase, the mean number of customers in the system increases. For table II we take the values of $\lambda = 13$ and $\beta = 6$ and varying the values of α as 4.0, 4.1, 4.2, 4.3, 4.4. From that table we observe that if the breakdown rate increase, the mean number of customers in the system increases. For table III we take the values of $\lambda = 13$ and $\alpha = 4.0$ and varying the values of β as 5.8, 5.9, 6.0, 6.1, 6.2. From that table we observe that if the repair rate increase, the mean number of customers in the system decreases. Also we can notice from tables I-III, if the service rate for the servers increases, the mean number of customers in the system decreases. In addition we observe that for any particular values, the mean number of customers in the system for heterogenous service rates is less than the mean number of customers in the homogenous service rates.

TABLE I: λ Vs E(L)

λ	E(L)					
	$\mu = 11$		$\mu = 11.5$		$\mu = 12$	
	Hetero.	Homo.	Hetero.	Homo.	Hetero.	Homo.
13	0.423477	0.694305	0.208174	0.464516	0.017432	0.262930
14	0.946101	1.235471	0.700748	0.970209	0.482742	0.736593
15	1.489247	1.803075	1.214066	1.502355	0.968846	1.236493
16	2.048675	2.392333	1.744174	2.056584	1.472068	1.758672
17	2.620646	2.998919	2.287510	2.628823	1.989047	2.299353

TABLE II: α Vs E(L)

α	E(L)					
	$\mu = 11$		$\mu = 11.5$		$\mu = 12$	
	Hetero.	Homo.	Hetero.	Homo.	Hetero.	Homo.
4.0	0.423477	0.694306	0.208175	0.464517	0.017433	0.262930
4.1	0.439359	0.714124	0.221909	0.481636	0.029351	0.277767
4.2	0.455387	0.734133	0.235767	0.498919	0.041377	0.292746
4.3	0.471562	0.754336	0.249751	0.516368	0.053510	0.307866
4.4	0.487883	0.774734	0.263860	0.533983	0.065751	0.323130

TABLE III: β Vs E(L)

β	E(L)					
	$\mu = 11$		$\mu = 11.5$		$\mu = 12$	
	Hetero.	Homo.	Hetero.	Homo.	Hetero.	Homo.
5.8	0.447962	0.724310	0.229536	0.490663	0.036135	0.285791
5.9	0.435469	0.709001	0.218636	0.477321	0.026591	0.274125
6.0	0.423477	0.694306	0.208175	0.464517	0.017433	0.262930
6.1	0.411958	0.680191	0.198127	0.452219	0.008637	0.252178
6.2	0.400887	0.666623	0.188470	0.440398	0.000183	0.241846

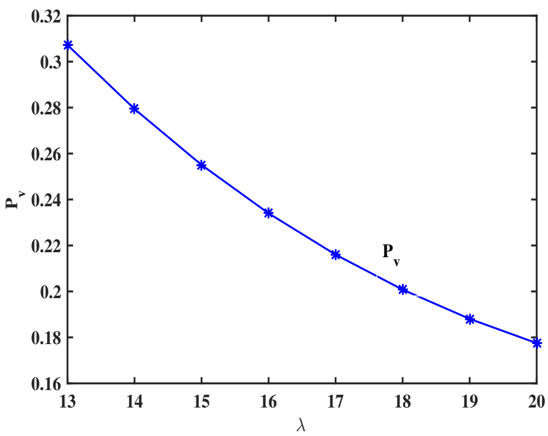

Fig. 1: λ Vs P_v

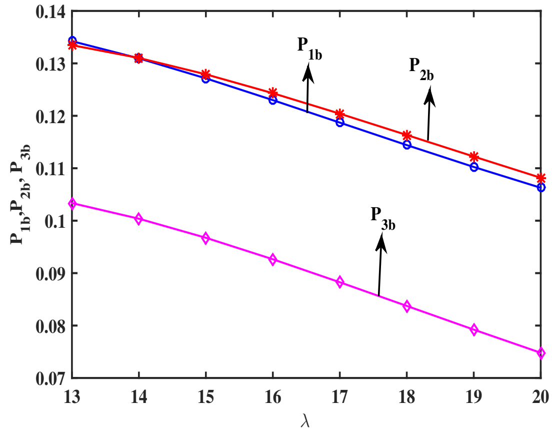

Fig. 2: λ Vs P_{1b}, P_{2b}, P_{3b}

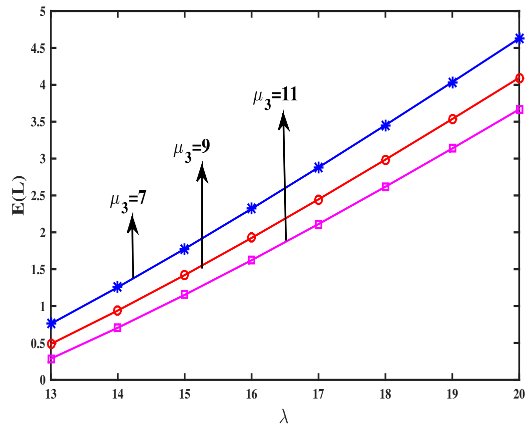

Fig. 5: λ Vs $E(L)$

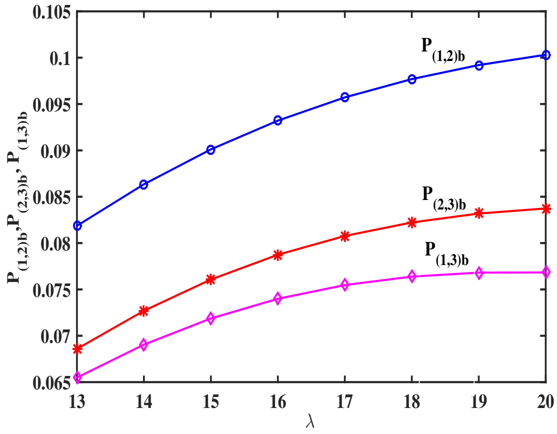

Fig. 3: λ Vs $P_{(1,2)b}, P_{(2,3)b}, P_{(1,3)b}$

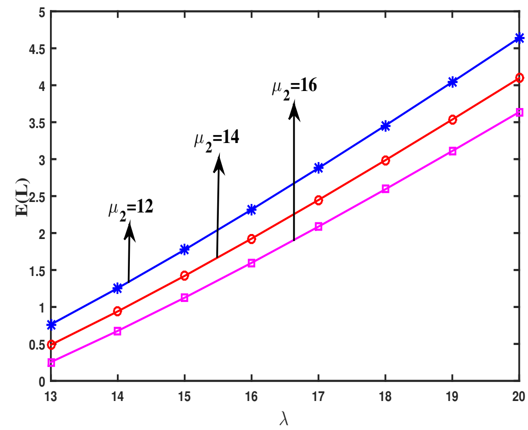

Fig. 6: λ Vs $E(L)$

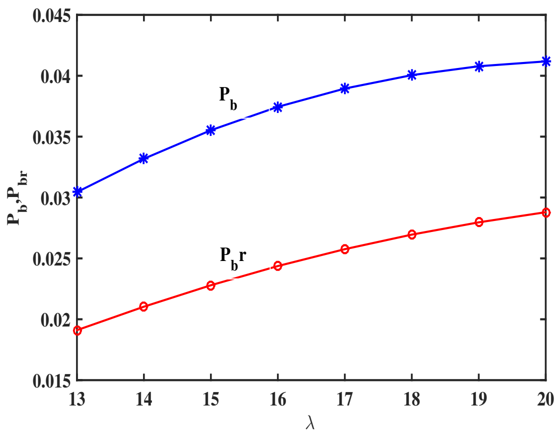

Fig. 4: λ Vs P_b, P_{br}

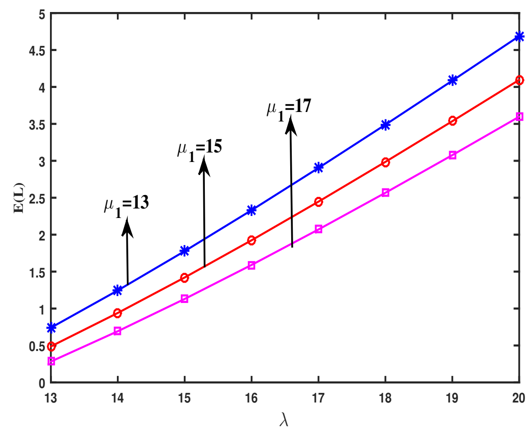

Fig. 7: λ Vs $E(L)$

VI. Conclusion

In this work, we have analysed a three-server heterogenous Markovian queueing system with multiple vacations and server breakdown. We have provided the stationary condition and boundary probability vectors for our model. The influence of λ on steady state probabilities and mean number of customers in the system for different values of service rates is also graphed in the numerical section. And also we have tested the effects of α and β on mean number of customers in the system for various service rates. In future we can find the explit expression of the rate matrix R and the conditional stochastic decomposition properties of the stationary queue length and the stationary waiting time for this proposed model. And also this work will be extended to heterogenous multi server multiple vacations queueing model with servers breakdown.

Acknowledgement

The authors appreciate and thank Dr. R. Padma Priya as a recognised research supervisor of English and Foreign

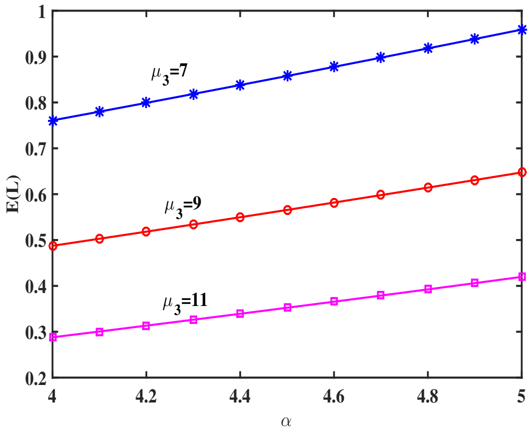

Fig. 8: α Vs $E(L)$

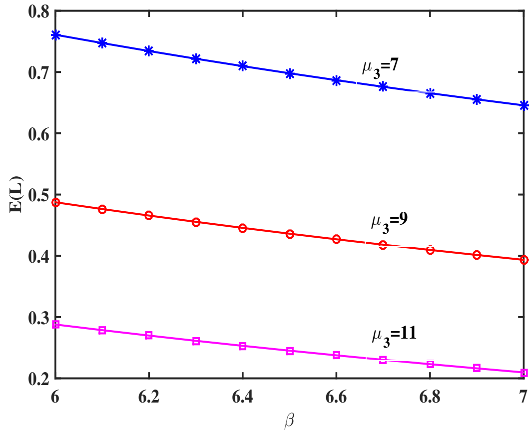

Fig. 11: β Vs $E(L)$

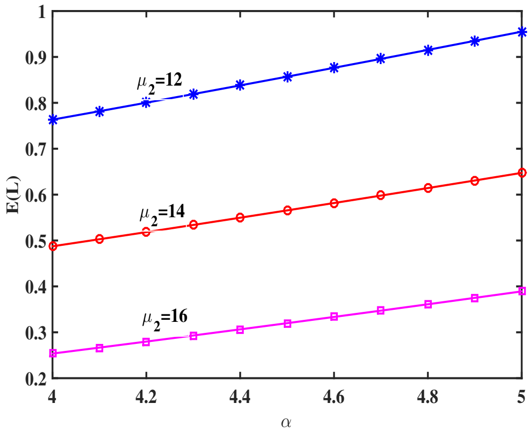

Fig. 9: α Vs $E(L)$

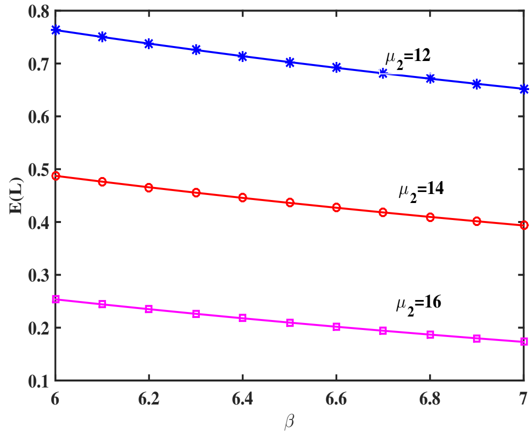

Fig. 12: β Vs $E(L)$

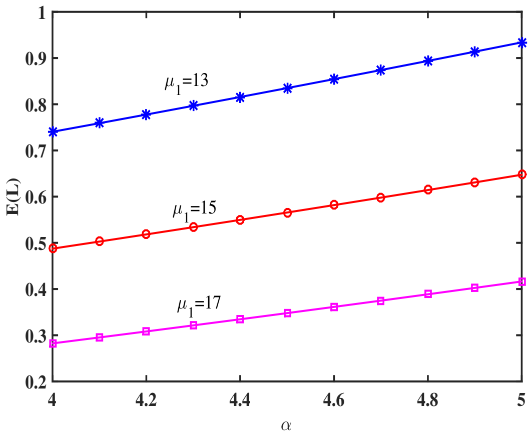

Fig. 10: α Vs $E(L)$

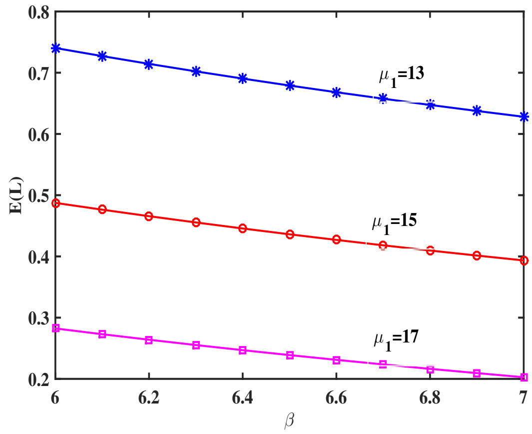

Fig. 13: β Vs $E(L)$

Languages and Assistant Professor from SRM Institute of Science and Technology, Chennai, India for proofreading and English language corrections.

REFERENCES

[1] B. T. Doshi, "Queueing systems with vacations – a survey", *Queueing Systems*, vol. 1, no. 1, pp. 29-66, 1986.
[2] H. Takagi, "Queueing Analysis, A Foundation of Performance Evaluation", *Vacation and Priority Systems*, vol. 1, no. 1, 1991.
[3] N. Tian and Z. G. Zhang, "Vacation Queueing Models: theory and applications", *Springer Science and Business Media*, vol. 93, 2006.
[4] J. C. Ke, C. H. Wu and Z. G. Zhang, "Recent Developments in Vacation Queueing Models: A Short Survey", *Operation Research*, vol. 7, no. 4, pp. 3-8, 2010.
[5] S. Upadhyaya, "Queueing Systems with Vacations: An Overview", *Int. Math. Oper. Res*, vol. 9, no. 2, pp. 167–213, 2016.
[6] Y. Levy and U. Yechiali, "An $M/M/c$ queue with server's vacations", *Information systems and operational research*, vol. 14, no. 2, pp. 153-163, 1976.
[7] B. Vinod, "Exponential queues with server vacations", *Journal of the Operational Research Society*, vol. 37, no. 10, pp. 1007-1014, 1986.
[8] N. Tian, Q. L. Li and J. Gao, "Conditional stochastic decompositions in the $M/M/c$ queue with server vacations", *Stochastic Models*, vol. 15, no. 2, pp. 367-377, 1999.

[9] E. P. Kao and K. S. Narayanan, “Analyses of an M/M/N/queue with server’s vacations”, *European Journal of Operational Research*, vol. 54, no. 2, pp. 256-266, 1991.
[10] N. Igaki, “Exponential two server queue withN-policy and general vacations”, *Queueing systems* vol. 10, no. 4, pp. 279-294, 1992.
[11] X. Chao and Y. Q. Zhao, “Analysis of multi-server queues with station and server vacations”, *European Journal of Operational Research*, vol. 110, no. 2, pp. 392-406, 2015.
[12] Z. G. Zhang and N. Tian, “Analysis on queueing systems with synchronous vacations of partial servers”, *Performance Evaluation*, vol. 52, no. 4, pp. 269-282, 2003.
[13] Z. G. Zhang and N. Tian, “An analysis of queueing systems with multi-task servers”, *European Journal of Operational Research*, vol. 156, no. 2, pp. 375-389. 2004.
[14] M. Houalef, A. A. Bouchentouf and L. Yahiaoui, “A Multi-server Queue in a Multi-phase Random Environment with Waiting Servers and Customers Impatience Under Synchronous Working Vacation Policy”, *Journal of the Operations Research Society of China*, pp. 1-29, 2022.
[15] J. T. Li and T. Li and M. An “An M/M/1 retrial queue with working vacation, orbit search and balking”, *Engineering Letters,* vol. 27, no. 1, pp. 97-102, 2019.
[16] K. C. Madan, W. Abu-Dayyeh and F. Taiyyan, “A two server queue with Bernoulli schedules and a single vacation policy”, *Applied Mathematics and Computation*, vol. 145, no. 1, pp. 59-71, 2003.
[17] V. P. Singh, “Two-server Markovian queues with balking: heterogeneous vs. homogeneous servers” *Operations Research*, vol. 18, no. 1, pp. 145-159, 1970.
[18] C. H. Lin and J. C. Ke, “Multi-server system with single working vacation”, *Appl Math Model*, vol. 33, no. 7, pp. 33-7, 2009.
[19] K. Kalidass, J. Gnanaraj, S. Gopinath and K. Ramanath, “Transient analysis of an M/M/1 queue with a repairable server and multiple vacations”, *International Journal of Mathematics in Operational Research*, vol. 4, no. 2, pp. 193-216, 2014.
[20] L.V. Shengli, W. U. Yueying, L. I. Jingbo and R. Wang, “The M/M/1 Repairable Queueing System with Variable Input Rates and Failure Rates”, *IAENG International Journal of Applied Mathematics*, vol. 52, no. 4, pp. 784-790, 2022.
[21] L.V. Shengli, “Two repairmen machine repairable system with flexible repair policy,” *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp. 336-341, 2020.
[22] L. I. Jing and L. I. Tao, “An M/G/1 G-queue with Server Breakdown, Working Vacations and Bernoulli Vacation Interruption”, *IAENG International Journal of Applied Mathematics,* vol. 50, no. 2, pp. 421-428, 2020.
[23] S. R. Chakravarthy and R. Kulshrestha, “A queueing model with server breakdowns, repairs, vacations, and backup server”, *Operations Research Perspectives*, vol. 7, pp. 100-131, (2020).
[24] M. Seenivasan and R. Abinaya, “Markovian Queueing Model with Single Working Vacation and Server Breakdown”, *Journal of Computational Analysis and Applications*, vol. 30, no. 2, pp. 409-421, 2022.
[25] R. R. Dasa, V. N. Devib, A. Rathore and K. Chandand, “Analysis of Markovian queueing system with server failures, N-policy and second optional service”, *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 1, pp. 3073-3083, 2022.
[26] I. Atencia, “A discrete-time queueing system with server breakdowns and changes in the repair times”, *Annals of Operations Research*, vol. 235, no. 1, pp. 37-49, 2015.
[27] G. Gupur, “Analysis of the M/G/1 retrial Queuing Model with server breakdowns”, *Journal of Pseudo-Differential Operators and Applications*, vol. 1, no. 3, pp. 313-340, 2010.
[28] M. Jain and A. Jain, “Working vacations queueing model with multiple types of server breakdowns”, *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 1-13, 2010.
[29] Y. L. Tsai, D. Yanagisawa and K. Nishinari, “Performance analysis of open queueing networks subject to breakdowns and repairs”, *Engineering Letters*, vol. 24, no. 2, pp. 207-214, 2016.
[30] D. Y. Yang, Y. H. Chen and C. H. Wu, “Modelling and optimisation of a two-server queue with multiple vacations and working breakdowns”, *International Journal of Production Research*, vol. 58, no. 10, pp. 3036-3048, 2020.
[31] M. F. Neuts, “Matrix-Geometric Solutions in Stochastic Models”, *Johns Hopkins University Press, Baltimore, Md.*, 1981.
[32] D. M. Lucantoni, “$AGI/M/C$ queue with a different service rate for customers who need not wait an algorithmic solution”, *Technical Rep. Uniersity of Delware, USA*,1979.
[33] M. F. Neuts and D. M. Lucantoni, “A Markovian queue with N servers subject to breakdowns and repairs”, *Management Science*, vol 25, no. 9, pp. 849-861, 1979.
[34] B. K. Kumar, and S. P. Madheswari, “An $M/M/2$ queueing system with heterogeneous servers and multiple vacations”, *Mathematical and Computer Modelling*, vol. 41, no. 13, pp. 1415-1429, 2005.