Application of Homotopy Analysis Shehu Transform Method for Fractional Black-Scholes Equation

C. Vijayan and R. Manimaran

Abstract—**The Black – Scholes Model is commonly used for option pricing, which is one of the most important applications in finance. In the absence of a transaction cost, the value of the option is determined by using B-S model. In the view of Caputo sense, this study proposes a result for the fractional Black–Scholes equation (FBSE) problem. The main goal of this paper is to show how to solve the FBSE by a semi-analytical method called the homotopy analysis shehu transform method (HASTM) and compare that homotopy analysis method (HAM), homotopy perturbation method (HPM), elzaki transform homotopy perturbation method (ETHPM). The HASTM result is quite similar to the HAM, HPM, ETHPM solution. The HASTM and other methods analytical solutions are also represented.**

Index Terms—**Fractional Black-Scholes Equation, Shehu Transform, Homotopy Analysis Shehu Transfrom Method, Homotopy Analysis Method, Homotopy Perturbation Method, Elzaki Transform Homotopy Perturbation Method.**

I. Introduction

In quantitative finance, determining the option pricing is a difficult barrier. Because, options are widely used in the financial sector today and these issues are both theoretical and practical. Fisher Black and Myron Scholes developed the well-known theoretical value model for option pricing [1] in 1973. Black-Scholes is modelled as a stochastic process with fluctuations in stock prices by including certain assumptions about the option markets. These assumptions are the absence of taxes, risk-free interest and always volatile. As a result, by calculating implied volatility, a key financial parameter for options in closed form, is found to be difficult. In order to address the challenge, many authors presented various types of B-S models from different perspectives. V. Gulkac (2010), calculates the analytical solution to the B-S problem in terms of convergent power series that is simple to compute using the homotopy perturbation method [4]. Elbeleze et al. (2013) solved the fractional B-S problem [6] using the Sumudu transform and He's polynomials in conjunction with the homotopy perturbation method. Sunil et al. (2014) compared two numerical algorithms for the time-fractional B-S equation for the European option using the homotopy perturbation method and the homotopy analysis method [8]. A.A. Hemeda (2014) extended the MHAM, to derive accurate solutions for the fractional order linear (nonlinear) ordinary differential equation. This technique reduces the amount of effort to only one iteration [9]. Ghandehari et al. (2016) used a combination of HPM and the separation of variables method to precisely solve option pricing problems based on the fractional B-S equation [10]. Khan and Ansari solved the fractional B-S equation in 2016, the analytical solution is based on sumudu transform and its differential and integral properties [11]. According to T. Allahviranloo and Sh.S. Behzadi (2013), the fractional B-S equation is solved using the adomain decomposition method, modified adomain decomposition method, variational iteration method, modified variational iteration method, HPM, MHPM and HAM. HAM converges faster than ADM, MADM, VIM, MVIM, HPM and MHPM [7]. Ouafoudi and Gao (2018) used the MHPM, HPM and Sumudu transforms for the fractional B-S equation. The outcome was the same for both approaches [13]. Yavuz and Ozdemir (2018) calculated the option price for fractional values [14] by redefining the B-S equation as a fractional mean and applying ADM to both the fractional and generalised B-S equations. S. Alfaqeih and T. Ozis (2020) used an Aboodh transform and an adomian decomposition method to solve a fractional B-S equation [16]. S.E. Fadugba and O.H. Edogbanya (2020) compared the fractional Laplace transform homotopy perturbation method to the fractional reduced differential transform method for solving the time-fractional B-S equation. FRDTM outperforms FLTHPM because its algorithms are shorter [17]. To obtain the analytical solution of the time space fractional B-S method, S.O. Edeki et al. (2020) used a coupled transform approach that combines the characteristics of the fractional complex transform and reduced differential transform methods [18]. P.R Bhadane et al.(2020) solved analytical solutions of the fractional B-S equation using a combination of HPM and the Elzaki transform, which is known as the Elzaki transform HPM [20]. To solve the fractional B-S problem, Ahmad et al. (2021) used a modified version of DTM called the fractional reduced differential transform method [21]. M. Yavuz and N. Ozdemir (2018) used an iterative method to derive an approximate solution of the fractional Black-Scholes models in confromable derivative sense [23]. Shehu Maitama and Weidong Zhao (2020) used the new semi-analytical method called the homotopy analysis Shehu transform method (HASTM) to solve the multidimensional fractional diffusion equation. This method reduces the need for iterative differentiation and integration while also overcomes the HAM restriction [22]. This prompted the application of the homotopy analysis shehu transform to the

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B-S problem. The solutions are extremely consistent with the existing results.

The following is the paper organization: **Section II** elaborates on the Black-Scholes partial differential equation. The preliminary procedures are covered in **Section III**. **Section IV**, provides a detailed description of the new hybrid methodology for solving non-linear fractional differential equations. **Sections V and VI** focuses on the HASTM, to solve the B-S equation and also the generalised fractional B-S equation. **Section VII**, Result and Discussion. **Section VIII**, some conclusion and presents the suggestions for future research.

II. Black-Scholes Equation

$$\frac{\partial E_C}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E_C}{\partial S^2} + r(t)S\frac{\partial E_C}{\partial S} - r(t)E_C = 0 \quad (1)$$

with $E_C(0,t) = 0$, $E_C(S,t) \sim S$ as S $\to \infty$

where $E_C = E_C(S,t)$ = European call option prices; S = Asset price; t = Time; r = Risk free rate; σ = Volatility; T = Exercise time; K = Exercise price. The following conversions are made in order to arrive at the fractional Black-Scholes equation:

$$S = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad E_C = K\upsilon(x,\tau). \quad (2)$$

Substituting the above Equation in Eq. (1), then Eq. (1) yields

$$\frac{\partial^\mu \upsilon}{\partial \tau^\mu} = \frac{\partial^2 \upsilon}{\partial x^2} + (A-1)\frac{\partial \upsilon}{\partial x} - A\upsilon, \ \ 0 < \mu \leq 1 \quad (3)$$

with the initial condition $\upsilon(x,0)$ = max$(e^x - 1, 0)$.

where A = $\frac{2r}{\sigma^2}$ and Eq. (2) is known as Fractional Black-Scholes Equation.

The Generalised Fractional Black-Scholes Equation (GF-BSE) was also discovered by Cen and Le (2011) [23] by taking into consideration r and σ in Eq. (3)

$$r = 0.06 \quad \text{and} \quad \sigma = 0.4(2 + sinx) \quad (4)$$

$$\frac{\partial^\mu \upsilon}{\partial \tau^\mu} = 0.08(2+sinx)^2 x^2 \frac{\partial^2 \upsilon}{\partial x^2} + 0.06x\frac{\partial \upsilon}{\partial x} - 0.06\upsilon, \quad 0 < \mu \leq 1 \quad (5)$$

Subject to the condition $\upsilon(x,0) = max(x - 25e^{-0.06}, 0)$.

III. Preliminaries

The definitions of certain basic topics are presented in this section.

Definition 1 The Riemann-Liouville fractional integral of f with order μ is defined by [15]

$$I^\mu f(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-\tau)^{\mu-1} f(\tau)d\tau. \quad (6)$$

Definition 2 The Riemann-Liouville fractional derivative of f with order μ is defined by [15]

$$D_x^\mu f(x) = \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_0^x (x-\tau)^{m-\mu-1} f(\tau)d\tau. \quad (7)$$

Definition 3 The function f with order μ in Caputo fractional derivative is defined as [15]

$$^C D_x^\mu f(x) = \frac{1}{\Gamma(m-\mu)}\int_0^x (x-\tau)^{m-\mu-1} f^m(\tau)d\tau. \quad (8)$$

Definition 4 The Mittag-Leffler is defined as [5], [3]

$$E_\alpha = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha > 0, z \in C, k = 0, 1..... \quad (9)$$

Shehu Transform The Laplace type integral transform (Shehu transform) the set of functions are defined under the exponential order [22].

$$B = \left[f(x) : \exists\, M, \eta_1, \eta_2 > 0, |f(x)| < M \; exp\left(\frac{|x|}{\eta_j}\right) if x \in (-1)^j \times [0,\infty) \right]$$

is defined by

$$\mathcal{S}[f(x)] = F(s,r) = \int_0^\infty exp\left(\frac{-sx}{r}\right) f(x)dx = \lim_{\mu\to\infty}\int_0^\mu exp\left(\frac{-sx}{r}\right) f(x)dx, \quad r > 0, s > 0 \quad (10)$$

Where r, s $\in (\eta_1, \eta_2)$, if the limit of the integral exists, it converges. Otherwise, it diverges.

Properties of Shehu Transform [22]

1) $\mathcal{S}[1] = \frac{r}{s}$
2) $\mathcal{S}[x] = \frac{r^2}{s^2}$
3) $\mathcal{S}[x^\mu] = \Gamma(\mu+1)\frac{r^{\mu+1}}{s^{\mu+1}}, \mu > -1$
4) $\mathcal{S}[\mu f(x) + \beta g(x)]$=$\mu\mathcal{S}[f(x)]$+$\beta\mathcal{S}[g(x)]$

Theorem 1 Shehu transform of Caputo fractional derivative is [22]

$$\mathcal{S}[^C D_x^\mu f(x)] = \left(\frac{s}{r}\right)^\mu \mathcal{S}[f(x)] - \sum_{k=0}^{m-1}\left(\frac{s}{r}\right)^{\mu-k-1} f^{(k)}(0), k = 0,1,2..... \quad (11)$$

IV. Homotopy Analysis Shehu Transform Method

Suppose the nonlinear fractional PDE be

$$(D^\mu \upsilon)(x,\tau) + R\upsilon(x,\tau) + \mathcal{N}\upsilon(x,\tau) = g(x,\tau); \quad 0 < \mu \leq 1. \quad (12)$$

where $(D^\mu\upsilon)$(x,τ) is the Caputo fractional derivative, R is the linear operator, $\mathcal{N}$ is the non-linear operator and g(x,τ) is the source term, to develop the core idea of the HASTM.

Step 1: Using the Shehu transform in equation (12)

$$\mathcal{S}[(D^\mu\upsilon)(x,\tau)] + \mathcal{S}[R\upsilon(x,\tau)] + \mathcal{S}[N\upsilon(x,\tau)] = \mathcal{S}[g(x,\tau)]. \quad (13)$$

Step 2: Use Theorem 1 to solve Eq. (13)

$$\left(\frac{s}{r}\right)^\mu \mathcal{S}[\upsilon(x,\tau)] - \sum_{k=0}^{m-1}\left(\frac{s}{r}\right)^{\mu-k-1}\upsilon^k(x,0) + \quad (14)$$

$$\mathcal{S}[R\upsilon(x,\tau)] + \mathcal{S}[\mathcal{N}\upsilon(x,\tau)] = \mathcal{S}[g(x,\tau)]$$

Equivalently,

$$\mathcal{S}[\upsilon(x,\tau)] - \left(\frac{r}{s}\right)^{\mu}\sum_{k=0}^{m-1}\left(\frac{s}{r}\right)^{\mu-k-1}\upsilon^{k}(x,0)+ \left(\frac{r}{s}\right)^{\mu}\left(\mathcal{S}[R\upsilon(x,\tau)] + \mathcal{S}[\mathcal{N}\upsilon(x,\tau)] - \mathcal{S}[g(x,\tau)]\right) = 0 \tag{15}$$

Non linear term

$$N[\Psi(x,\tau;q)] = \mathcal{S}[\Psi(x,\tau;q)] - \left(\frac{r}{s}\right)^{\mu}\sum_{k=0}^{m-1}\left(\frac{s}{r}\right)^{\mu-k-1}\Psi^{k}(x,0)+ \left(\frac{r}{s}\right)^{\mu}\left(\mathcal{S}[R\Psi(x,\tau)] + \mathcal{S}[\mathcal{N}\Psi(x,\tau)] - \mathcal{S}[g(x,\tau)]\right) \tag{16}$$

Step 3: We construct Homotopy as follows (35),

$$(1-q)\mathcal{S}[\Psi(x,\tau;q) - \upsilon_0(x,\tau)] = hqH(x,\tau) N[\Psi(x,\tau)] \tag{17}$$

Where $\Psi(x,\tau;q)$ is a real-valued of x, τ, q and q $\in$ [0,1] is the imbedding parameter, H(x,τ) denotes a non-zero auxiliary function, h$\neq$ 0 is an auxiliary parameter, $\upsilon_0(x,\tau)$ is the initial estimate of υ(x,τ) and Ψ(x,τ;q) is the unknown function.
The idea of HASTM allows for a lot of flexibility in terms of selecting an auxiliary parameter and an initial estimate. When q = 1 and q = 0 in Eq. (17), the conclusion was obtained as follows,

$$\Psi(x,\tau;0) = \upsilon_0(x,\tau) \; and \; \Psi(x,\tau;1) = \upsilon(x,\tau) \tag{18}$$

Thus q rises from 0 to 1, the solution $\Psi(x,\tau;q)$ shifts from the initial estimate $\upsilon_0(x,\tau)$ to the solution $\upsilon(x,\tau)$.

Step 4: Differentiating Eq. (17) w.r.t q = 0 and divide by $\Gamma(m+1)$, then m^{th}- order deformation equation

$$\mathcal{S}[\upsilon_m(x,\tau) - \chi_m\upsilon_{m-1}(x,\tau)] = hH(x,\tau)R_m(\overrightarrow{\upsilon}_{m-1},x,\tau) \tag{19}$$

where

$$R_m(\overrightarrow{\upsilon}_{m-1},x,\tau) = \left[\frac{1}{\Gamma(m)}\frac{\partial^{m-1}N[\Psi(x,\tau;q)]}{\partial q^{(m-1)}}\right]_{q=0} \tag{20}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases} \tag{21}$$

Step 5: Applying the inverse shehu transform in Eq. (19)

$$\upsilon_m(x,\tau) = \chi_m\upsilon_{m-1}(x,\tau) + \mathcal{S}^{-1}[hH(x,\tau) R_m(\overrightarrow{\upsilon}_{m-1},x,\tau)], \tag{22}$$

$$R_m(\overrightarrow{\upsilon}_{m-1},x,\tau) = (D^{\mu}\upsilon_{m-1}(x,\tau)(\tau)) + R\upsilon(x,\tau)_{m-1} + \mathcal{N}(\upsilon_{m-1}(x,\tau)) - (1-\chi_m)g(x,\tau) \tag{23}$$

Step 6: Compute $\upsilon_m(x,\tau)$ for $m \ge 1$.

$$\upsilon(x,\tau) = \lim_{M\to\infty}\sum_{m=0}^{M}\upsilon_m(x,\tau) \tag{24}$$

V. Fractional Black-Scholes equation

$$\frac{\partial^{\mu}\upsilon}{\partial\tau^{\mu}} = \frac{\partial^2\upsilon}{\partial x^2} + (A-1)\frac{\partial\upsilon}{\partial x} - A\upsilon, \;\; 0<\mu\le 1 \tag{25}$$

with the initial condition $\upsilon(x,0)$ = max$(e^x - 1, 0)$.

By HASTM, applying Theorem 1 to Eq. (25)

$$\mathcal{S}[\upsilon(x,\tau)] - \frac{r}{s}\upsilon^{k}(x,0) - \frac{r^{\mu}}{s^{\mu}}\mathcal{S}\left[\frac{\partial^2\upsilon(x,\tau)}{\partial x^2} + (A-1)\frac{\partial\upsilon(x,\tau)}{\partial x} - A\upsilon(x,\tau)\right] = 0 \tag{26}$$

Non linear term

$$N[\Psi(x,\tau;q)] = \mathcal{S}[\Psi(x,\tau,q)] - \frac{r}{s}max(e^x-1,0) - \frac{r^{\mu}}{s^{\mu}}\mathcal{S}\left[\frac{\partial^2\upsilon(x,\tau)}{\partial x^2} + (A-1)\;\frac{\partial\upsilon(x,\tau)}{\partial x} - A\upsilon(x,\tau)\right] = 0, 0\le q\ge 1, \tau>0. \tag{27}$$

Thus

$$R_m(\overrightarrow{\upsilon}_{m-1},x,\tau) = \mathcal{S}[\upsilon_{m-1}(x,\tau)] - (1-\chi_m)\frac{r}{s}max(e^x-1,0) - \frac{r^{\mu}}{s^{\mu}} \mathcal{S}\left[\frac{\partial^2\upsilon(x,\tau)}{\partial x^2} + (A-1)\;\frac{\partial\upsilon(x,\tau)}{\partial x} - A\upsilon(x,\tau)\right] = 0. \tag{28}$$

The Mth-order deformation equation is define as

$$\mathcal{S}\left[\upsilon_m(x,\tau) - \chi_m\upsilon_{m-1}(x,\tau)\right] = [hH(x,\tau) R_m(\overrightarrow{\upsilon}_{m-1},x,\tau)] \tag{29}$$

Computing the inverse Shehu transform of the Eq. (29), we deduce

$$\upsilon_m(x,\tau) = \chi_m\upsilon_{m-1}(x,\tau) + \mathcal{S}^{-1}[hH(x,\tau)R_m(\overrightarrow{\upsilon}_{m-1},x,\tau)] \tag{30}$$

We solve above equation iteratively for m $\ge$ 1 using H(x,τ) = 1 and obtain the following results.

$$\upsilon_0(x,\tau) = max(e^x-1,0)$$

$$\upsilon_1(x,\tau) = -hA\frac{\tau^{\mu}}{\Gamma(\mu+1)}(max(e^x,0) - max(e^x-1,0))$$

$$\upsilon_2(x,\tau) = -h(h+1)A\frac{\tau^{\mu}}{\Gamma(\mu+1)}(max(e^x,0) - max(e^x-1,0)) - h^2A^2\frac{\tau^{2\mu}}{\Gamma(2\mu+1)} (max(e^x,0) - max(e^x-1,0))$$

$$\upsilon_3(x,\tau) = (h+1)\nu_2(x,\tau) - h^2(h+1)A^2\frac{\tau^2\mu}{\Gamma(2\mu+1)} (max(e^x,0) - max(e^x-1,0)) - h^3A^3 \frac{\tau^{3\mu}}{\Gamma(3\mu+1)}(max(e^x,0) - max(e^x-1,0))$$

and so on

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If h = -1, we obtain

$$\begin{aligned}\upsilon(x,\tau) &= max(e^x-1,0)E_\mu(-A\tau^\mu)+max(e^x,0)\\ &(1-E_\mu(-A\tau^\mu))\\ &= max(e^x-1,0)e^{-A\tau^\mu}+max(e^x,0)\\ &(1-e^{-A\tau^\mu})\end{aligned} \tag{31}$$

with Substituting Eq. (2) into Eq. (31)

$$\begin{aligned}E_C(S,t) &= K\upsilon(x,\tau)\\ &= K\left[max\left(ln\frac{S}{K}-1,0\right)e^{-\frac{2r}{\sigma^2}\left(\frac{\sigma^2}{2}(T-t)\right)^\mu}\right]\\ &+K\left[max\left(ln\frac{S}{K},0\right)\left(1-e^{-\frac{2r}{\sigma^2}\left(\frac{\sigma^2}{2}(T-t)\right)^\mu}\right)\right]\end{aligned} \tag{32}$$

VI. Generalized Fractional Black-Scholes Equation

$$\frac{\partial^\mu\upsilon}{\partial\tau^\mu}=0.08(2+sinx)^2x^2\frac{\partial^2\upsilon}{\partial x^2}+0.06x\frac{\partial\upsilon}{\partial x}-0.06\upsilon,\quad 0<\mu\leq 1 \tag{33}$$

Subject to the condition $\upsilon(x,0)=max(x-25e^{-0.06},0)$

By HASTM, applying Theorem 1 to Eq. (33)

$$\mathcal{S}[\upsilon(x,\tau)]-\frac{r}{s}\upsilon^k(x,0)-\frac{r^\mu}{s^\mu}\mathcal{S}\left[\frac{\partial^\mu\upsilon}{\partial\tau^\mu}+0.08(2+sinx)^2x^2+\frac{\partial^2\upsilon}{\partial x^2}+0.06x\frac{\partial\upsilon}{\partial x}-0.06\upsilon\right]=0 \tag{34}$$

Non linear term

$$\begin{aligned}N[\Psi(x,\tau;q)] &= \mathcal{S}[\Psi(x,\tau,q)]-\frac{r}{s}max(e^x-1,0)-\\ &\frac{r^\mu}{s^\mu}\mathcal{S}\left[\frac{\partial^2\upsilon(x,\tau)}{\partial x^2}+(A-1)\ \frac{\partial\upsilon(x,\tau)}{\partial x}\right.\\ &\left.-A\upsilon(x,\tau)\right]=0, 0\leq q\geq 1,\tau>0.\end{aligned} \tag{35}$$

Thus

$$\begin{aligned}&R_m(\overrightarrow{\upsilon}_{m-1},x,\tau)=\\ &\mathcal{S}[\upsilon_{m-1}(x,\tau)]-(1-\chi_m)\frac{r}{s}max(e^x-1,0)-\frac{r^\mu}{s^\mu}\\ &\mathcal{S}\left[\frac{\partial^2\upsilon(x,\tau)}{\partial x^2}+(A-1)\ \frac{\partial\upsilon(x,\tau)}{\partial x}-A\upsilon(x,\tau)\right]=0.\end{aligned} \tag{36}$$

The Mth-order deformation equation is define as

$$\begin{aligned}&\mathcal{S}\left[\upsilon_m(x,\tau)-\chi_m\upsilon_{m-1}(x,\tau)\right.=\\ &\qquad [hH(x,\tau)R_m(\overrightarrow{\upsilon}_{m-1},x,\tau)]\end{aligned} \tag{37}$$

Computing the inverse Shehu transform of the Eq. (37), we deduce

$$\begin{aligned}\upsilon_m(x,\tau) &= \chi_m\upsilon_{m-1}(x,\tau)+\\ &\mathcal{S}^{-1}[hH(x,\tau)R_m(\overrightarrow{\upsilon}_{m-1},x,\tau)]\end{aligned} \tag{38}$$

we solve the above equation iteratively for $m\geq 1$ using H(x,τ) = 1 and obtain the following results.

$$\begin{aligned}\upsilon_0(x,\tau) &= max(x-25e^{-0.06},0)\\ \upsilon_1(x,\tau) &= hx\frac{0.06\tau^\mu}{\Gamma(\mu+1)}-hmax(x-25e^{-0.06},0)\frac{0.06\tau^\mu}{\Gamma(\mu+1)}\\ \upsilon_2(x,\tau) &= h(h+1)\frac{0.06\tau^\mu}{\Gamma(\mu+1)}(x-max(x-25e^{-0.06},0))\\ &\quad -h^2\frac{(0.06)^2\tau^{2\mu}}{\Gamma(2\mu+1)}(x-max(x-25e^{-0.06},0))\\ \upsilon_3(x,\tau) &= (h+1)\nu_2(x,\tau)-h^2(h+1)\frac{(0.06)^2\tau^{2\mu}}{\Gamma(2\mu+1)}\\ &(x-max(x-25e^{-0.06},0))+h^3\\ &\frac{(0.06)^3\tau^{3\mu}}{\Gamma(3\mu+1)}(x-max(x-25e^{-0.06},0))\end{aligned}$$

and so on

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if h = -1

$$\begin{aligned}\upsilon(x,\tau) &= x[(1-E_\mu(-0.06\tau^\mu))]+max(x-25e^{-0.06},0)\\ &E_\mu(-0.06\tau^\mu)\\ &= x[(1-e^{-0.06\tau^\mu})]+max(x-25e^{-0.06},0)\\ &e^{-0.06\tau^\mu}\end{aligned} \tag{39}$$

with Substituting Eq. (4) into Eq. (39),

$$\begin{aligned}E_C(S,t) &= K\upsilon(x,\tau)\\ E_C(S,t) &= K\left[ln\frac{S}{K}\left(1-e^{-r\left(\frac{\sigma^2}{2}(T-t)\right)^\mu}\right)\right]\\ &+K\left[max\left(ln\frac{S}{K}-25e^{-r}\right)\right]e^{-r\left(\frac{\sigma^2}{2}(T-t)\right)^\mu}\end{aligned} \tag{40}$$

VII. Result and discusion

Consider valuing the Eq. (25) using the parameters indicated in Table I. When comparing Figures 1 and 2 with the fractional Black Scholes Model and the BSM over two time periods. The solution of the fractional B-S Equation for μ = 0.92, 0.94, 0.96, 0.98, over two time periods respectively is plotted in Figure 3 and 4. Figure 5-8 depicts, the financial pricing derivatives as a function of the fractional parameter values μ = 0.92, 0.94, 0.96, 0.98. The result of FBSE by HASTM, HAM, HPM and ETHPM with fractional parameters μ = 0.92, 0.94, 0.98 are shown in Table II-IV. The absolute error for the exact solution and fractional solution of Eq. (25) is shown in Table V. Figure 17 displays h-curves of Eq. (25) using 4th-order approximations.

Similarly for the evaluation of GFBSE, using the parameters indicated in Table VI along with the Eq. (33). Figures 9 and 10 compares the generalized fractional Black Scholes Model and the BSM over two time periods. Figure 11 and 12 displays, the solution of the GFBSE for μ = 0.92, 0.94, 0.96, 0.98, over two time periods respectively. Figure 13-16 depicts the financial pricing derivatives as a function of the fractional parameter values μ = 0.92, 0.94, 0.96, 0.98. Hence the result of GFBSE by HASTM, HAM, HPM and ETHPM with fractional parameters μ = 0.92, 0.94, 0.98 are shown in Table VII-IX. The absolute error for the exact solution and fractional solution of Eq. (33) is shown in Table X. Figure 18 displays h-curves of Eq. (33) using 4th-order approximations.

TABLE I
THE VARIABLES VALUES

Variables	Describtion	Value
μ	Fractional order	0.92, 0.94, 0.96, 0.98
S	Stock price	125, 150, 175, 200, 225
K	Exercise price	160
r	Risk-free interest rate	25%
T	Time to expiry in month	6, 15
σ	Voltality	4%

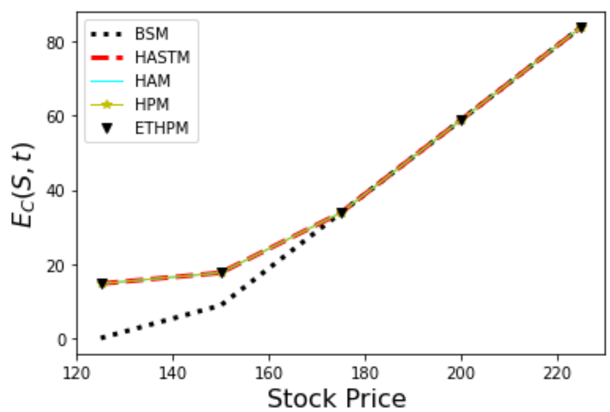

Fig. 1. The result of HASTM, HAM, HPM, ETHPM and BSM with μ = 1 and T = 6 Months.

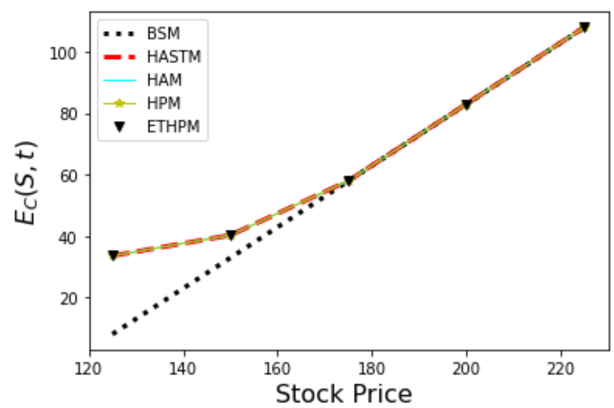

Fig. 2. The result of HASTM, HAM, HPM, ETHPM and BSM with μ = 1 and T = 1.3 Years.

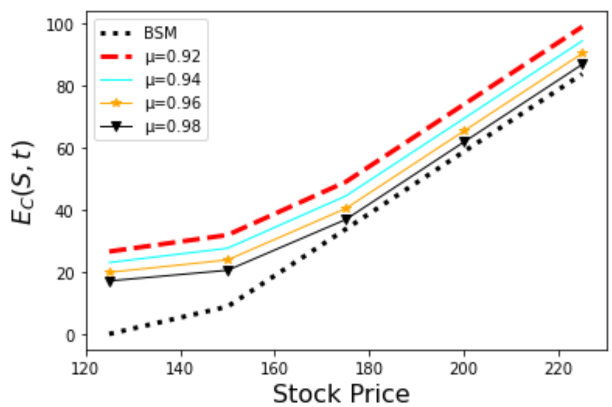

Fig. 3. The result of HASTM and BSM with μ = 0.92, 0.94, 0.96, 0.98 and T = 6 Months.

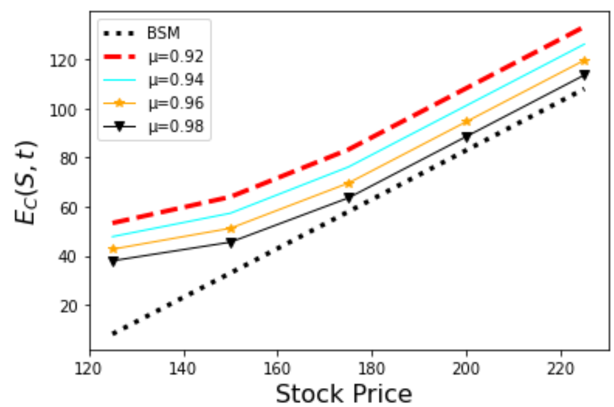

Fig. 4. The result of HASTM and BSM with μ = 0.92, 0.94, 0.96, 0.98 and T = 1.3 Years.

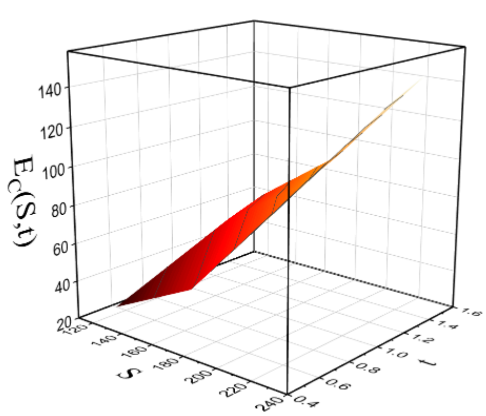

Fig. 5. The result of the B-S equation, when $\mu = 0.92$.

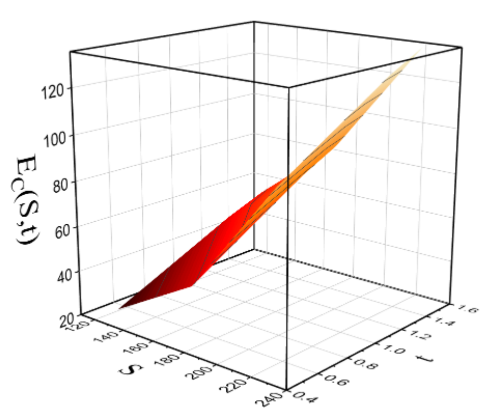

Fig. 6. The result of the B-S equation, when $\mu = 0.94$.

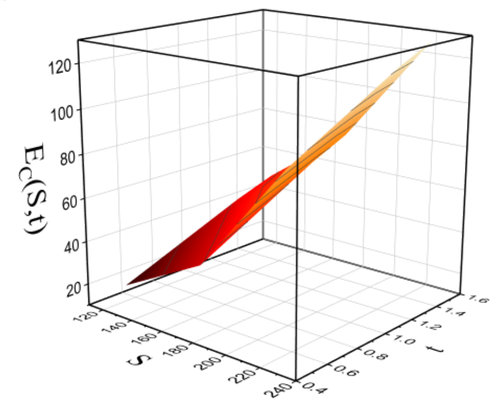

Fig. 7. The result of the B-S equation, when $\mu = 0.96$

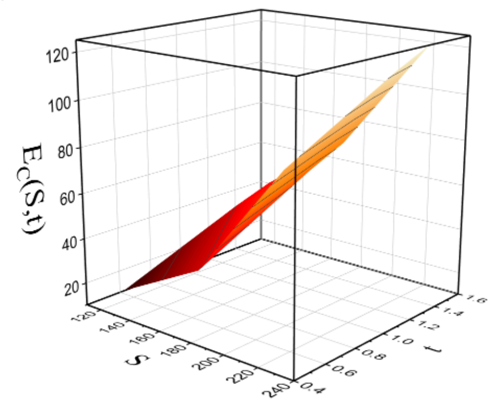

Fig. 8. The result of B-S the equation, when $\mu = 0.98$

TABLE II
THE COMPARISON OF HASTM WITH OTHER METHODS. WHEN $\mu = 0.92$.

	HASTM			HAM		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	26.579	49.021	99.021	26.579	49.021	99.021
t=15	53.288	83.209	133.209	53.288	83.209	133.209
t=18	60.538	92.489	142.489	60.538	92.489	142.489
	HPM			**ETHPM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	26.579	49.021	99.021	26.579	49.021	99.021
t=15	53.288	83.209	133.209	53.288	83.209	133.209
t=18	60.538	92.489	142.489	60.538	92.489	142.489

TABLE III
THE COMPARISON OF HASTM WITH OTHER METHODS. WHEN $\mu = 0.94$.

	HASTM			HAM		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	23.023	44.470	94.470	23.023	44.470	94.470
t=15	47.747	76.116	126.116	47.747	76.116	126.116
t=18	54.552	84.827	134.827	54.552	84.827	134.827
	HPM			**ETHPM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	23.023	44.470	94.470	23.023	44.470	94.470
t=15	47.747	76.116	126.116	47.747	76.116	126.116
t=18	54.552	84.827	134.827	54.552	84.827	134.827

TABLE IV
THE COMPARISON OF HASTM WITH OTHER METHODS. WHEN $\mu = 0.98$.

	HASTM			HAM		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	17.109	36.900	86.900	17.109	36.900	86.900
t=15	37.915	63.532	113.532	37.915	63.532	113.532
t=18	43.913	71.209	121.209	43.913	71.209	121.209
	HPM			**ETHPM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	17.109	36.900	86.900	17.109	36.900	86.900
t=15	37.915	63.532	113.532	37.915	63.532	113.532
t=18	43.913	71.209	121.209	43.913	71.209	121.209

TABLE V
ABSOLUTE ERROR FOR THE EXACT SOLUTION AND FRACTIONAL SOLUTION OF EQ. 25.

τ	$\nu(x,t)$	Exact	Absolute error
0	147.413159102577	147.4131591025766	4×10^{-13}
0.2	147.743825769243	147.7428390565409	$9.8671270204\times10^{-4}$
0.4	147.978492435910	147.9638301384593	0.0146622974506
0.6	148.181159102577	148.1119648906644	0.0691942119126
0.8	148.415825769243	148.2112625845819	0.2045631846610
1	148.746492435910	148.2778238193399	0.4686686165700

TABLE VI
THE VARIABLES VALUES

Variables	Describtion	Value
μ	Fractional order	0.92, 0.94, 0.96, 0.98
S	Stock price	125, 150, 175, 200, 225
K	Exercise price	160
r	Risk-free interest rate	0.06
T	Time to expiry in month	6, 15

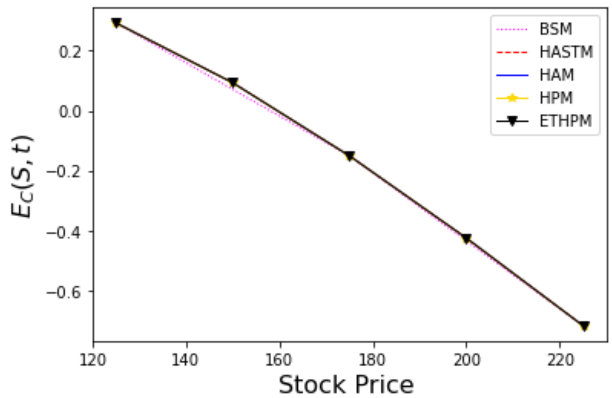

Fig. 9. The result of HASTM, HAM, HPM, ETHPM and BSM with μ = 1 and T = 6 Months.

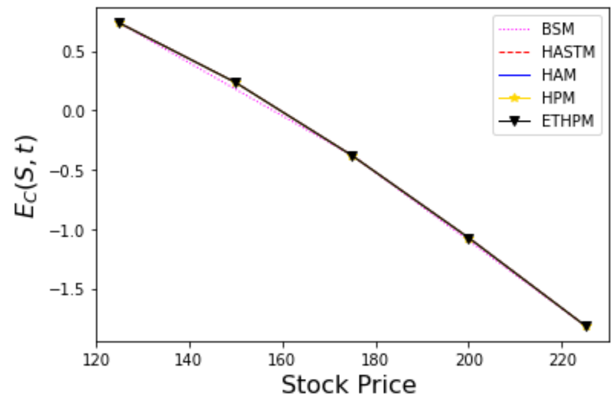

Fig. 10. The result of HASTM, HAM, HPM, ETHPM and BSM with μ = 1 and T = 1.3 Years.

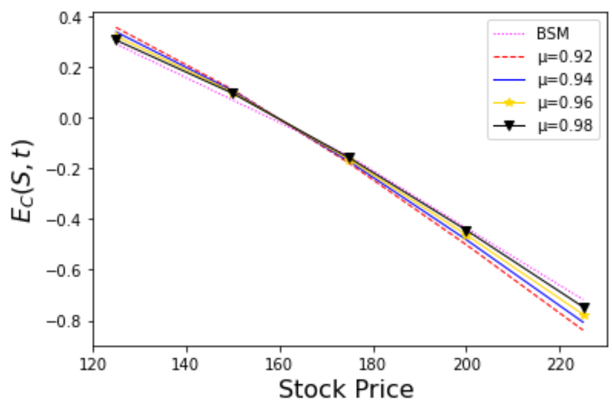

Fig. 11. The result of HASTM and BSM with μ = 0.92, 0.94, 0.96, 0.98 and T = 6 Months.

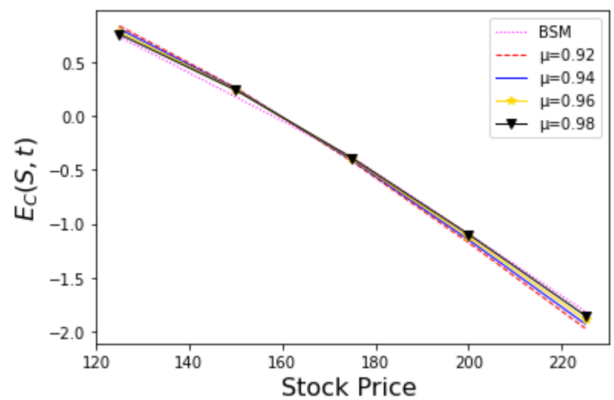

Fig. 12. The result of HASTM and BSM with μ = 0.92, 0.94, 0.96, 0.98 and T = 1.3 Years.

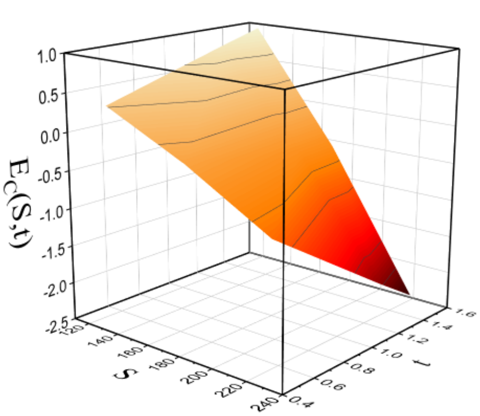

Fig. 13. The result of the B-S equation, when $\mu = 0.92$.

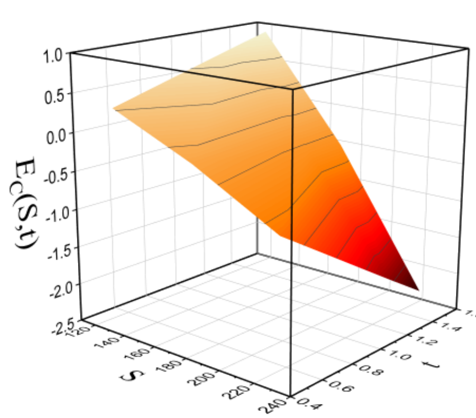

Fig. 14. The result of the B-S equation, when $\mu = 0.94$.

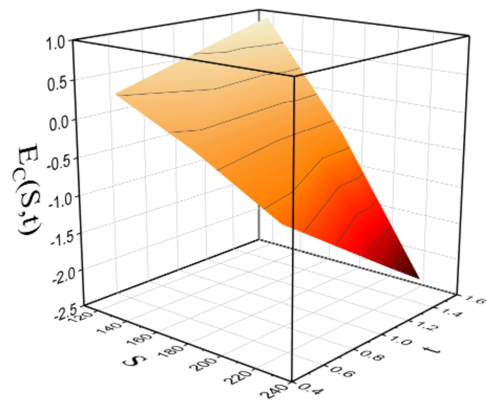

Fig. 15. The result of the B-S equation, when $\mu = 0.96$

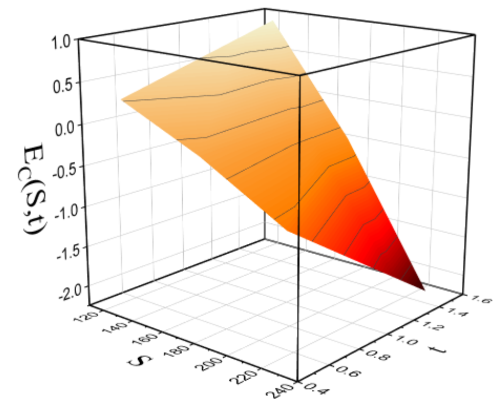

Fig. 16. The result of B-S the equation, when $\mu = 0.98$

TABLE VII
THE COMPARISON OF HASTM WITH OTHER METHODS. WHEN $\mu = 0.92$.

	HASTM			**HAM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	0.3469	-0.1737	-0.8118	0.3469	-0.1737	-0.8118
t=15	0.8379	-0.4208	-1.9701	0.8379	-0.4208	-1.9701
t=18	0.9930	-0.4991	-2.3381	0.9930	-0.4991	-2.3381
	HPM			**ETHPM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	0.3469	-0.1737	-0.8118	0.3469	-0.1737	-0.8118
t=15	0.8379	-0.4208	-1.9701	0.8379	-0.4208	-1.9701
t=18	0.9930	-0.4991	-2.3381	0.9930	-0.4991	-2.3381

TABLE VIII
THE COMPARISON OF HASTM WITH OTHER METHODS. WHEN $\mu = 0.94$.

	HASTM			**HAM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	0.3326	-0.1677	-0.7872	0.3326	-0.1677	-0.7872
t=15	0.8118	-0.4105	-1.9303	0.8118	-0.4105	-1.9303
t=18	0.9655	-0.4886	-2.2991	0.9655	-0.4886	-2.2991
	HPM			**ETHPM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	0.3326	-0.1677	-0.7872	0.3326	-0.1677	-0.7872
t=15	0.8118	-0.4105	-1.9303	0.8118	-0.4105	-1.9303
t=18	0.9655	-0.4886	-2.2991	0.9655	-0.4886	-2.2991

TABLE IX
THE COMPARISON OF HASTM WITH OTHER METHODS. WHEN $\mu = 0.98$.

	HASTM			**HAM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	0.3058	-0.1564	-0.7404	0.3058	-0.1564	-0.7404
t=15	0.7614	-0.3903	-1.8517	0.7614	-0.3903	-1.8517
t=18	0.9122	-0.4680	-2.2213	0.9122	-0.4680	-2.2213
	HPM			**ETHPM**		
Data	S = 125	S = 175	S=225	S = 125	S = 175	S=225
t=6	0.3058	-0.1564	-0.7404	0.3058	-0.1564	-0.7404
t=15	0.7614	-0.3903	-1.8517	0.7614	-0.3903	-1.8517
t=18	0.9122	-0.4680	-2.2213	0.9122	-0.4680	-2.2213

TABLE X
ABSOLUTE ERROR FOR THE EXACT SOLUTION AND FRACTIONAL SOLUTION OF EQ. 33.

τ	$\nu(x,t)$	**Exact**	**Absolute error**
0	0	0	0
0.2	-0.00603614443303890	-0.0060361440000000	$-4.33038824\times10^{-10}$
0.4	-0.01214515894531076	-0.0121451520000000	-6.94531076×10^{-9}
0.6	-0.01832792324546184	-0.0183278880000000	-3.52454618×10^{-8}
0.8	-0.02458532766223531	-0.0245852160000001	-1.11662230×10^{-7}
1	-0.03091827327267982	-0.0309180000000000	-2.73272679×10^{-7}

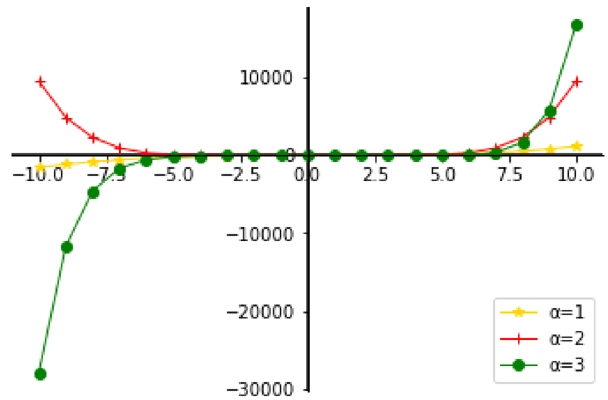

Fig. 17. h-curves of Eq. (25) using 4th-order approximations.

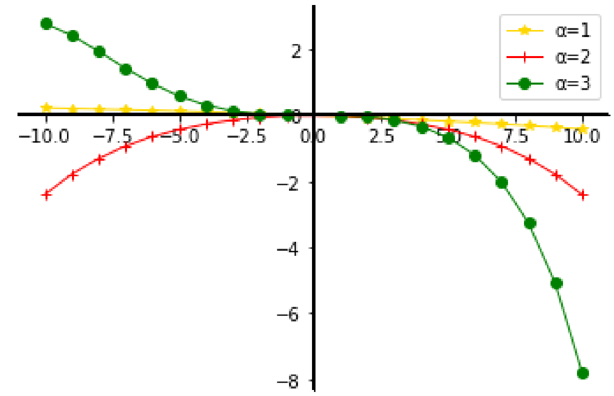

Fig. 18. h-curves of Eq. (33) using 4th-order approximations.

VIII. Conclusion

In this study, the fractional Black-Scholes equation and the generalised fractional Black-Scholes equation were solved using HASTM with the help of a convergence control parameter. The findings exhibited that HASTM and BSM are in agreement. Additionally, it is noted that for the FBSE and GFBSE, HASTM results are comparable to other existing approaches, like HAM, HPM and ETHPM are same. The physical behaviour of option pricing has been depicted using plots for different values of μ. The convergence region is obtained by the h-curve for the B–S equation. Therefore, HASTM is found to be precise, efficient, and suitable for obtaining both exact and approximative solutions for FBSE and GFBSE. Furthermore, the results obtained from this study using HASTM are better compared to the classic Black-Scholes model because they have smoother graphics. Also, HASTM can be used to solve challenging non-linear differential equations and fractional differential equations that arise in a range of science and engineering disciplines.

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