Fixed Point Results in Partially Ordered Ultrametric Space via p-adic Distance

Balaanandhan Radhakrishnan and Uma Jayaraman*

Abstract—**In this paper, using the rational contraction we prove certain fixed-point theorems (FPTs) via p-adic distance over partially ordered ultrametric spaces. Further, these results are explored with suitable examples.**

Index Terms—**Fixed point; p-adic distance; Partially ordered metric space; Rational contraction; Ultrametric space;**

I. Introduction

Fixed-point theory plays one of the key roles in the advancement of functional analysis. The research in fixed point theory was initiated by Poincare in the 19th century. Fixed point theory and its applications are an emerging field of research as they have applications in solving a growing number of nonlinear problems. A famous principle known as the Banach contraction principle was introduced by Banach [4] in 1922 and played a major role in obtaining the sufficient conditions for the existence of fixed points and further proving its uniqueness in various algebraic spaces. The author of [4] demonstrated that every contraction mapping has a unique fixed point in complete metric spaces. Several fascinating extensions and generalizations have been obtained for the Banach contraction principle.

A new contractive principle known as rational contraction, was developed by Dass and Gupta [7] in 1975 for the existence of fixed points which was stated as follows: Let (χ, d) be a complete metric space and $\top$ is a self map on χ such that there exists $\varrho, \vartheta \geq 0$ with $\varrho + \vartheta < 1$ satisfy

$$d(\top x, \top y) \leq \varrho \frac{d(x, \top x)(1 + d(y, \top y))}{1 + d(x, y)} + \vartheta d(x, y),$$

for all $x, y \in \chi$, then $\top$ has a fixed point.

Further, in 1997, Jaggi [12] introduced a new rational type contractive condition which also helped to demonstrate the uniqueness of fixed points in metric spaces and the theorem is stated as follows:

Suppose $\top$ is a continuous self-map defined on a complete metric space (χ, d). Let $\top$ satisfies the following contractive condition:

$$d(\top x, \top y) \leq \varrho \frac{d(x, \top x) d(y, \top y)}{d(x, y)} + \vartheta d(x, y),$$

for all $x, y \in \chi$, for some $\varrho, \vartheta \in [0, 1)$ and $\varrho + \vartheta < 1$, then $\top$ has a unique fixed point in χ.

The existence of fixed points for self-mappings defined over partially ordered sets was initially discussed by Ran and Reurings [27]. Further, they gave some applications for matrix equations. Thereafter, the results of [27] were generalized to partially ordered sets [28]. Some related results about partially ordered sets can be found in [1] and the references therein.

Later, Cabrera [6] proved the results of Dass and Gupta over partially ordered metric spaces. The same results have been presented by Poom Kumam [19] through rational contractions in ordered metric spaces. Harjani et al., [10] proved a fixed point theorem in partially ordered metric spaces, meeting a rational type contractive condition attributed to Jaggi [12].

The concept of weakly increasing property on maps was investigated by Nashine and Samet [17]. In 1996, Junck [13] generalized the notion of weakly commuting maps by introducing the concept of compatible maps. In 1998, Pant [18] initiated the notion of reciprocally continuous maps and obtained some fixed point results. This idea has been well utilized in checking the compatibility between the mappings.

In 1897, German mathematician Hensel [11] introduced the concept of p-adic numbers. The number theory involves significant use of p-adic numbers. The completion of the field Q of rational numbers concerning a p-adic valuation $|\cdot|_p$ is called the field of p-adic numbers denoted by Q_p.

Further, the concept of ultrametric spaces was introduced by Van Rooij [26] in 1978. Using generalized contractive mappings, Gajic [8] proved some fixed point theorems in a spherically complete ultrametric spaces. Rao et al., [25] discussed some coincidence point theorems for three and four self-maps using generalized contractive conditions. Some fixed point theorems in ultrametric spaces have been investigated by Kirk and Shahzad [14]. There are numerous studies on this topic have been conducted including [5], [9], [16], [20], [21], [22], [29]. In the year 2017, Hamid Mamghaderi et al., [15] proved some fixed point theorems in partially ordered ultrametric and non-Archimedean normed spaces which he considered single-valued and strongly contractive mappings. Also, Ramesh Kumar and Pitchaimani [23], [24] analyzed some set-valued contractions and Prešić-Reich types of mappings in ultrametric spaces.

Motivated by the above results, in this paper, we investigate the various fixed point results in ultrametric spaces using p-adic distance under rational-type contractive conditions.

Manuscript received October 20, 2022; revised May 08, 2023.

Balaanandhan Radhakrishnan is a Research Scholar in the Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chengalpattu, Tamilnadu-603203, India. (e-mail:br9214@srmist.edu.in).

Uma Jayaraman (Corresponding Author) is an Assistant Professor in the Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chengalpattu, Tamilnadu-603203, India. (e-mail:umaj@srmist.edu.in).

II. Preliminaries

Definition 2.1. [2] Consider a fixed prime number p. Also, let $c \in \mathbb{R}$, where $0 < c < 1$ and c will be fixed. If $\varkappa$ is any rational number other than zero, we can write $\varkappa$ in the form

$$\varkappa = p^{\alpha}\frac{a}{b},$$

where $\alpha \in \mathbb{Z}$, $a, b \in \mathbb{Z}$ and $p \nmid a$,$p \nmid b$. Clearly, α may be positive, negative or zero depending on $\mathbf{X}$. We now define

$$|\varkappa|_p = c^{\alpha} \text{ and } |0|_p = 0,$$

it follows immediately from the definition that, $|\varkappa|_p \geq 0$ and equals 0 if and only if $\varkappa = 0$.

Example 2.1. [2] Take $\varkappa = \frac{19}{216}$. Suppose if we want to find its 2-adic absolute value (where p=2), first, we write $\varkappa$ in the following form

$$\varkappa = \frac{19}{8 \times 27} = 2^{-3} \times \frac{19}{27},$$

which implies that $|\varkappa|_2 = 2^3 = 8$.
Then, what about its 19-adic absolute value? It will simply be $|\varkappa|_{19} = \frac{1}{19}$ because

$$\varkappa = 19 \times \frac{1}{216} \text{ thus } |\varkappa|_{19} = \frac{1}{19}.$$

Also, it is trivial that the p-adic absolute value of a rational number when p divides neither the numerator nor the denominator is 1, since $p^0 = 1$.

Definition 2.2. [26] A non-Archimedean metric known as an ultrametric is a function $d_p : \mathbf{X}^2 \to R^+$ such that

$$(i)\ d_p(\varkappa, \mathsf{y}) \geq 0 \text{ and } d_p(\varkappa, \mathsf{y}) = 0 \text{ iff } \varkappa = \mathsf{y},$$
$$(ii)\ d_p(\varkappa, \mathsf{y}) = d_p(\mathsf{y}, \varkappa),$$
$$(iii)\ d_p(\varkappa, \mathsf{y}) \preceq \max\ \{d_p(\varkappa, z), d_p(z, \mathsf{y})\} \text{ (stronger triangle inequality)},$$

for all $\varkappa, \mathsf{y}, z \in \mathbf{X}$. The p-adic valuation $|.|_p$ induces the above metric d_p and so it can be defined by $d_p(\varkappa, \mathsf{y}) = |\varkappa - \mathsf{y}|_p$.

Definition 2.3. [15] Let $\mathbf{X}$ be a non-void set. A partially ordered relation $\preceq$ over $\mathbf{X}$ is a relation satisfying the following conditions:

(i) for all $\varkappa \in \mathbf{X}$, $\varkappa \preceq \varkappa$, (Reflexive)

(ii) for all $\varkappa, \mathsf{y} \in \mathbf{X}$, $\varkappa \preceq \mathsf{y}$ *and* $\mathsf{y} \preceq \varkappa$ *imply* $\varkappa = \mathsf{y}$, (anti-symmetry)

(iii) for all $\varkappa, \mathsf{y}, z \in \mathbf{X}, \varkappa \preceq \mathsf{y}$ *and* $\mathsf{y} \preceq z$ imply $\varkappa \preceq z$. (transitivity)

Then, the pair $(\mathbf{X}, \preceq)$ is called a partially ordered set. If $(\mathbf{X}, \preceq)$ is a partially ordered set, then $\varkappa$ and y are called comparable elements of $\mathbf{X}$ if either $\varkappa \preceq \mathsf{y}$ or $\mathsf{y} \preceq \varkappa$.

Partial ordered sets have been extensively studied by Ran and Reurings [27]. Here we introduce the concept of partially ordered ultrametric spaces as follows.

Definition 2.4. Let $(\mathbf{X}, d_p, \preceq)$ is said to be partially ordered ultrametric spaces, if d_p is defined over the partially ordered set $(\mathbf{X}, \preceq)$.

Definition 2.5. Let $(\mathbf{X}, d_p)$ be a complete ultrametric spaces, then the triple $(\mathbf{X}, d_p, \preceq)$ is said to be partially ordered complete ultrametric spaces.

Definition 2.6. Let $(\mathbf{X}, d_p, \preceq)$ be an ultrametric space. Assume that $\mathbf{X}$ is regular if and only if there is a non-decreasing sequence $\{\varkappa_n\}$ in $\mathbf{X}$ such that

$$\lim_{n\to\infty} \varkappa_n = \varkappa,$$

then $\varkappa_n \preceq \varkappa$, for all $n \in \mathcal{N}$.

Definition 2.7. Let $(\mathbf{X}, d_p, \preceq)$ be an ultrametric spaces and $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the mappings such that $\top\mathbf{X} \subseteq \mathsf{R}\mathbf{X}$. Then $\top$ is called weakly increasing with respect to R if and only if

$$\top\varkappa \preceq \top\mathsf{y}, \quad \forall\ \varkappa \in \mathbf{X}, \ \ \mathsf{y} \in \mathsf{R}^{-1}(\top\varkappa).$$

Definition 2.8. Let R,$\top$ be the self maps on $\mathbf{X}$ with an ultrametric d_p, then the pair $\{\mathsf{R},\top\}$ is called reciprocally continuous if and only if

$$\lim_{n\to\infty} \mathsf{R}\top\varkappa_n = \mathsf{R}z \quad and \quad \lim_{n\to\infty} \top\mathsf{R}\varkappa_n = \top z,$$

for every sequence $\{\varkappa_n\}$ in $\mathbf{X}$ satisfying

$$\lim_{n\to\infty} \mathsf{R}\varkappa_n = z = \lim_{n\to\infty} \top\varkappa_n, \qquad \text{for some } z \in \mathbf{X}.$$

Definition 2.9. Let R,$\top$ be the self maps on $\mathbf{X}$ with an ultrametric d_p.Then, the pair $\{\mathsf{R},\top\}$ is called weakly reciprocally continuous if and only if

$$\lim_{n\to\infty} \mathsf{R}\top\varkappa_n = \mathsf{R}z,$$

for every sequence $\{\varkappa_n\} \in \mathbf{X}$ satisfying

$$\lim_{n\to\infty} \mathsf{R}\varkappa_n = z = \lim_{n\to\infty} \top\varkappa_n \qquad \text{for some } z \in \mathbf{X}.$$

Definition 2.10. Let R,$\top$ be the self maps on $\mathbf{X}$ with metric d_p. Then the ultrametric spaces $(\mathbf{X}, d_p)$ is called compatible if and only if

$$\lim_{n\to\infty} d_p(\mathsf{R}(\top(\varkappa_n)), \top(\mathsf{R}(\varkappa_n))) = 0,$$

whenever a sequence $\{\varkappa_n\}$ in $\mathbf{X}$ such that

$$\lim_{n\to\infty} \mathsf{R}\varkappa_n = \lim_{n\to\infty} \top\varkappa_n = z,$$

where $z \in \mathbf{X}$.

Example 2.2. Let $\mathbf{X} = [0, 1]$ and d_p be the ultrametric on $\mathbf{X}$. The mappings $\mathsf{A}, \mathsf{S} : \mathbf{X} \to \mathbf{X}$ defined by

$$\mathsf{S}(\varkappa) = \begin{cases} 3\varkappa - 2, & if \ \ \varkappa \in [0, 1], \\ 0, & \text{otherwise} \end{cases}$$

$$\mathsf{A}(\varkappa) = \begin{cases} \varkappa^2, & if \ \ \varkappa \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Now, consider the sequence $\{\varkappa_n\} = \{1 - \frac{1}{n}\}$ in $\mathbf{X}$. Then

$$\lim_{n\to\infty} \mathsf{A}\varkappa_n = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^2 = 1$$

$$1 = \lim_{n\to\infty} \mathsf{S}\varkappa_n = \lim_{n\to\infty} 3\left(1-\frac{1}{n}\right) - 2$$

$$\begin{aligned}\lim_{n\to\infty} \mathsf{AS}\varkappa_n &= \lim_{n\to\infty} \mathsf{A}\big(3\left(1-\frac{1}{n}\right) - 2\big)\\ &= \lim_{n\to\infty} \mathsf{A}\big(1-\frac{3}{n}\big) = \lim_{n\to\infty} \big(1-\frac{3}{n}\big)^2\\ &= 1.\end{aligned}$$

Similarly,
$\lim_{n\to\infty} \mathsf{SA}\varkappa_n = \lim_{n\to\infty} \mathsf{S}\big(1-\frac{1}{n}\big)^2 = \lim_{n\to\infty} 3\big(1-\frac{1}{n}\big)^2 - 2 = 1.$
Therefore, the pair (A,S) is compatible.

Inspired by the notions of rational type contraction, we introduce a new p-adic rational type contractive condition and prove some fixed point theorems in partially ordered ultrametric spaces.

III. Main results

In this section, we use our new p-adic rational type contractive condition and prove some fixed point results which has been discussed by Poom Kumam et al. [19] for the classical case.

Theorem 3.1. Let $(\mathbf{X},\preceq,d_p)$ be a partially ordered complete ultrametric spaces. Let $\top,\mathsf{R}:\mathbf{X}\to\mathbf{X}$ be the two functions with

$$d_p(\top\varkappa,\top \mathsf{y}) \preceq \frac{1}{|\vartheta|}\frac{d_p(\mathsf{R}\varkappa,\top\varkappa)d_p(\mathsf{R}\mathsf{y},\top\mathsf{y})}{d_p(\mathsf{R}\varkappa,\mathsf{R}\mathsf{y})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa,\mathsf{R}\mathsf{y}), \tag{1}$$

where $\varrho,\vartheta\in[0,1)$ with $\varrho+\vartheta<1$ and assume that

(i) the pair $(\mathsf{R},\top)$ is both weakly reciprocally continuous and commuting,
(ii) $\mathbf{X}$ is regular and $\top$ is weakly increasing with R.

Then there exist a coincidence point $\mathsf{u}\in\mathbf{X}$ of $\top$ and R such that $\mathsf{Ru}=\top\mathsf{u}$.

Proof: Let $\varkappa_0$ be an arbitrary point in $\mathbf{X}$. Since $\top\mathbf{X}\subseteq \mathsf{R}\mathbf{X}$, we construct a sequence $\{\varkappa_n\}$ in $\mathbf{X}$ by

$$\top\varkappa_{\mathrm{n-1}} = \mathsf{R}\varkappa_n. \tag{2}$$

As $\varkappa_1\in\mathsf{R}^{-1}(\top\varkappa_0)$ and $x_2\in\mathsf{R}^{-1}(\top\varkappa_1)$, from definition 2.7, we obtain

$$\mathsf{R}\varkappa_1 = \top x_0 \preceq \top\varkappa_1 = \mathsf{R}\varkappa_2 \preceq \top\varkappa_2 = \mathsf{R}\varkappa_3 \preceq \top\varkappa_3 = \mathsf{R}\varkappa_4.$$

Continuing this process indefinitely, we get

$$\mathsf{R}\varkappa_1 \preceq \mathsf{R}\varkappa_2 \preceq \mathsf{R}\varkappa_3 \preceq .. \preceq \mathsf{R}\varkappa_{n-1} \preceq \mathsf{R}\varkappa_n \preceq \mathsf{R}\varkappa_{n+1} \preceq \cdots.$$

Now, to prove that $\mathsf{R}(\varkappa_n)$ is a Cauchy sequence. Since $\mathsf{R}(\varkappa_1)\geq\mathsf{R}(\varkappa_0)$, using (1),we have

$$\begin{aligned}d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) &= d_p(\top\varkappa_0,\top\varkappa_1)\\ &\preceq \frac{1}{|\vartheta|}\frac{d_p(\mathsf{R}\varkappa_0,\top\varkappa_0)d_p(\mathsf{R}\varkappa_1,\top\varkappa_1)}{d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1)}\\ &\quad + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1),\end{aligned}$$

$$d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) \preceq \frac{1}{|\vartheta|}d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1)$$

$$(1-\frac{1}{|\vartheta|})d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) \preceq \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1)$$

$$|\frac{\vartheta-1}{\vartheta}|d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) \preceq \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1)$$

$$d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) \preceq |\frac{\vartheta}{\varrho(1-\vartheta)}|\ d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1).$$

Where $\kappa=\frac{\vartheta}{\varrho(1-\vartheta)}<1$,

$$\implies d_p(\mathsf{R}\varkappa_1,\mathsf{R}\varkappa_2) \preceq |\kappa|\ d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1). \tag{3}$$

For $n>0$, As $\mathsf{R}(\varkappa_{n+1})\geq\mathsf{R}(\varkappa_n)$, using(1), we have

$$\begin{aligned}d_p(\mathsf{R}\varkappa_{n+1},\mathsf{R}\varkappa_{n+2}) &= d_p(\top\varkappa_n,\top\varkappa_{n+1})\\ &\preceq \frac{1}{|\vartheta|}\frac{d_p(\mathsf{R}\varkappa_n,\top\varkappa_n)\ d_p(\mathsf{R}\varkappa_{n+1},\top\varkappa_{n+1})}{d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1})}\\ &\quad + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1}),\\ &\preceq \frac{1}{|\vartheta|}\frac{d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1})\ d_p(\mathsf{R}\varkappa_{n+1},\mathsf{R}\varkappa_{n+2})}{d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1})}\\ &\quad + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1}),\end{aligned}$$

$$\begin{aligned}d_p(\mathsf{R}\varkappa_{n+1},\mathsf{R}\varkappa_{n+2}) &\preceq \frac{1}{|\vartheta|}d_p(\mathsf{R}\varkappa_{n+1},\mathsf{R}\varkappa_{n+2})\\ &\quad + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1}),\end{aligned}$$

$$d_p(\mathsf{R}\varkappa_{n+1},\mathsf{R}\varkappa_{n+2}) \preceq |\frac{\vartheta}{\varrho(1-\vartheta)}|d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1}).$$

$$d_p(\mathsf{R}\varkappa_{n+1},\mathsf{R}\varkappa_{n+2}) \preceq |\kappa|^{n+1}d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1), \tag{4}$$

with $\kappa=\frac{\vartheta}{\varrho(1-\vartheta)}$.

Now, let $\varkappa_n\in\mathbf{X}$, then using (4), we get

$$d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1}) \preceq |\kappa^n|\,d_p(\mathsf{R}\varkappa_0,\mathsf{R}\varkappa_1), \tag{5}$$

this implies that,

$$d_p(\mathsf{R}\varkappa_n,\mathsf{R}\varkappa_{n+1})\to 0 \text{ as } n\to\infty$$
$$\text{since } 0<\kappa=\frac{\vartheta}{\varrho(1-\vartheta)}<1.$$

Therefore, the sequence $\{\mathsf{R}\varkappa_n\}$ is Cauchy. Further, since $\mathbf{X}$ is complete, there exists a $\varkappa\in\mathbf{X}$ such that

$$\lim_{n\to\infty}\top(\varkappa_n) = \lim_{n\to\infty}\mathsf{R}(\varkappa_n) = \varkappa. \tag{6}$$

Also, by commutativity of $\top$ and R, we have

$$\mathsf{R}(\mathsf{R}\varkappa_{n+1}) = \mathsf{R}(\top\varkappa_n) = \top(\mathsf{R}\varkappa_n),$$

this implies that $\mathsf{R}^{-1}(\top(\mathsf{R}\varkappa_n)) = \mathsf{R}\varkappa_{n+1}$.
Since $\top$ is weakly increasing with R, we can write

$$\mathsf{R}(\mathsf{R}\varkappa_{n+2}) = \top(\mathsf{R}\varkappa_{n+1}) \geq \top(\mathsf{R}\varkappa_n) = \mathsf{R}(\mathsf{R}\varkappa_{n+1}). \tag{7}$$

So that, $\mathsf{R}(\mathsf{R}\varkappa_n)$ is non decreasing. Since R and $\top$ are weakly reciprocally continuous,

$$\lim_{n\to\infty}\mathsf{R}(\top\varkappa_{n-1}) = \lim_{n\to\infty}\mathsf{R}(\mathsf{R}\varkappa_n) = \mathsf{R}\varkappa,$$

hence by the regularity of $\mathbf{X}$, we obtain that

$$\mathsf{R}(\mathsf{R}\varkappa_n) \preceq \mathsf{R}\varkappa. \tag{8}$$

i.e., $\mathsf{R}(\mathsf{R}\varkappa_n)$ and $\mathsf{R}\varkappa$ are comparable.
Now, by using the triangle inequality and using equation (1),

we have

$$\begin{aligned} d_p(\mathsf{R}\varkappa, \top\varkappa) &\preceq \max\Big\{ d_p(\mathsf{R}\varkappa, \mathsf{R}(\mathsf{R}\varkappa_{n+1})), d_p(\mathsf{R}(\mathsf{R}\varkappa_{n+1}), \top\varkappa) \Big\} \\ &\preceq \max\Big\{ d_p(\mathsf{R}\varkappa, \mathsf{R}(\mathsf{R}\varkappa_{n+1})), d_p(\mathsf{R}(\top\varkappa_n), \top\varkappa) \Big\} \\ &\preceq \max\Big\{ d_p(\mathsf{R}\varkappa, \mathsf{R}(\mathsf{R}\varkappa_{n+1})), d_p(\top(\mathsf{R}\varkappa_n), \top\varkappa) \Big\} \\ &\preceq \max\Big\{ d_p(\mathsf{R}\varkappa, \mathsf{R}(\mathsf{R}\varkappa_{n+1})), \\ &\quad \frac{1}{|\vartheta|} \frac{d_p(\mathsf{R}(\mathsf{R}\varkappa_n), \top(\mathsf{R}\varkappa_n)) d_p(\mathsf{R}\varkappa, \top\varkappa)}{d_p(\mathsf{R}(\mathsf{R}\varkappa_n), \mathsf{R}\varkappa)} \\ &\quad + \frac{1}{|\varrho|} d_p(\mathsf{R}(\mathsf{R}\varkappa_n), \mathsf{R}\varkappa) \Big\}. \end{aligned}$$

On taking limit as $n \to \infty$, and using equation (8), we get

$$d_p(\mathsf{R}\varkappa, \top\varkappa) = 0,$$

so that $\mathsf{R}\varkappa = \top\varkappa$. As a result, we have demonstrated that R and $\top$ have a coincidence point. ■

Theorem 3.2. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|} \frac{d_p(\mathsf{R}\varkappa, \top\varkappa) d_p(\mathsf{R}y, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})} + \frac{1}{|\varrho|} d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y}), \tag{9}$$

where $\varrho, \vartheta \in [0, 1)$ with $\varrho + \vartheta < 1$ and presume that

(i) $\top$ is weakly increasing with R,
(ii) the pair $(\top, \mathsf{R})$ is compatible and reciprocally continuous.

Then there exist a coincidence point $\varkappa \in \mathbf{X}$ of $\top$ and R such that $\mathsf{R}\varkappa = \top\varkappa$.

Proof: Proceeding in a similar way of the above Theorem 3.1, there exist a sequence $\{\varkappa_n\}$ such that

$$\lim_{n\to\infty} \mathsf{R}(\varkappa_n) = \lim_{n\to\infty} \top(\varkappa_n) = \varkappa. \tag{10}$$

We now prove that, $\varkappa$ is the coincidence point of R and $\top$. Since $\{\top,\mathsf{R}\}$ is compatible and reciprocally continuous, we have

$$\lim_{n\to\infty} d_p(\mathsf{R}(\top(\varkappa_n)), \top(\mathsf{R}(\varkappa_n))) = 0, \tag{11}$$

$$\mathsf{R}(\varkappa) = \lim_{n\to\infty} \mathsf{R}\top\varkappa_n, \qquad \top(\varkappa) = \lim_{n\to\infty} \top\mathsf{R}\varkappa_n, \tag{12}$$

whenever,

$$\lim_{n\to\infty} \mathsf{R}(\varkappa_n) = \lim_{n\to\infty} \top(\varkappa_n) = \varkappa. \tag{13}$$

Further, using equation (12) in (11), we get,

$d_p(\top\varkappa, \mathsf{R}\varkappa) = 0$, so that $\top\varkappa = \mathsf{R}\varkappa$. ■

As a consequence of Theorems 3.1 and 3.2, we have the following.

Theorem 3.3. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top : \mathbf{X} \to \mathbf{X}$ be a non decreasing mapping satisfying

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|} \frac{d_p(\varkappa, \top\varkappa) d_p(\mathsf{y}, \top\mathsf{y})}{d_p(\varkappa, \mathsf{y})} + \frac{1}{|\varrho|} d_p(\varkappa, \mathsf{y}), \tag{14}$$

where $\varrho, \vartheta \in [0, 1)$ such that $\varrho + \vartheta < 1$ and presume that

(i) $\top\varkappa \preceq \top(\top\varkappa)$, $\forall\ \varkappa \in \mathbf{X}$,
(ii) either $\top$ is continuous or $\mathbf{X}$ is regular.

Then $\top$ has a fixed point.

Proof: In equation (1) of Theorem 3.1, taking R to be an identity mapping on $\mathbf{X}$, we get the proof of the theorem. ■

Property (A): If $\mathsf{R}(\varkappa_n)$ is a non decreasing sequence in $\mathbf{X}$ such that $\lim\limits_{n\to\infty} \mathsf{R}(\varkappa_n) = \varkappa$, then $\mathsf{R}(\varkappa_n)$ is comparable to $\mathsf{R}\varkappa$, for all $n \in \mathcal{N}$.

Theorem 3.4. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|} \frac{d_p(\mathsf{R}\varkappa, \top\varkappa) d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})} + \frac{1}{|\varrho|} d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y}), \tag{15}$$

where $\varrho, \vartheta \in [0, 1)$ such that $\varrho + \vartheta < 1$ and presume that

(i) $\varkappa$ is regular and $\top$ is weakly increasing with R,
(ii) the pair $(\top, \mathsf{R})$ is both commuting and weakly reciprocally continuous,
(iii) R satisfies the property (A).

Then $\top, \mathsf{R}$ have a common fixed point.

Proof: Proceeding in a similar way as discussed in Theorem 3.1, one can construct a non decreasing sequence $\{\mathsf{R}\varkappa_n\}$ such that $\lim\limits_{n\to\infty} \mathsf{R}\varkappa_{n+1} = \lim\limits_{n\to\infty} \top\varkappa_n = \varkappa$ and $\top(\varkappa) = \mathsf{R}(\varkappa)$.

Since $\mathsf{R}(\varkappa_n)$ and $\mathsf{R}(\varkappa)$ are comparable, by using equation (15), we have

$$\begin{aligned} d_p(\mathsf{R}\varkappa, \mathsf{R}\varkappa_{n+1}) &= d_p(\top\varkappa, \top\varkappa_n) \\ &\preceq \frac{1}{|\vartheta|} \frac{d_p(\mathsf{R}\varkappa, \top\varkappa) d_p(\mathsf{R}\varkappa_n, \top\varkappa_n)}{d_p(\mathsf{R}\varkappa, \mathsf{R}\varkappa_n)} + \frac{1}{|\varrho|} d_p(\mathsf{R}\varkappa, \mathsf{R}\varkappa_n). \end{aligned}$$

Now, taking the limit as $n \to \infty$, one can get $\varkappa = \mathsf{R}\varkappa = \top\varkappa$. Hence the proof. ■

Theorem 3.5. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|} \frac{d_p(\mathsf{R}\varkappa, \top\varkappa) d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})} + \frac{1}{|\varrho|} d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y}) \tag{16}$$

where $\varrho,\ \vartheta \in [0, 1)$ such that $\varrho + \vartheta < 1$ and presume that

(i) $\top$ is weakly increasing with R,
(ii) the pair $(\top, \mathsf{R})$ is compatible and reciprocally continuous,
(iii) R satisfies the property (A).

Then $\top$ and R have a common fixed point.

Proof: Proof is similar by the way of Theorems 3.2 and 3.4. ■

Remarks 3.1. The above theorems cannot be proved when ϱ or ϑ is equal to zero.

Theorem 3.6. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$\begin{aligned} d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|} &\frac{(1 + d_p(\mathsf{R}\varkappa, \top\varkappa)) d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y})}{1 + d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})} \\ &+ \frac{1}{|\varrho|} d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y}), \end{aligned} \tag{17}$$

where $\varrho, \vartheta \in [0, 1)$ with $\varrho + \vartheta < 1$ and presume that

(i) the pair $(\mathsf{R}, \top)$ is both weakly reciprocally continuous and commuting,
(ii) $\mathbf{X}$ is regular and $\top$ is weakly increasing with R.

Then there exist a coincidence point $\mathsf{u} \in \mathbf{X}$ of $\top$ and R such that $\mathsf{Ru} = \top\mathsf{u}$.

Theorem 3.7. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|}\frac{(1 + d_p(\mathsf{R}\varkappa, \top\varkappa))d_p(\mathsf{Ry}, \top\mathsf{y})}{1 + d_p(\mathsf{R}\varkappa, \mathsf{Ry})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa, \mathsf{Ry}), \tag{18}$$

where $\varrho, \vartheta \in [0, 1)$ with $\varrho + \vartheta < 1$ and presume that

(i) $\top$ is weakly increasing with R,
(ii) the pair $(\top, \mathsf{R})$ is compatible and reciprocally continuous.

Then there exist a coincidence point $\varkappa \in \mathbf{X}$ of $\top$ and R such that $\mathsf{R}\varkappa = \top\varkappa$.

Theorem 3.8. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric spaces. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|}\frac{(1 + d_p(\mathsf{R}\varkappa, \top\varkappa))d_p(\mathsf{Ry}, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{Ry})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa, \mathsf{Ry}), \tag{19}$$

where $\varrho, \vartheta \in [0, 1)$ such that $\varrho + \vartheta < 1$ and presume that

(i) $\varkappa$ is regular and $\top$ is weakly increasing with R,
(ii) the pair $(\top, \mathsf{R})$ is both commuting and weakly reciprocally continuous,
(iii) R satisfies the property (A).

Then $\top, \mathsf{R}$ have a common fixed point.

Theorem 3.9. Let $(\mathbf{X}, \preceq, d_p)$ be a partially ordered complete ultrametric space. Let $\top, \mathsf{R} : \mathbf{X} \to \mathbf{X}$ be the two functions with

$$d_p(\top\varkappa, \top\mathsf{y}) \preceq \frac{1}{|\vartheta|}\frac{(1 + d_p(\mathsf{R}\varkappa, \top\varkappa))d_p(\mathsf{Ry}, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{Ry})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa, \mathsf{Ry}), \tag{20}$$

where $\varrho,\ \vartheta \in [0, 1)$ such that $\varrho + \vartheta < 1$ and presume that

(i) $\top$ is weakly increasing with R,
(ii) the pair $(\top, \mathsf{R})$ is compatible and reciprocally continuous,
(iii) R satisfies the property (A).

Then $\top$ and R have a common fixed point.

Example 3.1. Let $\mathbf{X}$ be a partially ordered ultrametric space and $\top, \mathsf{R}$ be the self maps on $\mathbf{X}$ defined by

$$\top\varkappa = \frac{\varkappa}{2} + \frac{1}{8} \quad \text{and} \quad \mathsf{R}\varkappa = 2\varkappa - \frac{1}{4}$$

with the distance function d_p defined in equation(15) , then $\top$ and R have a common fixed point.

Solution: From the Definition 2.1, of $\top\varkappa$ and $\mathsf{R}\varkappa$, we get

$$d_p(\top\varkappa, \top\mathsf{y}) = \tfrac{1}{2}|\varkappa - \mathsf{y}|_p, \quad d_p(\mathsf{R}\varkappa, \top\varkappa) = \tfrac{3}{2}|\varkappa - \tfrac{1}{4}|_p$$

$$d_p(\mathsf{Ry}, \top\mathsf{y}) = \tfrac{3}{2}|\mathsf{y} - \tfrac{1}{4}|_p, \quad d_p(\mathsf{R}\varkappa, \mathsf{Ry}) = 2|\varkappa - \mathsf{y}|_p$$

consider the values of ϑ and ϱ lies between 0 and 1, with $\vartheta + \varrho < 1$.

Let $\varkappa$ and y be fixed such that $\varkappa = \frac{1}{3}$, $\mathsf{y} = \frac{1}{2}$ and using the inequality (15) we obtain the following results:
In table,

$$R = \frac{1}{|\vartheta|}\frac{d_p(\mathsf{R}\varkappa, \top\varkappa)d_p(\mathsf{Ry}, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{Ry})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa, \mathsf{Ry})$$

Table I: p-adic calculation of (15) of the Theorem 3.5

p-adic	ϱ	ϑ	$d_p(\top\varkappa, \top\mathsf{y})$	R
2-adic	0.1	0.8	4	90
	0.2	0.7		96.42857143
	0.3	0.6		110
	0.4	0.5		130.5
	0.5	0.4		162
	0.6	0.3		215
	0.7	0.2		321.4285714
	0.8	0.1		641.25
3-adic	0.1	0.8	3	30.13888889
	0.2	0.7		15.15873016
	0.3	0.6		10.18518519
	0.4	0.5		7.722222222
	0.5	0.4		6.277777778
	0.6	0.3		5.37037037
	0.7	0.2		4.841269841
	0.8	0.1		4.861111111
5-adic 7-adic 11-adic 13-adic	0.1	0.8	1	11.25
	0.2	0.7		6.428571429
	0.3	0.6		5
	0.4	0.5		4.5
	0.5	0.4		4.5
	0.6	0.3		5
	0.7	0.2		6.428571429
	0.8	0.1		11.25

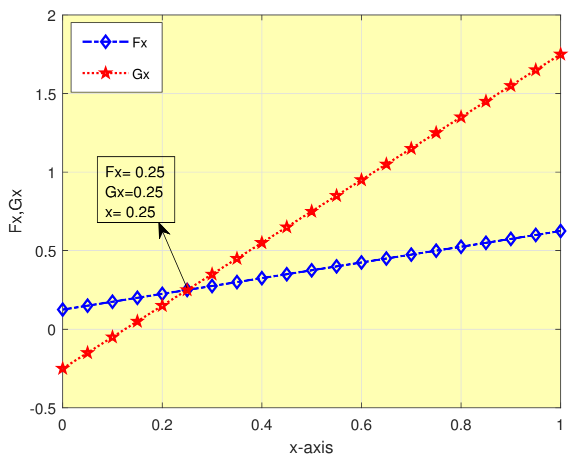

Figure 1: Existence of common fixed point.

From the above table, we obtain the common fixed point as $\mathsf{R}(\frac{1}{4}) = \top(\frac{1}{4}) = \frac{1}{4}$, which is clearly shown in figure 1.

Example 3.2. Let $\mathbf{X}$ be a partially ordered ultrametric spaces and $\top, \mathsf{R}$ be the self maps on $\mathbf{X}$ defined by

$$\top\varkappa = \varkappa^3 + 2 \qquad \text{and} \qquad \mathsf{R}\varkappa = 2\varkappa^3 + 1$$

with the distance function d_p defined in equation (1), then $\top$,R have a point of coincidence .
Solution: From the Definition 2.1, of $\top\varkappa$ and $\mathsf{R}\varkappa$, we get

$$d_p(\top\varkappa, \top\mathsf{y}) = |\varkappa^3 - \mathsf{y}^3|_p, \quad d_p(\mathsf{R}\varkappa, \top\varkappa) = |\varkappa^3 - 1|_p$$

$$d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y}) = |\mathsf{y}^3 - 1|_p, \quad d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y}) = 2|\varkappa^3 - \mathsf{y}^3|_p.$$

Consider the values of ϑ and ϱ lies between 0 and 1, with $\vartheta + \varrho < 1$.

Let $\varkappa$ and y be fixed such that $\varkappa = \frac{1}{3}$ and $\mathsf{y} = \frac{1}{2}$ and using the inequality (1) we obtain the following results:
In table,

$$R_1 = \frac{1}{|\vartheta|}\frac{d_p(\mathsf{R}\varkappa, \top\varkappa)d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y})}{d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})$$

Table II: p-adic calculation of (1) of the Theorem 3.1.

p-adic	ϱ	ϑ	$d_p(\top\varkappa, \top\mathsf{y})$	R_1
3-adic	0.1	0.8	27	271.25
	0.2	0.7		136.4285714
	0.3	0.6		91.66666667
	0.4	0.5		69.5
	0.5	0.4		56.5
	0.6	0.3		48.33333333
	0.7	0.2		43.57142857
	0.8	0.1		43.75
5-adic 11-adic	0.1	0.8	1	11.25
	0.2	0.7		6.428571429
	0.3	0.6		5
	0.4	0.5		4.5
	0.5	0.4		4.5
	0.6	0.3		5
	0.7	0.2		6.428571429
	0.8	0.1		11.25
7-adic	0.1	0.8	1	10.17857143
	0.2	0.7		5.204081633
	0.3	0.6		3.571428571
	0.4	0.5		2.785714286
	0.5	0.4		2.357142857
	0.6	0.3		2.142857143
	0.7	0.2		2.142857143
	0.8	0.1		2.678571429
13-adic	0.1	0.8	1	10.09615385
	0.2	0.7		5.10989011
	0.3	0.6		3.461538462
	0.4	0.5		2.653846154
	0.5	0.4		2.192307692
	0.6	0.3		1.923076923
	0.7	0.2		1.813186813
	0.8	0.1		2.019230769

From the above table, we obtain the coincidence point as $\mathsf{R}(1) = \top(1) = 3$, which is clearly shown in figure 2.

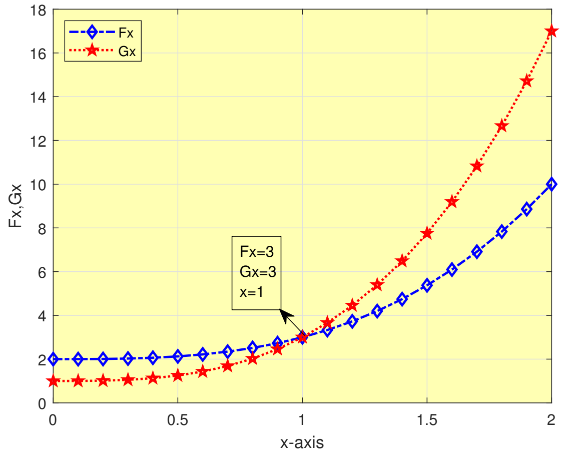

Figure 2: Existence of coincidence point.

Example 3.3. Let $\mathbf{X}$ be a partially ordered ultrametric spaces and $\top, \mathsf{R}$ be the self maps on $\mathbf{X}$ defined by

$$\top\varkappa = \varkappa^2 \qquad \text{and} \qquad \mathsf{R}\varkappa = \varkappa^3$$

with the distance function d_p defined in equation (19) , then $\top$ and R have a common fixed points.

Solution: From the Definition 2.1, of $\top\varkappa$ and $\mathsf{R}\varkappa$, we get

$$d_p(\top\varkappa, \top\mathsf{y}) = |\varkappa^2 - \mathsf{y}^2|_p, \quad d_p(\mathsf{R}\varkappa, \top\varkappa) = |\varkappa^3 - \varkappa^2|_p$$

$$d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y}) = |\mathsf{y}^3 - \mathsf{y}^2|_p, \quad d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y}) = |\varkappa^3 - \mathsf{y}^3|_p.$$

Consider the values of ϑ and ϱ lies between 0 and 1, with $\vartheta + \varrho < 1$.

Let $\varkappa$ and y be fixed such that $\varkappa = \frac{1}{3}$, $\mathsf{y} = \frac{1}{2}$ and using the inequality (19), we obtain the following results:
In table,

$$R_2 = \frac{1}{|\vartheta|}\frac{(1 + d_p(\mathsf{R}\varkappa, \top\varkappa))d_p(\mathsf{R}\mathsf{y}, \top\mathsf{y})}{1 + d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})} + \frac{1}{|\varrho|}d_p(\mathsf{R}\varkappa, \mathsf{R}\mathsf{y})$$

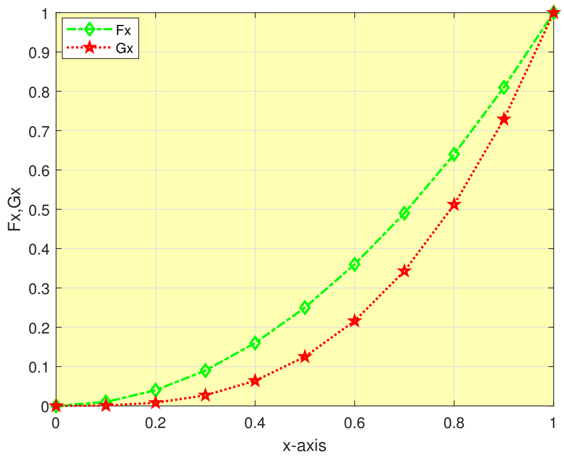

Figure 3: Existence of common fixed point.

From the below table, we obtain the common fixed point as $\mathsf{R}(0) = \top(0) = 0$, and $\mathsf{R}(1) = \top(1) = 1$ which is clearly shown in figure 3.

Table III:p-adic calculation of (19) of the Theorem 3.8.

p-adic	ϱ	ϑ	$d_p(\top\varkappa, \top\mathbf{y})$	$\mathbf{R}_2$
2-adic	0.1	0.8	4	81.66666667
	0.2	0.7		41.9047619
	0.3	0.6		28.88888889
	0.4	0.5		22.66666667
	0.5	0.4		19.33333333
	0.6	0.3		17.77777778
	0.7	0.2		18.0952381
	0.8	0.1		23.33333333
3-adic	0.1	0.8	9	271.25
	0.2	0.7		136.4285714
	0.3	0.6		91.66666667
	0.4	0.5		69.5
	0.5	0.4		56.5
	0.6	0.3		48.33333333
	0.7	0.2		43.57142857
	0.8	0.1		43.75
5-adic **7-adic** **11-adic** **13-adic**	0.1	0.8	1	11.25
	0.2	0.7		6.428571429
	0.3	0.6		5
	0.4	0.5		4.5
	0.5	0.4		4.5
	0.6	0.3		5
	0.7	0.2		6.428571429
	0.8	0.1		11.25

IV. CONCLUSION

In this paper, we established some new fixed point results using rational type contraction with the help of p-adic distance over partially ordered complete ultrametric spaces. Our results are the extensions of the fixed point results discussed recently by Poom kumam [19]. Further, we justified our main results by suitable examples. The uniqueness of fixed points is still an open problem to discuss in future.

REFERENCES

[1] R. P. Agarwal, M.A. El-Gebeily, D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol 87, no. 1, pp. 109–116, 2008.

[2] G. Bachman, "Introduction to p-Adic Numbers and Valuation Theory," *Academic Press*, New York, 1964.

[3] A. P. Baisnab, P. Das, S. Pal, "Fixed points of operators satisfying various contractive conditions in complete partial metric spaces," *Fixed Point Theory*, vol. 20, no. 2, pp. 451–468, 2019.

[4] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.

[5] T. G. Bhaskar, V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[6] I. Cabrera, J. Harjani, K. Sadarangani, "A fixed point theorem for contractions of rational type in partially ordered metric spaces," *Annali Dell'Universita'Di Ferrara*, vol. 59, pp. 251–258, 2013.

[7] B. K. Dass, S. Gupta, "An extension of Banach contraction principle through rational expression," *Indian Journal of Pure and Applied Mathematics*, vol. 6, no. 12, pp. 1455–1458, 1975.

[8] L. Gajic, "On ultrametric space," *Novi Sad Journal of Mathematics*, vol. 31, no. 2, pp. 69-71, 2001.

[9] J. Harjani, K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 7-8, pp. 3403–3410, 2009.

[10] J. Harjani, B. L´opez, K. Sadarangani, "A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space," *Abstract and Applied Analysis*, Article ID 190701, 8 pages, 2010.

[11] K. Hensel, "Über eine neue Begründung der Theorie der algebraischen Zahlen," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 6, pp. 83-88, 1897.

[12] D. S. Jaggi, "Some unique fixed point theorems," *Indian Journal of Pure and Applied Mathematics*, vol. 8, no. 2, pp. 223–230, 1977.

[13] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.

[14] W. A. Kirk, N. Shahzad, "Some fixed point results in ultrametric spaces," *Topology and its Applications*, vol. 159, no.15, pp. 3327-3334, 2012.

[15] H. Mamghadert, H. Parvaneh Masiha, M. Hosseini, "Some fixed point theorems for single valued strongly contractive mapping in partially ordered ultrametric space and non-Archimedean normed spaces," *Turkish Journal of Mathematics*, vol. 41, no. 1, pp. 9-14, 2017.

[16] S. N. Mishra, R. Pant, "Generalization of some fixed point theorems in ultrametric spaces," *Advanced Fixed Point Theory*, vol. 4, no. 1, pp. 41-47, 2013.

[17] H. K. Nashine, B. Samet, "Fixed point results for mappings satisfying (ϕ, ψ)-weakly contractive condition in partially ordered metric space," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 6, pp. 2201–2209, 2011.

[18] R. P. Pant, "Common fixed points of four mappings," *Bulletin of the Calcutta Mathematical Society*, vol. 90, no. 4, pp. 281–286, 1998.

[19] P. Kumam, F. Rouzkard, M. Imdad, D. Gopal, "Fixed point theorems on ordered metric spaces through a rational contraction," *Abstract and Applied Analysis*, Article ID 206515, 9 pages, 2013.

[20] P. Worapun, A. Kangtunyakarn, "An Approximation Method for Solving Fixed Points of General System of Variational Inequalities with Convergence Theorem and Application," *IAENG International Journal of Applied Mathematics*, vol. 51, no.3, pp. 751–756, 2021.

[21] S. Priess-Crampe, P. Ribenboim, "Generalized ultrametric spaces I," *Abhandlungen ausdem Mathematischen Seminar der Universität Hamburg*, vol. 66, pp. 55–73, 1996.

[22] S. Priess-Crampe, P. Ribenboim, "Generalized ultrametric spaces II," *Abhandlungen ausdem Mathematischen Seminar der Universität Hamburg*, vol. 67, no. 1, pp. 19–31, 1997.

[23] D. Ramesh Kumar, M. Pitchaimani, "Set-valued contraction mappings of Presic-Reich type in ultrametric spaces," *Asian-European Journal of Mathematics*, vol. 10, no. 4, Article ID 1750065, 2017.

[24] D. Ramesh Kumar, M. Pitchaimani, "A generalization of set-valued Presic-Reich type contractions in ultrametric spaces with applications," *Journal of Fixed Point Theory and Applications* , vol. 19, pp. 1871–1887, 2017.

[25] K. P. R. Rao, G.N.V. Kishore, T.R. Rao, "Some coincidence point theorems in ultrametric spaces," *International Journal of Mathematical Analysis*, vol. 1, no. 18, pp. 897-902, 2007.

[26] A. C. M Van Rooij, "Non Archimedean Functional Analysis," *Marcel Dekker*, New York, no. 51, 1978.

[27] A. Ran, M. Reurings, "A fixed point theorem in partially ordered sets and some application to matrix equations," *Proceedings of the American Mathematical Society*, vol 132, no. 5, pp. 1435–1443, 2004.

[28] J. J. Nieto, R. R. López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol 22, no. 3, pp. 223–239, 2005.

[29] X. Gao, L. Chen, "Existence of Solutions for a System of Coupled Hybrid Fractional Integro-differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 52, no.1, pp. 149–154, 2022.