A Minimum Problem of Cepstral-based Clustering for Time Series

Yu-Lan Wang   Chin Lin Wen   Ming-Li Chen

Abstract—**We study the minimum problem proposed by a recently published paper to point out that their formulated optimal solution is incomplete since in their formulations variables are mingled together. For a reduced version of their minimum problem, we convert the objective function from six variables into a variable minimum problem. For a special case of the fuzzy controlling parameter, we derive an approximated formulated solution that attains the minimum value within 0.0208% average estimation error. In the direction of future research, we propose a minimum problem under symmetric expressions.**

Index Terms—**Minimum problem, Financial time series, Weighting system, Fuzzy c-medoids**

I. Introduction

We study the paper of D'Urso et al. [1] that was published in Expert System with Applications to develop a clustering model for financial time series to examine the minimum problem of cepstral-based clustering for time series. We recall that the cepstrum is denoted as the inverse Fourier transform of the method of the Fourier transform with the time series, and then the cepstral analysis is developed as a non-linear signal processing method. A time series is an assembling of data collected from sequential quantity over time. Researchers tried to image the shape of information by data mining those data from time series and to avoid small fluctuations in time series through similarities between patterns. There are many types of time series, for example, clustering: Lin and Keogh [2]; motif discovery: Lin et al. [3]; intrusion detection: Liu et al. [4]; segmentation: Wu and Leahy [5]; classification: Bakshi and Stephanopoulos [6]; prediction: Golub et al. [7], and query by content: Faloutsos et al. [8]. In the following, we provide a brief review of recent articles that dealt with time series by clustering data. Kalpakis et al. [9] applied the Auto-Regressive Integrated Moving Average (ARIMA) models to handle economic and environmental time series by the Linear Predictive Coding (LPC) cepstrum with Euclidean distance measure to cluster data. Zhong and Ghosh [10] examined an integrated framework to cluster data and models under a bipartite graph view to discover differences and similarities among previously published clustering algorithms. Laxman and Sastry [11] constructed a survey for recent research with unstructured and large-volume information to find relationships and regularities among data for temporal data mining. Wang and Hyndman [12] considered handling noisy or missing data by their structural characteristics of very long time series in finance and medicine to study the underlying characteristics: chaos, self-similarity, nonlinearity, serial correlation, kurtosis, seasonality, skewness, and periodicity. Lin et al. [13] developed a new symbolic representation of time series to reduce the dimensionality of the symbolic representation for the original time series and then they constructed distance measures to operate data mining algorithms. Buchin et al. [14] generalized curve analysis to the free space algorithm to study the single file movement with theoretical and experimental validation. Zhang et al. [15] constructed a new method for shape-based time series clustering to reduce the size and to improve efficiency by similarity with triangle distance and then validated by synthetic and real data to illustrate its efficiency and effectiveness. Rani and Sikka [16] provided a literature survey for clustering time series to realize its insight and then to forecast the future values for the coming time series with the following areas: government, economics, business, health care, finance, and engineering. D'Urso and Maharaj [17] developed the fuzzy relational method with univariate and multivariate wavelet features to cluster time series under different error correlation structures. Kini and Sekhar [18] combined several margins autoregressive models to construct a large margin autoregressive model to apply their new model to electroencephalogram time series data, the simulated time-series data, speech data for E-set in the English alphabet, and electrocardiogram data. They compared their new model with a support vector machine to obtain a better classification performance. Montero and Vilar [19] studied the dissimilarity among time series for cluster analysis with the R package TSclust to examine forecast behaviors, underlying parametric models, complexity levels, extracted features, and raw data. Bagnall et al. [20] examined time-series classification (TSC) to obtain improved precise algorithms by simple ensemble schemes to derive improved accuracy and transform time-series data into another space to find discriminatory features. D'Urso et al. [21] established several fuzzy clustering models for analyzing unstable time series to neutralize the negative effects of the noise and to obtain robustness. Marjani et al. [22] presented a literature review for the state-of-the-art research efforts with the Internet of things (IoT) and big data analytics by providing a new

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structure for big IoT data analytics. Murray et al. [23] tried to develop an approach for each customer with individual historical transaction data under noise and imprecision to group them by segmentation and then provided segment-level forecasting. Under available data, they compared their results with other traditional methods to show improved accuracy for a large population of customers. D'Urso et al. [24] examined three issues to consider the assignment of a geographical unit under uncertainty, to cluster multivariate time trajectories under their space characteristics, and to group the units under their spatial nature. They developed a fuzzy partitioning around medoids method for multivariate time series with different lengths. Soheily-Khah and Marteau [25] applied Dynamic Time Warping (DTW) to deal with temporal data to mining and analyze by similarity and distance measures to avoid the quadratic computational cost resulting from a large-scale application and fit with Support Vector Machine (SVM) under a direct positive definite kernel. Chintalapudi et al. [26] studied a real case of the COVID-19 virus outbreak in Italy for the sixty-day lockdown and predict the influence of further action to extend another sixty-day lockdown with self isolation by data-driven model analysis. Righi et al. [27] provided a proactive elasticity system to merge high performance computing and the cloud to study the IoT scalability issue under globally compliant architecture to be compatible with the model load.

Based on the above literature review, we can claim that clustering time series is a hot research topic among researchers. We will point out their solution approach for the minimum problems of is D'Urso et al. [1] incomplete and then we provide our improvements.

II. Notation

To be compatible with D'Urso et al. [1], we follow them to use the following notation.

C denotes the index set for clusters, with $c = 1,2,\dots,C$.

I denotes the index set for units, with $i = 1,2,\dots,I$.

K denotes the index set for unconditional moments, with $k = 1,2,\dots,K$.

u_{ic} denotes the membership degree of the i-th unit to the c-th cluster, under the restriction $u_{ic} \geq 0$, and $\sum_{c=1}^{C} u_{ic} = 1$.

w_k denotes the weight of the k-th estimated unconditional moments coefficient, under the restriction $w_k \geq 0$, and $\sum_{k=1}^{K} w_k = 1$.

m is the controlling parameter for the fuzziness of the partition, with $m > 1$.

cp_{ik} is the k-th unconditional moments estimated for the i-th time series according to the dynamic conditional score (generalized autoregressive score).

$\widetilde{cp}_{ck}$ represents the c-th medoid.

III. Review of D'Urso et al. [1]

We cite their clustering model as follows:

$$\min \sum_{i=1}^{I} \sum_{c=1}^{C} u_{ic}^{m} \sum_{k=1}^{K} [w_k(cp_{ik} - \widetilde{cp}_{ck})]^2 \tag{3.1}$$

under the restrictions $\sum_{k=1}^{K} w_k = 1$, $w_k \geq 0$, $\sum_{c=1}^{C} u_{ic} = 1$, $u_{ic} \geq 0$, for $i = 1,2,\dots,I$, and $m > 1$ which is developed by D'Urso et al. [28].

Based on the Lagrangian method, D'Urso et al. [1] mentioned their minimum solutions,

$$u_{ic} = \frac{1}{\sum_{x=1}^{C}\left(\frac{\sum_{k=1}^{K}[w_k(cp_{ik}-\widetilde{cp}_{ck})]^2}{\sum_{k=1}^{K}[w_k(cp_{ik}-\widetilde{cp}_{xk})]^2}\right)^{\frac{1}{m-1}}}, \tag{3.2}$$

for $i = 1,2,\dots,I$, and $c = 1,2,\dots,C$, and

$$w_k = \frac{1}{\sum_{y=1}^{K}\left(\frac{\sum_{i=1}^{I}\sum_{c=1}^{C}\left[u_{ic}^{m}(cp_{ik}-\widetilde{cp}_{ck})\right]^2}{\sum_{i=1}^{I}\sum_{c=1}^{C}\left[u_{ic}^{m}(cp_{iy}-\widetilde{cp}_{cy})\right]^2}\right)}, \tag{3.3}$$

for $k = 1,2,\dots,K$.

We will provide a simplified version of Equation (3.1) to demonstrate that the solutions of Equations (3.2) and (3.3) proposed by D'Urso et al. [1] are incomplete.

We assume that $I = 2$, $C = 2$, and $K = 2$. To further simplify the expression, we assume that $(cp_{ik} - \widetilde{cp}_{ck})^2 = \Delta_{ick}$. Hence, we will use the following minimum problem, which we denote as $M(u_{11}, u_{12}, u_{21}, u_{22}, w_1, w_2)$, with

$$M(u_{11}, u_{12}, u_{21}, u_{22}, w_1, w_2) = \sum_{i=1}^{2} \sum_{c=1}^{2} u_{ic}^{m} \sum_{k=1}^{2} \Delta_{ick} w_k^2, \tag{3.4}$$

under the restrictions $w_1 + w_2 = 1$, $w_1 \geq 0$, $w_2 \geq 0$, $u_{11} + u_{12} = 1$, $u_{21} + u_{22} = 1$, $u_{11} \geq 0$, $u_{12} \geq 0$, $u_{21} \geq 0$, $u_{22} \geq 0$, and $m > 1$, such that the solution proposed by D'Urso et al. [1] of Equations (3.2) and (3.3), can be expressed as follows

$$u_{11} = \frac{1}{\sum_{x=1}^{2}\left(\frac{\sum_{k=1}^{2}[w_k(cp_{1k}-\widetilde{cp}_{1k})]^2}{\sum_{k=1}^{2}[w_k(cp_{1k}-\widetilde{cp}_{xk})]^2}\right)^{\frac{1}{m-1}}}, \tag{3.5}$$

$$u_{12} = \frac{1}{\sum_{x=1}^{2}\left(\frac{\sum_{k=1}^{2}[w_k(cp_{1k}-\widetilde{cp}_{2k})]^2}{\sum_{k=1}^{2}[w_k(cp_{1k}-\widetilde{cp}_{xk})]^2}\right)^{\frac{1}{m-1}}}, \tag{3.6}$$

$$u_{21} = \frac{1}{\sum_{x=1}^{2}\left(\frac{\sum_{k=1}^{2}[w_k(cp_{2k}-\widetilde{cp}_{1k})]^2}{\sum_{k=1}^{2}[w_k(cp_{2k}-\widetilde{cp}_{xk})]^2}\right)^{\frac{1}{m-1}}}, \tag{3.7}$$

$$u_{22} = \frac{1}{\sum_{x=1}^{2}\left(\frac{\sum_{k=1}^{2}[w_k(cp_{2k}-\widetilde{cp}_{2k})]^2}{\sum_{k=1}^{2}[w_k(cp_{2k}-\widetilde{cp}_{xk})]^2}\right)^{\frac{1}{m-1}}}, \tag{3.8}$$

$$w_1 = \frac{1}{\sum_{y=1}^{2}\left(\frac{\sum_{i=1}^{2}\sum_{c=1}^{2}\left[u_{ic}^{m}(cp_{i1}-\widetilde{cp}_{c1})\right]^2}{\sum_{i=1}^{2}\sum_{c=1}^{2}\left[u_{ic}^{m}(cp_{iy}-\widetilde{cp}_{cy})\right]^2}\right)}, \tag{3.9}$$

and

$$w_2 = \frac{1}{\sum_{y=1}^{2}\left(\frac{\sum_{i=1}^{2}\sum_{c=1}^{2}\left[u_{ic}^{m}(cp_{i2}-\widetilde{cp}_{c2})\right]^2}{\sum_{i=1}^{2}\sum_{c=1}^{2}\left[u_{ic}^{m}(cp_{iy}-\widetilde{cp}_{cy})\right]^2}\right)}, \tag{3.10}$$

that did not solve the minimum problem of Equation (3.4).

To denote our example more clearly, we further write down Equations (3.4-3.10) in a detailed presentation as follows,

$$\min(\Delta_{111}u_{11}^{m}w_1^2 + \Delta_{112}u_{11}^{m}w_2^2 + \Delta_{121}u_{12}^{m}w_1^2 + \Delta_{122}u_{12}^{m}w_2^2 + \Delta_{211}u_{21}^{m}w_1^2 + \Delta_{212}u_{21}^{m}w_2^2 + \Delta_{221}u_{22}^{m}w_1^2 + \Delta_{222}u_{22}^{m}w_2^2), \tag{3.11}$$

under the restrictions $w_1 + w_2 = 1$, $w_1 \geq 0$, $w_2 \geq 0$, $u_{11} + u_{12} = 1$, $u_{21} + u_{22} = 1$, $u_{11} \geq 0$, $u_{12} \geq 0$, $u_{21} \geq 0$, $u_{22} \geq 0$, and $m > 1$, such that the authors obtained the following solutions,

$$u_{11} = \frac{1}{\left(\frac{\Delta_{111}w_1^2+\Delta_{112}w_2^2}{\Delta_{111}w_1^2+\Delta_{112}w_2^2}\right)^{\frac{1}{m-1}} + \left(\frac{\Delta_{111}w_1^2+\Delta_{112}w_2^2}{\Delta_{121}w_1^2+\Delta_{122}w_2^2}\right)^{\frac{1}{m-1}}}, \tag{3.12}$$

$$u_{12} = \frac{1}{\left(\frac{\Delta_{121}w_1^2+\Delta_{122}w_2^2}{\Delta_{111}w_1^2+\Delta_{112}w_2^2}\right)^{\frac{1}{m-1}} + \left(\frac{\Delta_{121}w_1^2+\Delta_{122}w_2^2}{\Delta_{121}w_1^2+\Delta_{122}w_2^2}\right)^{\frac{1}{m-1}}}, \tag{3.13}$$

$$u_{21}=\frac{1}{\left(\frac{\Delta_{211}w_1^2+\Delta_{212}w_2^2}{\Delta_{211}w_1^2+\Delta_{212}w_2^2}\right)^{\frac{1}{m-1}}+\left(\frac{\Delta_{211}w_1^2+\Delta_{212}w_2^2}{\Delta_{221}w_1^2+\Delta_{222}w_2^2}\right)^{\frac{1}{m-1}}}, \quad (3.14)$$

$$u_{22}=\frac{1}{\left(\frac{\Delta_{221}w_1^2+\Delta_{222}w_2^2}{\Delta_{211}w_1^2+\Delta_{212}w_2^2}\right)^{\frac{1}{m-1}}+\left(\frac{\Delta_{221}w_1^2+\Delta_{222}w_2^2}{\Delta_{221}w_1^2+\Delta_{222}w_2^2}\right)^{\frac{1}{m-1}}}, \quad (3.15)$$

$$w_1=\frac{1}{P_1+P_2}, \quad (3.16)$$

and

$$w_2=\frac{1}{P_3+P_4}, \quad (3.17)$$

where $P_1=\frac{\Delta_{111}u_{1,1}^m+\Delta_{121}u_{1,2}^m+\Delta_{211}u_{2,1}^m+\Delta_{221}u_{22}^m}{\Delta_{111}u_{11}^m+\Delta_{121}u_{12}^m+\Delta_{211}u_{21}^m+\Delta_{221}u_{22}^m}$,

$P_2=\frac{\Delta_{111}u_{11}^m+\Delta_{121}u_{12}^m+\Delta_{211}u_{21}^m+\Delta_{221}u_{22}^m}{\Delta_{112}u_{11}^m+\Delta_{122}u_{12}^m+\Delta_{212}u_{21}^m+\Delta_{222}u_{22}^m}$,

$P_3=\frac{\Delta_{112}u_{11}^m+\Delta_{122}u_{12}^m+\Delta_{212}u_{21}^m+\Delta_{222}u_{22}^m}{\Delta_{111}u_{11}^m+\Delta_{121}u_{12}^m+\Delta_{211}u_{21}^m+\Delta_{221}u_{22}^m}$, and

$P_4=\frac{\Delta_{112}u_{11}^m+\Delta_{122}u_{12}^m+\Delta_{212}u_{21}^m+\Delta_{222}u_{22}^m}{\Delta_{112}u_{11}^m+\Delta_{122}u_{12}^m+\Delta_{212}u_{21}^m+\Delta_{222}u_{22}^m}$.

We can further simplify the expressions of Equations (3.12-3.17) as follows,

$$u_{11}=\frac{1}{1+\left(\frac{\Delta_{111}w_1^2+\Delta_{112}w_2^2}{\Delta_{121}w_1^2+\Delta_{122}w_2^2}\right)^{\frac{1}{m-1}}}, \quad (3.18)$$

$$u_{12}=\frac{1}{\left(\frac{\Delta_{121}w_1^2+\Delta_{122}w_2^2}{\Delta_{111}w_1^2+\Delta_{112}w_2^2}\right)^{\frac{1}{m-1}}+1}, \quad (3.19)$$

$$u_{21}=\frac{1}{1+\left(\frac{\Delta_{211}w_1^2+\Delta_{212}w_2^2}{\Delta_{221}w_1^2+\Delta_{222}w_2^2}\right)^{\frac{1}{m-1}}}, \quad (3.20)$$

$$u_{22}=\frac{1}{\left(\frac{\Delta_{221}w_1^2+\Delta_{222}w_2^2}{\Delta_{211}w_1^2+\Delta_{212}w_2^2}\right)^{\frac{1}{m-1}}+1}, \quad (3.21)$$

$$w_1=\frac{1}{1+\frac{\Delta_{111}u_{11}^m+\Delta_{121}u_{12}^m+\Delta_{211}u_{21}^m+\Delta_{221}u_{22}^m}{\Delta_{112}u_{11}^m+\Delta_{122}u_{12}^m+\Delta_{212}u_{21}^m+\Delta_{222}u_{22}^m}}, \quad (3.22)$$

and

$$w_2=\frac{1}{\frac{\Delta_{112}u_{11}^m+\Delta_{122}u_{12}^m+\Delta_{212}u_{21}^m+\Delta_{222}u_{22}^m}{\Delta_{111}u_{11}^m+\Delta_{121}u_{12}^m+\Delta_{211}u_{21}^m+\Delta_{221}u_{22}^m}+1}. \quad (3.23)$$

We observe Equations (3.18-3.21) to find out the following issue. If we try to find the value of u_{11}, u_{12}, u_{21}, and u_{22}, and then we must know the exact values of w_1 and w_2 in advance such that we can evaluate the values of w_1^2 and w_2^2.

Similarly, we observe Equations (3.22-3.23), if we try to find the value of w_1 and w_2 and then we must know the exact values of u_{11}, u_{12}, u_{21}, and u_{22}, such that we can evaluate the values of u_{11}^m, u_{12}^m, u_{21}^m, u_{22}^m.

Hence, the solution procedure provided by D'Urso et al. [1] is incomplete. In Section 5, numerical examples, we will apply the incomplete solution procedure proposed by D'Urso et al. [1] to develop an iterative approach to derive the optimal solution.

IV. Our Improvements

Before we reconsider the minimum problem of Equation (3.4), we have to discuss some material for our future derivations.

We try to solve the following minimum problem

$$\min f(x,y)=ax^m+by^m, \quad (4.1)$$

under the condition $x+y=1$, $m>1$, and $0\le x\le 1$.

We substitute $x+y=1$ into $f(x,y)$ to convert the objective function from $f(x,y)$ to $f(x)$ such that

$$\min f(x)=ax^m+b(1-x)^m, \quad (4.2)$$

with $m>1$, and $0\le x\le 1$.

We derive that

$$\frac{d}{dx}f(x)=amx^{m-1}-bm(1-x)^{m-1}, \quad (4.3)$$

and

$$\frac{d^2}{dx^2}f(x)=am(m-1)x^{m-2}+bm(m-1)(1-x)^{m-2}. \quad (4.4)$$

To find critical points, we solve $\frac{d}{dx}f(x)=0$ to derive that

$$a^{1/(m-1)}x=b^{1/(m-1)}(1-x). \quad (4.5)$$

Hence, we obtain that

$$x=\frac{b^{1/(m-1)}}{a^{1/(m-1)}+b^{1/(m-1)}}, \quad (4.6)$$

and then

$$y=\frac{a^{1/(m-1)}}{a^{1/(m-1)}+b^{1/(m-1)}}. \quad (4.7)$$

From $\frac{d^2}{dx^2}f(x)>0$, we know that the solution of Equation (4.6) is the global minimum point, and then the minimum value is

$$f\left(\frac{b^{1/(m-1)}}{a^{1/(m-1)}+b^{1/(m-1)}},\frac{a^{1/(m-1)}}{a^{1/(m-1)}+b^{1/(m-1)}}\right)=\frac{ab}{\left(a^{1/(m-1)}+b^{1/(m-1)}\right)^{m-1}}. \quad (4.8)$$

Based on Equation (3.11), we can rewrite the expression of Equation (3.4) as

$$\min F(u_{11},u_{12})+F(u_{21},u_{22}), \quad (4.9)$$

where we construct two auxiliary functions $F(u_{11},u_{12})$ and $F(u_{21},u_{22})$, with

$$F(u_{11},u_{12})=A(w_1,w_2)u_{11}^m+B(w_1,w_2)u_{12}^m, \quad (4.10)$$

and

$$F(u_{21},u_{22})=C(w_1,w_2)u_{21}^m+D(w_1,w_2)u_{22}^m. \quad (4.11)$$

where $A(w_1,w_2)=\Delta_{111}w_1^2+\Delta_{112}w_2^2$, $B(w_1,w_2)=\Delta_{121}w_1^2+\Delta_{122}w_2^2$, $C(w_1,w_2)=\Delta_{211}w_1^2+\Delta_{212}w_2^2$, and $D(w_1,w_2)=(\Delta_{221}w_1^2+\Delta_{222}w_2^2)$.

We apply our findings of Equations (4.6-4.8) to $F(u_{11},u_{12})$ to derive the minimum points, u_{11}^* and u_{12}^*, as follows

$$u_{11}^*=\frac{B(w_1,w_2)^{1/(m-1)}}{A(w_1,w_2)^{1/(m-1)}+B(w_1,w_2)^{1/(m-1)}}, \quad (4.12)$$

and

$$u_{12}^*=\frac{A(w_1,w_2)^{1/(m-1)}}{A(w_1,w_2)^{1/(m-1)}+B(w_1,w_2)^{1/(m-1)}}, \quad (4.13)$$

and then the minimum value,

$$F^*(u_{11},u_{12})=\frac{A(w_1,w_2)B(w_1,w_2)}{\left(A(w_1,w_2)^{1/(m-1)}+B(w_1,w_2)^{1/(m-1)}\right)^{m-1}}. \quad (4.14)$$

By the same argument, we apply our findings of Equations (4.6-4.8) to $F(u_{21},u_{22})$ to obtain the minimum points, u_{21}^* and u_{22}^*, as follows

$$u_{21}^*=\frac{D(w_1,w_2)^{1/(m-1)}}{C(w_1,w_2)^{1/(m-1)}+D(w_1,w_2)^{1/(m-1)}}, \quad (4.15)$$

and

$$u_{22}^{*}=\frac{C(w_1,w_2)^{1/(m-1)}}{C(w_1,w_2)^{1/(m-1)}+D(w_1,w_2)^{1/(m-1)}}, \quad (4.16)$$

and then the minimum value,

$$F^{*}(u_{21},u_{22})=\frac{C(w_1,w_2)D(w_1,w_2)}{\left(C(w_1,w_2)^{1/(m-1)}+D(w_1,w_2)^{1/(m-1)}\right)^{m-1}}. \quad (4.17)$$

Based on our results of Equations (4.14) and (4.17), we convert our minimum problem from a six-variable problem, $M(u_{11},u_{12},u_{21},u_{22},w_1,w_2)$ to a two-variable problem, $M(w_1,w_2)$ as

$$M(w_1,w_2)=\frac{A(w_1,w_2)B(w_1,w_2)}{(A(w_1,w_2)^{1/(m-1)}+B(w_1,w_2)^{1/(m-1)})^{m-1}} +\frac{C(w_1,w_2)D(w_1,w_2)}{\left(C(w_1,w_2)^{1/(m-1)}+D(w_1,w_2)^{1/(m-1)}\right)^{m-1}}, \quad (4.18)$$

under $w_1+w_2=1$, $w_1\geq 0$, and $w_2\geq 0$.

We plug $w_2=1-w_1$ into Equation (4.18), and further simplify the expression with $w_1=x$, to convert our objective function from $M(w_1,w_2)$ to $M(x)$ and then we will face the following minimum problem

$$M(x)=\frac{P(x)Q(x)}{(P(x)^{1/(m-1)}+Q(x)^{1/(m-1)})^{m-1}} +\frac{R(x)S(x)}{\left(R(x)^{1/(m-1)}+S(x)^{1/(m-1)}\right)^{m-1}}, \quad (4.19)$$

where $P(x)=\Delta_{111}x^2+\Delta_{112}(1-x)^2$, $Q(x)=\Delta_{121}x^2+\Delta_{122}(1-x)^2$, $R(x)=\Delta_{211}x^2+\Delta_{212}(1-x)^2$, and

$$S(x)=\Delta_{221}x^2+\Delta_{222}(1-x)^2. \quad (4.20)$$

For a general setting of m as expressed as Equation (4.19), to prove the existence and uniqueness of the minimum solution of $M(x)$ is beyond our ability. Hence, in the following, we will consider those special parameters of m mentioned in D'Urso et al. [1].

For the controlling parameter for the fuzziness of the partition, m, we check the numerical examples in D'Urso et al. [1] to find that under the restriction of $m>1$, then there are two candidates $m=1.5$ and m=2.

When $m=2$, $M(x)$ of Equation (4.19) is changed to

$$M_{m=2}(x)=\frac{T(x)Q(x)}{T(x)+Q(x)}+\frac{R(x)S(x)}{R(x)+S(x)}. \quad (4.21)$$

We slightly change the expression of $M_{m=2}(x)$ as follows,

$$M_{m=2}(x)=\frac{1}{(1/T(x))+(1/Q(x))}+\frac{1}{(1/R(x))+(1/S(x))}. \quad (4.22)$$

For $M_{m=2}(x)$, we assume an auxiliary function, denoted as $P(x)$, where

$$P(x)=\Delta_{111}x^2+\Delta_{112}(1-x)^2 +\Delta_{121}x^2+\Delta_{122}(1-x)^2+\Delta_{211}x^2 +\Delta_{212}(1-x)^2+\Delta_{221}x^2+\Delta_{222}(1-x)^2, \quad (4.23)$$

which is motivated by the following approximation

$$\frac{1}{1/A}+\frac{1}{1/B}\sim\frac{1}{(1/A)+(1/B)}. \quad (4.24)$$

We can rewrite Equation (4.23) as

$$P(x)=(\Delta_{111}+\Delta_{121}+\Delta_{211}+\Delta_{221})x^2 +(\Delta_{112}+\Delta_{122}+\Delta_{212}+\Delta_{222})(1-x)^2, \quad (4.25)$$

and then the minimum solution of $P(x)$ is derived as

$$x^{P}=\frac{\Delta_{112}+\Delta_{122}+\Delta_{212}+\Delta_{222}}{\Delta_{111}+\Delta_{121}+\Delta_{211}+\Delta_{221}+\Delta_{112}+\Delta_{122}+\Delta_{212}+\Delta_{222}}. \quad (4.26)$$

We will treat x^P as a formulated approximated solution for $M_{m=2}(x)$.

V. Numerical Examples

We will construct several hypothetical examples to illustrate our approximated formulated solution of Equation (4.23) is very close to the optimal solution by numerical approach.

For our first numerical example, we assume that $\Delta_{111}=4$, $\Delta_{112}=9$, $\Delta_{121}=16$, $\Delta_{122}=25$, $\Delta_{211}=36$, $\Delta_{212}=49$, $\Delta_{221}=64$, and $\Delta_{222}=81$. Based on Equation (4.23), we find our formulated approximated solution, $x^P=0.577$ and the approximated minimum value

$$M(x^P=0.577)=15.400218. \quad (5.1)$$

We list numerical results for our first numerical example in the following tables 1-3.

Based on Table 1, we can claim that $M(x)$ is a convex function and the minimum point lies in the interval [0.5,0.6]. Hence, we check values among [0.5,0.6] and then list them in the following table 2.

Based on Table 2, we can claim that $M(x)$ is a convex function and the minimum point lies in the interval [0.59,0.6]. Hence, we check values among [0.59,0.6] and then list them in the following table 3.

Based on Table 3, we must extend the decimal expressions of $M(0.590)$ and $M(0.591)$ to find that

$$M(0.590)=15.395779, \quad (5.2)$$

and

$$M(0.591)=15.396322, \quad (5.3)$$

such that the minimum point is $x^*=0.590$ and the minimum value is $M(0.590)=15.395779$.

Next, we compare our formulated approximated value with the numerical approach to obtain that

$$\frac{15.400218-15.395779}{15.395779}=0.0253\%. \quad (5.4)$$

Based on our findings of Equation (5.4), we can claim that our approximated value is very close to the numerical approach result that has a 0.0253% estimation error.

For our second numerical example, we assume that $\Delta_{111}=100$, $\Delta_{112}=81$, $\Delta_{121}=64$, $\Delta_{122}=49$, $\Delta_{211}=36$, $\Delta_{212}=25$, $\Delta_{221}=16$, and $\Delta_{222}=9$. Based on Equation (4.26), we find our formulated approximated solution, $x^P=0.432$ and the approximated minimum value

$$M(x^P=0.432)=21.347733. \quad (5.5)$$

Table 1. List of values for our first example to the first decimal place.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
M(x)	37.15	30.35	24.83	20.57	17.59	15.86	15.41	16.21	18.29	21.63	26.24

Table 2. List of values for our first example to the second decimal place.

x	0.5	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.6
M(x)	15.86	15.76	15.67	15.59	15.53	15.48	15.44	15.41	15.397	15.396	15.41

Table 3. List of values for our first example to the third decimal place.

x	0.59	0.591	0.592	0.593	0.594	0.595	0.596	0.597	0.598	0.599	0.6
M(x)	15.396	15.396	15.397	15.398	15.399	15.400	15.401	15.402	15.404	15.405	15.407

Table 4. List of values for our second example to the first decimal place.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
M(x)	37.15	30.59	25.78	22.72	21.40	21.82	23.99	27.70	33.55	40.95	50.10

Table 5. List of values for our second example to the second decimal place.

x	0.4	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.5
M(x)	21.40	21.36	21.347	21.346	21.362	21.396	21.447	21.515	21.601	21.704	21.82

Table 6. List of values for our second example to the third decimal place.

x	0.42	0. 421	0. 422	0. 423	0. 424	0. 425	0. 426	0. 427	0. 428	0. 429	0.43
M(x)	21.347	21.346	21.3455	21.3449	21.3445	21.3443	21.3443	21.3444	21.3447	21.3452	21.346

We list numerical results for our first numerical example in the above tables 4-6.

Based on Table 4, we can claim that $\mathrm{M(x)}$ is a convex function and the minimum point lies in the interval $[0.4,0.5]$. Hence, we check values among $[0.5,0.6]$ and then list them in the above table 5.

Based on Table 5, we can claim that $\mathrm{M(x)}$ is a convex function and the minimum point lies in the interval $[0.42,0.43]$. Hence, we check values among $[0.42,0.43]$ and then list them in the above table 6.

Based on Table 6, we must extend the decimal expressions of $\mathrm{M}(0.425)$ and $\mathrm{M}(0.426)$ to find that

$$\mathrm{M}(0.425) = 21.344312, \tag{5.6}$$

and

$$\mathrm{M}(0.426) = 21.344278, \tag{5.7}$$

such that the minimum point is $\mathrm{x}^* = 0.426$ and the minimum value is $\mathrm{M}(0.426) = 21.344278$.

Next, we compare our formulated approximated value with the numerical approach to obtain that

$$\frac{21.347733-21.344278}{21.344278} = 0.000162. \tag{5.8}$$

Based on our findings of Equation (5.8), we can claim that our approximated value is very close to the numerical approach result that has a 0.0162% estimation error.

From our results of Equations (5.4) and (5.8), we find the average estimation error

$$\frac{0.0253\%+0.0162\%}{2} = 0.0208\%. \tag{5.9}$$

For our third numerical example, we reconsider solving our second numerical example by an iterative approach for the solution system of Equations (3.18-3.23) proposed by D'Urso et al. [1] as follows, with $\mathrm{m} = 2$,

$$\mathrm{u}_{11}(\mathrm{k}+1) = \frac{1}{1+\frac{\Delta_{111}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{112}(\mathrm{w}_2(\mathrm{k}))^2}{\Delta_{121}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{122}(\mathrm{w}_2(\mathrm{k}))^2}}, \tag{5.10}$$

$$\mathrm{u}_{12}(\mathrm{k}+1) = \frac{1}{\frac{\Delta_{121}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{122}(\mathrm{w}_2(\mathrm{k}))^2}{\Delta_{111}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{112}(\mathrm{w}_2(\mathrm{k}))^2}+1}, \tag{5.11}$$

$$\mathrm{u}_{21}(\mathrm{k}+1) = \frac{1}{1+\frac{\Delta_{211}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{212}(\mathrm{w}_2(\mathrm{k}))^2}{\Delta_{221}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{222}(\mathrm{w}_2(\mathrm{k}))^2}}, \tag{5.12}$$

$$\mathrm{u}_{22}(\mathrm{k}+1) = \frac{1}{\frac{\Delta_{221}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{222}(\mathrm{w}_2(\mathrm{k}))^2}{\Delta_{211}(\mathrm{w}_1(\mathrm{k}))^2+\Delta_{212}(\mathrm{w}_2(\mathrm{k}))^2}+1}, \tag{5.13}$$

$$\mathrm{w}_1(\mathrm{k}+1) = \frac{1}{1+\frac{\mathrm{M}_1+\mathrm{M}_3}{\mathrm{M}_2+\mathrm{M}_4}}, \tag{5.14}$$

and

$$\mathrm{w}_2(\mathrm{k}+1) = \frac{1}{\frac{\mathrm{M}_2+\mathrm{M}_4}{\mathrm{M}_1+\mathrm{M}_3}+1}, \tag{5.15}$$

where $\mathrm{M}_1 = \Delta_{111}\left(\mathrm{u}_{11}(\mathrm{k}+1)\right)^2 + \Delta_{121}\left(\mathrm{u}_{12}(\mathrm{k}+1)\right)^2$, $\mathrm{M}_2 = \Delta_{112}\left(\mathrm{u}_{11}(\mathrm{k}+1)\right)^2 + \Delta_{122}\left(\mathrm{u}_{12}(\mathrm{k}+1)\right)^2$, $\mathrm{M}_3 = \Delta_{211}\left(\mathrm{u}_{21}(\mathrm{k}+1)\right)^2 + \Delta_{221}\left(\mathrm{u}_{22}(\mathrm{k}+1)\right)^2$, and $\mathrm{M}_4 = \Delta_{212}\left(\mathrm{u}_{21}(\mathrm{k}+1)\right)^2 + \Delta_{222}\left(\mathrm{u}_{22}(\mathrm{k}+1)\right)^2$ are abbreviations to simplify the expressions.

Table 7. Results for our iterative procedure with the solution system of D'Urso et al. [1]

	$\mathrm{u}_{11}(\mathrm{k})$	$\mathrm{u}_{12}(\mathrm{k})$	$\mathrm{u}_{21}(\mathrm{k})$	$\mathrm{u}_{22}(\mathrm{k})$	$\mathrm{w}_1(\mathrm{k})$	$\mathrm{w}_2(\mathrm{k})$
k=0					0	1
k=1	0.377	0.623	0.265	0.735	0.425	0.575
k=2	0.382	0.618	0.284	0.716	0.426	0.574
k=3	0.382	0.618	0.284	0.716	0.426	0.574

We assume that $w_1(0) = 0$ and $w_2(0) = 1$ to execute our iterative procedure and then we list our findings in the above table 7.

Based on Table 7, we show that the system proposed by D'Urso et al. [1] generates six sequences that will converge to the same result as we derived in Equation (5.7).

However, we recall that Wu and Ouyang [29] considered inventory models with a distribution-free approach and defective items, and then Wu and Ouyang [29] took the partial derivatives concerning (a) Q, the order quantity, and (b) k, the safety factor to construct a first partial derivative system where the order quantity and the safety factor are mingled together. Wu and Ouyang [29] claimed that by an iterative method they can derive the optimal solution of the order quantity and the safety factor. However, Tung et al. [30] pointed out that the solutions provided by Wu and Ouyang [29] are not consistent with the findings executed by Tung et al. [30] through an iterative method. Lin [31] reconsidered this inventory model to develop a first partial derivative system with three variables and then she operated her new system to derive the optimal solutions. Luo et al. [32] examined the numerical example proposed by Lin [31] to show that the results provided by Lin [31] are dependent on the decimal expressions, owing to the three sequences have different convergent ratios. The above two discussions reveal that (i) operating an iterative system is not an easy task for researchers, and (ii) several sequences have different convergent ratios that may be influent researchers when to stop the iterative process.

In the following, we will provide a further discussion to illustrate that applying iterative approaches could arouse severe debate among researchers. Gallego [33] constructed an inventory model where the demand during the shortage period was estimated by a minimax distribution-free method. Moon and Gallego [34] provided theoretical proof for Gallego [33] to show that their approach is the best estimation for a two-point distribution model. Chu [35] verified that the interior solution derived by the partial derivative system proposed by Moon and Gallego [34] is the optimal solution. Lin et al. [36] found the criterion to guarantee the uniqueness and existence of the optimal solution and the convergence of the sequence proposed by Gallego [33]. Tuan [37] provided a further revision for Gallego [33] and Moon and Gallego [34] for the convergence of the sequence generated by the iterative method proposed by Gallego [33] and Moon and Gallego [34]. Hu et al. [38] presented a detailed discussion for the convergence of the sequence by an alternative sequence, and then Hu et al. [38] discussed the convergent ratio and the dominant factors to influence the convergent ratio. Hence, we can claim that to present a well-defined iterative algorithm, researchers needed to consider (a) conditions for convergence, (b) convergent ratio, and (c) the dominant factors to influence the convergent ratio.

Hence, our approach reducing a multiple variables problem to a single variable problem that can avoid applying the iterative method has its merit.

VI. A RELATED PROBLEM

We examine a problem that had been studied by Chakraborty and Banik [39], Cho and Cho [40], Karapetrovic and Rosenbloom [41], Kwiesielewicz and van Uden [42], Ma et al. [43], Pramod et al. [44], Saaty [45], and VanDeWater and DeVries [46] to study the paradoxical examples to check the consistency of comparison matrices in the Analytical Hierarchy Process. The original paper prepared three examples that provides unreasonable results may appear in apply the consistency test of AHP. The second paper considered the derivation of their three examples to point out that contained questionable procedure such that these three examples did not provide counterexamples. In this section, we reexamine the third example to demonstrate that there indeed exists paradox in applying the consistency test of AHP.

For some cases, pairwise comparison matrices are inconsistent and cannot pass the consistency test proposed by Saaty [44], and then Saaty [44] advised that decision makers modified their entries of comparison matrices and then tried to pass the consistency test. Karapetrovic and Rosenbloom [41] prepared three examples to illustrate that their judgments are neither random nor illogical such that the entries are not subject to revise. However, these three pairwise comparison matrices failed to pass the consistency test. VanDeWater and DeVries [46] provided a detailed explanation to investigate how these comparison matrices are generated, and then prepared more reasonable construction to slightly modify these three examples such that these three examples in Karapetrovic and Rosenbloom [41] do not represent paradoxes in the consistency test of AHP. We agree the viewpoint of VanDeWater and DeVries [46] for the first and the second example such that after the revision of VanDeWater and DeVries [46], paradoxes disappeared. However, the third example still provides a counterexample for a reasonable pairwise comparison matrix but failed the consistency test of AHP. Hence, in this paper, we will only reconsider the third example.

VII. REVIEW OF VANDEWATER AND DE VRIES [46]

Karapetrovic and Rosenbloom [41] considered the following dice game. Two players each have 4 dice where Dice A: 0-0-4-4-4-4, Dice B: 3-3-3-3-3-3, Dice C: 2-2-2-2-7-7, Dice D: 1-1-1-5-5-5. Each player chooses one of his dice and rolls it. The player with the higher number wins. Dice A beats dice B 2/3 of the time, dice C 4/9 of the time, and dice D 1/3 of the time. Hence, the pairwise comparison matrix is

$$A = \begin{bmatrix} 1 & 2 & 4/5 & 1/2 \\ 1/2 & 1 & 2 & 1 \\ 5/4 & 1/2 & 1 & 2 \\ 2 & 1 & 1/2 & 1 \end{bmatrix}, \tag{7.1}$$

with the maximum eigenvalue

$$\lambda_{\max} = 4.507, \tag{7.2}$$

and consistency index

$$CI = 0.169. \tag{7.3}$$

where the random index is 0.90 so that A did not pass the consistency test. Karapetrovic and Rosenbloom [41] mentioned that the priority vector

$$\left(w_A, w_B, w_C, w_D\right) = \left(0.239, 0.253, 0.262, 0.246\right) \quad (7.4)$$

is quite reasonable in this scenario.

VanDeWater and DeVries [46] pointed out that entries $a_{13} = 4/5$ and $a_{31} = 5/4$ are not in the permissible range of a 1-9 bounded set, $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 9\}$ proposed by Saaty [44]. Therefore, the revised entries should be $a_{13} = 1$, and $a_{31} = 1$.

Moreover, they reconsider the four dice problem under the restriction of 1-9 bounded scale proposed by Saaty [44] and Equation (~~6~~7.1), it yields the following ratios among dices A, B, C and D, where w_A , w_B , w_C and w_D denote their priority weight, respectively,

$$w_A/w_A = 1,\ w_A/w_B = 2,\ w_A/w_C = 1,$$

and

$$w_A/w_D = 1/2, \quad (7.5)$$

$$w_B/w_B = 1,\ w_B/w_C = 2,\ w_B/w_D = 1,$$

and

$$w_B/w_A = 1/2, \quad (7.6)$$

$$w_C/w_C = 1,\ w_C/w_D = 2,\ w_C/w_A = 1,$$

and

$$w_C/w_B = 1/2, \quad (7.7)$$

$$w_D/w_D = 1,\ w_D/w_A = 2,\ w_D/w_B = 1,$$

and

$$w_D/w_C = 1/2. \quad (7.8)$$

VanDeWater and DeVries [46] listed these four dices, repeated in a row, C, D, A, B, C, D, A, B, C, D, A, B, …, to discover that a dice will beat the right dice by 2, and lose to the left dice by 1/2, and make it even with the next two right (that is the next two left) dice, and then those four dices should be equally important such that the priority vector should be directly implied

$$\left(w_A, w_B, w_C, w_D\right) = \left(0.25, 0.25, 0.25, 0.25\right). \quad (7.9)$$

VanDeWater and DeVries [46] also prepare an intuitive explanation by probability approach to derive Equation (7.9) to claim that there is no need to use the consistency test of AHP.

VIII. Our revisions

After the revision of VanDeWater and DeVries [46], the winning rates of A over B, B over C, C over D, and D over A are all the same so that, we assume that the winning rate of A over B is "a" . Hence, the winning rates of A over D, B over A, C over B, and D over C are assumed as "$1/a$". One the other hand, A over C, B over D, C over A, and D over B are tie so the ratio should be 1. It implies that the revised pairwise comparison matrix can be abstractly expressed as

$$A = \begin{bmatrix} 1 & a & 1 & 1/a \\ 1/a & 1 & a & 1 \\ 1 & 1/a & 1 & a \\ a & 1 & 1/a & 1 \end{bmatrix}, \quad (8.1)$$

with the maximum eigenvalue $\lambda_{\max} = 2 + a + (1/a)$, where $CI = (a + (1/a) - 2)/3$ and the priority vector

$$\left(w_A, w_B, w_C, w_D\right) = \left(0.25, 0.25, 0.25, 0.25\right). \quad (8.2)$$

Since the random index for 4 by 4 comparison matrices is 0.90, our abstract pairwise comparison matrix will pass the consistency test, if

$$(a + (1/a) - 2)/3 \le (0.1)(0.90) \quad (8.3)$$

holds.

From the condition of Equation (8.3), then

$$a^2 - 2.27a + 1 \le 0. \quad (8.4)$$

We imply that

$$0.598 \le a \le 1.672. \quad (8.5)$$

Because winning is preferred by almost everyone, the natural restriction of "a" should be that $a \ge 1$. Hence, we know that when

$$1 \le a \le 1.672, \quad (8.6)$$

then our revised pairwise comparison matrix will pass the consistency test.

For any four dice game where each die having the same winning ratio, 1/2 as the priority vector is

$$\left(w_A, w_B, w_C, w_D\right) = \left(0.25, 0.25, 0.25, 0.25\right), \quad (8.7)$$

to select any one of them will have the same winning rate. If we still create the comparison matrix where entries is the ratio and apply the consistency test, when $a \ge 2$ then the consistency test failed. It indicates that the consistency test contains questionable results that are beyond doubt (unbelievable by ordinary person). From the original example, we can derive that the winning rate of A over B is found as 2.

In the following, we try to obtain that the winning rate of A over B is estimated as 2, such that the ratio is exact

$$A = \begin{bmatrix} 1 & 2 & 1 & 1/2 \\ 1/2 & 1 & 2 & 1 \\ 1 & 1/2 & 1 & 2 \\ 2 & 1 & 1/2 & 1 \end{bmatrix}. \quad (8.8)$$

We explained in detail that the third example in Karapetrovic and Rosenbloom [41] indeed provided evidence that paradoxes exist in the consistency test of Saaty [44]. On the basis of our findings, we may advise researchers with care when applying the consistency tests of AHP.

IX. Direction for future research

When $\mathrm{m} = 1.5$, $\mathrm{M(x)}$ of Equation (4.19) is changed to

$$\mathrm{M}_{\mathrm{m}=1.5}(\mathrm{x}) =$$

$$\frac{(\Delta_{111}x^2+\Delta_{112}(1-x)^2)(\Delta_{121}x^2+\Delta_{122}(1-x)^2)}{\sqrt{(\Delta_{111}x^2+\Delta_{112}(1-x)^2)^2+(\Delta_{121}x^2+\Delta_{122}(1-x)^2)^2}},$$
$$+\frac{(\Delta_{211}x^2+\Delta_{212}(1-x)^2)(\Delta_{221}x^2+\Delta_{222}(1-x)^2)}{\sqrt{(\Delta_{211}x^2+\Delta_{212}(1-x)^2)^2+(\Delta_{221}x^2+\Delta_{222}(1-x)^2)^2}}. \quad (9.1)$$

We slightly modify the expression of $M_{m=1.5}(x)$ in an addition to symmetric representation as follows,

$$M_{m=1.5}(x) = \frac{1}{\sqrt{\frac{\Delta_{111}x^2+\Delta_{112}(1-x)^2}{\Delta_{121}x^2+\Delta_{122}(1-x)^2}+\frac{\Delta_{121}x^2+\Delta_{122}(1-x)^2}{\Delta_{111}x^2+\Delta_{112}(1-x)^2}}},$$
$$+\frac{1}{\sqrt{\frac{\Delta_{211}x^2+\Delta_{212}(1-x)^2}{\Delta_{221}x^2+\Delta_{222}(1-x)^2}+\frac{\Delta_{221}x^2+\Delta_{222}(1-x)^2}{\Delta_{211}x^2+\Delta_{212}(1-x)^2}}}. \quad (9.2)$$

We can further simplify the result of Equation (9.2) with

$$\frac{\Delta_{111}x^2+\Delta_{112}(1-x)^2}{\Delta_{121}x^2+\Delta_{122}(1-x)^2} = E(x), \quad (9.3)$$

and

$$\frac{\Delta_{211}x^2+\Delta_{212}(1-x)^2}{\Delta_{221}x^2+\Delta_{222}(1-x)^2} = F(x), \quad (9.4)$$

and then we rewrite $M_{m=1.5}(x)$ as follows

$$M_{m=1.5}(x) = \sqrt{\frac{1}{E(x)+[1/E(x)]}} + \sqrt{\frac{1}{F(x)+[1/F(x)]}}. \quad (9.5)$$

Based on our derivation of Equation (9.5), we claim that the minimum problem of $M_{m=1.5}(x)$ can be classified as one of the symmetric expressions.

To derive some progress for $M_{m=1.5}(x)$ concerning the existence and uniqueness of the minimum solutions will be an interesting research topic for future research.

Moreover, how to derive the minimum solution for $M(x)$ of Equation (4.19) will be another important issue for further studies.

There are several related papers that are worthy to mention to indicate the current research trend. for example, Yen [47] studied inventory systems solving by algebraic process and considered the open question proposed by Chang et al. [48]. Yen [49] pointed out that Çalışkan [50], Çalışkan [51], Wee et al. [52], and Çalışkan [53] contained severe questionable findings and then provided improvements for them. Yang and Chen [54] examined Yen [47], Osler [55], Çalışkan [56], and Çalışkan [57], and then presented their revisions.

Wang and Chen [58] showed that Aguaron and Moreno-Jimenez [59] contained questionable results and then offered revisions. Moreover, Wang and Chen [58] provided further comments for Yen [47] and Aguaron and Moreno-Jimenez [59].

There are four papers: Atatalab, and Najafabadi [60], Raghu and Prameela [61], Zhang et al. [62], and Wichapa and Sodsoon [63] that are valuable for researchers to consider the direction for the future study. On the other hand, there are other four articles: Tan et al. [64], Zhu et al. [65], Patil et al. [66], and Pappalardo et al. [67] that are important for practitioners to examine for their further research topics.

X. CONCLUSION

We show that the existing solution procedure only derived a system that the membership degrees and the weights are mixed such that only an iterative method generates several sequences that may converge to the desired optimal solution. However, the starting point for iteration and conditions to guarantee the convergence did not provide in D'Urso et al. [1]. Hence, their solution approach is incomplete. For a simplified version for this kind of minimum problem, we transfer a six-variable problem into a one-variable such that we can use a numerical method to find its minimum and we also obtain a formulated approximated solution that attains the minimum value within a very small estimation error. Our results will help researchers to study this kind of cepstral-based clustering for financial time series.

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