Almost Surely and Pth Moment Exponential Stability of Nonlinear Stochastic Delayed Systems Driven by G-Brownian Motion

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***Abstract*—This article is devoted to analyze the stability of nonlinear stochastic delayed systems driven by G-Brownian motion. Firstly, we study the existence of global unique solution. Secondly, by using G-Itô formula, G-Lyapunov function, Gronwall's inequality and Borel-Cantelli lemma, we discuss the pth moment exponential stability and almost surely exponential stability of stochastic delayed systems. Finally, we provide an example to verify the results.**

***Index Terms*—Nonlinear stochastic delayed systems; pth moment exponential stability; almost surely exponential stability; G-Brownian motion; existence and uniqueness**

I. Introduction

Most systems do not satisfy the principle of linear superposition. Thence, except for a small part that can be approximately regarded as linear systems, most of them are nonlinear systems, such as simple pendulum systems ([12]), gravitational three-body systems ([20]) and turbulent system of fluids ([6]). The nonlinear system is the essence and the linear system is the approximation or part of the nonlinear system. Therefore, it is necessary to discuss the properties of nonlinear systems. The research of nonlinear system has always been a hot issue in the field of control. For example, Liu et al. ([9]) used an improved dwell time technique to establish the framework for nonlinear Markovian switched systems. Sun et al. ([19]) studied finite time feedback control problem for nonlinear systems. Liu et al. ([10]) used the adaptive control means to analyze stochastic feedback Markovian switched system. Generally, because of uncertain communication environment, the time delay is always unavoidable. Therefore, it is required to be taken into consideration for stochastic systems. Chen et al. ([3]) provided two different types of mean square exponential stability analysis methods for stochastic systems with aperiodic sampling and multiple time-delays. Feng et al. ([5]) investigated the exponential stability for highly nonlinear hybrid neutral stochastic systems with time varying delays by the novel approach of multiple degenerate functionals. Plonis et al. ([17]) presented the procedure of synthesis of the meander delay system using the Pareto-optimal multilayer perceptron network and multiple linear regression model with the M5 descriptor. Zhao and Zhu ([24]) discussed the existence and boundedness of unique global solution for highly nonlinear switched stochastic systems with time delays.

In recent years, Peng pioneered the concept of G-expectation and established a corresponding theoretical system ([13]–[15]). This topic has attracted wide attention. For example, Chen and Yang ([4]) analyzed time-varying delay Hopfield neural networks. By applying aperiodically intermittent adaptive control, Li et al. ([8]) discussed the stabilisation for stochastic complicated systems. By using feedback control, Ren et al. ([18]) studied stability for discrete-time stochastic differential equations. Sun et al. ([19]) discussed stability for delay impulsive stochastic C-G neural networks.

The nonlinear characteristic of the systems make the performance of the systems more complicated, which brings difficulties to the analysis of stability of systems. Stability has always been the most fundamental and core issue in system analysis. In recent years, lots of results about stability has been reported in the literature ([1], [7], [21], [22]). For example, by applying Lyapunov techniques, Caraballo et al. ([2]) analyzed the stability of stochastic perturbed singular systems. Ngoc ([11]) used a new method to analyze stability of delay stochastic system. Zhang et al. ([23]) discussed stability for time-varying stochastic system driven by multiplicative noise. Since G-Brownian motion has been widely used in uncertainty problems and switching systems are an important class of hybrid dynamic systems, it is necessary to consider these factors. In this article, the existence of global unique solution for the nonlinear stochastic delayed systems is proved. The pth moment exponential stability and almost surely exponential stability of system are investigated with the help of G-Itô formula, Borel-Cantelli lemma, Gronwall's inequality, Hölder inequality and Chebyshev inequality.

The rest of this paper is organized as follows. The system, some definitions and assumptions are introduced in Section 2. The existence of global unique solution is derived in Section 3. Moreover, the pth moment exponential stability and almost surely exponential stability of the system are studied as well. We provide an example in Section 4. We make the conclusion and give some future works in Section 5.

II. Problem Formulation and Preliminaries

Denote $(\{\mathcal{F}_t\}_{t\geq 0})$ is a filtration generated by G-Brownian motion $\{B(t), t \geq 0\}$, $\mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$ is the family of $V(x,y,t) > 0$. Define $M_G^{p,0}([0,t],\mathbb{R}^n,\mathbb{S}) = \{\alpha_t(\omega) = \Sigma_{j=1}^{N-1}\gamma_{ij}\beta_{t_j}(\omega)I_{[t_j,t_{j+1})};\beta_{t_j} \in L^p_{F_t}(\Omega;\mathbb{R}^n), t > 0\}$, $M_G^p([0,t],\mathbb{R}^n,\mathbb{S})$:= the completion of $M_G^{p,0}([0,t],\mathbb{R}^n,\mathbb{S})$

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under the norm $\|\alpha\|_{M_G^p([0,t],\mathbb{R}^n,\mathbb{S})} = (\int_0^t \widehat{\mathbb{E}}|\alpha_s|^p ds)^{\frac{1}{p}}$ where $L^p_{F_t}(\Omega;\mathbb{R}^n :=$ the family of all F_t measurable $\mathbb{R}^n$-valued stochastic variables β satisfies $\widehat{\mathbb{E}}|\beta|^p < \infty$.

The nonlinear stochastic delayed systems driven by G-Brownian motion is introduced as follows:

$$\begin{aligned} dx(t) \quad &= f(x(t), x(t-\tau(t)), t)dt \\ &+g(x(t), x(t-\tau(t)), t)d<B>(t) \\ &+h(x(t), x(t-\tau(t)), t)dB(t), \end{aligned} \tag{1}$$

where $0 \le \tau(t) \le \tau$, the nonrandom initial data $\{x(t) = \xi(t) : -\tau \le t \le 0\} = \xi \in \mathcal{C}([-\tau, 0];\mathbb{R}^n)$, $B(t)$ is a one dimensional G-Brownian motion with $G(a) := \frac{1}{2}\widehat{\mathbb{E}}[aB^2(1)] = \frac{1}{2}(\overline{\sigma}^2 a^+ + \underline{\sigma} a^-)$, for $a \in \mathbb{R}$, where $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, $\overline{\sigma}^2 = \widehat{\mathbb{E}}[B^2(1)]$, $\underline{\sigma}^2 = -\widehat{\mathbb{E}}[-B^2(1)]$, $<B>(t)$ is the quadratic variation process of $B(t)$, $\widehat{\mathbb{E}}$ stands for the G-expectation. The mapping rules of f, g and h are: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$.

Firstly, we give some assumptions, definitions and lemmas.

Assumption 1: $\exists L_K > 0$, for $\forall t \ge 0$, $|x| \vee |x_1| \vee |y| \vee |y_1| \le K$,

$$\begin{aligned} &|f(x,y,t) - f(x_1,y_1,t)| \vee |g(x,y,t) - g(x_1,y_1,t)| \\ &\vee |h(x,y,t) - h(x_1,y_1,t)| \le L_K(|x - x_1| + |y - y_1|). \end{aligned}$$

Assumption 2:

$$f(0,t,i) \equiv 0, \quad g(0,t,i) \equiv 0, \quad h(0,t,i) \equiv 0.$$

Assumption 3:

$$\lim_{|x|\to\infty} \inf_{t\ge 0} V(x,y,t) = \infty,$$

$$\mathcal{L}V(x,y,t) \le -a_3 V(x,y,t),$$

where $V(x,y,t) \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$, $a_3 > 0$

Definition 1: The system (1) is said to be pth moment exponentially stable if there is a pair of positive constants λ and C such that

$$|x(t;t_0,x_0|^p \le C|x_0|^p e^{-\lambda(t-t_0)}, \quad t \ge t_0,$$

for all $x_0 \in \mathbb{R}^n$. When $p = 2$, it is said to be exponentially stable in mean square.

Definition 2: The system (1) is said to be almost sure exponentially stable if

$$\lim_{t\to\infty} \sup \frac{1}{t}\log(|x(t;t_0,x_0)|) < 0, a.s.$$

for all $x_0 \in \mathbb{R}^n$.

Given $V \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$. The expression of operator $\mathcal{L}V$ is as follows:

$$\begin{aligned} \mathcal{L}V(x,y,t) \;&=\; V_t(x,y,t) + V_x(x,y,t)f(x,y,t) \\ &+\; G(2V_x(x,y,t)g(x,y,t) \\ &\quad +h^T(x,y,t)V_{xx}(x,y,t)h(x,y,t)). \end{aligned}$$

Lemma 1: ([14]) (G-Itô formula): Let $\varphi \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$ and

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t g_s d<B>_s + \int_0^t h_s dB_s,$$

where $f, g, h \in M_G^2(0,T;\mathbb{R}^n)$. Then, for $\forall t > 0$,

$$\begin{aligned} &\varphi(X_t,t) - \varphi(X_0,t) \\ &= \int_0^t [\partial_s\varphi(X_s,s) + \partial_x\varphi(X_s,s)f_s \\ &+G(2\partial_x\varphi(X_s,s)g_s) + \partial_{xx}\varphi(X_s,s)h_s^2]ds \\ &+ \int_0^t \partial_x\varphi(X_s,s)h_s dB_s + \int_0^t [\partial_x\varphi(X_s,s)g_s \\ &+\frac{1}{2}\partial_{xx}\varphi(X_s,s)h_s^2]d<B>_s \\ &- \int_0^t G(2\partial_x\varphi(X_s,s)g_s + \partial_{xx}\varphi(X_s,s)h_s^2)ds. \end{aligned}$$

III. Main Results and Proofs

Theorem 1: When Assumptions 1-3 hold, the global unique solution of system (1) exists.

Proof: Let the initial value $|x_0| \le \xi$. For $m \ge \xi$, $m \in \mathbb{N}$, define the truncation function

$$f_m(x,y,t) = \begin{cases} f(x,y,t) & if|x|,|y| \le m, \\ f(\frac{mx}{|x|}, \frac{my}{|y|}, t) & if|x|,|y| > m, \end{cases} \tag{2}$$

$$g_m(x,y,t) = \begin{cases} g(x,y,t) & if|x|,|y| \le m, \\ g(\frac{mx}{|x|}, \frac{my}{|y|}, t) & if|x|,|y| > m. \end{cases} \tag{3}$$

$$h_m(x,y,t) = \begin{cases} h(x,y,t) & if|x|,|y| \le m, \\ h(\frac{mx}{|x|}, \frac{my}{|y|}, t) & if|x|,|y| > m. \end{cases} \tag{4}$$

It can be checked that $f^{(m)}$, $g^{(m)}$ and $h^{(m)}$ fulfill the linear growth condition and Lipschitz condition. Thus,

$$\begin{aligned} dx_m(t) \quad &= f_m(x_m(t), x_m(t-\tau(t)), t)dt \\ &+g_m(x_m(t), x_m(t-\tau(t)), t)d<B>(t) \\ &+h_m(x_m(t), x_m(t-\tau(t)), t)dB(t), \end{aligned} \tag{5}$$

has the global unique solution.

Let

$$\eta_m = \inf\{t \ge 0 : |x_m(t)| \ge m\}, \tag{6}$$

where $\inf \phi = \infty$.

When $0 \le t \le \eta_m$, $x_m(t) = x_{m+1}$. Then, $\{\eta_m\}$ is an increasing sequence. Thus, $\exists\eta$ satisfies

$$\eta = \lim_{m\to\infty} \eta_m. \tag{7}$$

Let

$$x(t) = \lim_{m\to\infty} x_m(t), \quad 0 \le t < \eta. \tag{8}$$

We get that $x(t)$ is unique.

With the help of G-Itô formula, when $t \geq 0$, we obtain

$$\begin{aligned}
&V(x_m(t\wedge\eta_m), x_m(t\wedge\eta_m-\tau(t\wedge\eta_m)), t\wedge\eta_m)\\
&= V(\xi(0), x_m(-\tau(0)), 0)\\
&+ \int_0^{t\wedge\eta_m} \mathcal{L}_m V(x_m(s), x_m(s-\tau(s)), s)ds\\
&+ \int_0^{t\wedge\eta_m} V_x(x_m(s), x_m(s-\tau(s)), s)\\
&h(x_m(s), x_m(s-\tau(s)), s)dB(s)\\
&+ \int_0^{t\wedge\eta_m} [V_x(x_m(s), x_m(s-\tau(s)), s)\\
&g(x_m(s), x_m(s-\tau(s)), s)\\
&+\frac{1}{2}h^T(x_m(s), x_m(s-\tau(s)), s)\\
&V_{xx}(x_m(s), x_m(s-\tau(s)), s)\\
&h(x_m(s), x_m(s-\tau(s)), s)]d<B>(s)\\
&- \int_0^{t\wedge\eta_m} G(2V_x(x_m(s), x_m(s-\tau(s)), s)\\
&g(x_m(s), x_m(s-\tau(s)), s)\\
&+h^T(x_m(s), x_m(s-\tau(s)), s)\\
&V_{xx}(x_m(s), x_m(s-\tau(s)), s)\\
&h(x_m(s), x_m(s-\tau(s)), s)ds,
\end{aligned}$$

where $\mathcal{L}_m V(x_m(s), x_m(s-\tau(s)), s) = \mathcal{L}V(x_m(s), x_m(s-\tau(s)), s)$ when $0 \leq s \leq t\wedge\eta_m$.

According to ([16]),

$$\begin{aligned}
&\widehat{\mathbb{E}}[\int_0^{t\wedge\eta_m} V_x(x_m(s), x_m(s-\tau(s)), s)\\
&h(x_m(s), x_m(s-\tau(s)), s)dB(s)]\\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{\mathbb{E}}[\int_0^{t\wedge\eta_m} [V_x(x_m(s), x_m(s-\tau(s)), s)\\
&g(x_m(s), x_m(s-\tau(s)), s)\\
&+\frac{1}{2}h^T(x_m(s), x_m(s-\tau(s)), s)\\
&V_{xx}(x_m(s), x_m(s-\tau(s)), s)\\
&h(x_m(s), x_m(s-\tau(s)), s)]d<B>(s)\\
&- \int_0^{t\wedge\eta_m} G(2V_x(x_m(s), x_m(s-\tau(s)), s)\\
&g(x_m(s), x_m(s-\tau(s)), s)\\
&+h^T(x_m(s), x_m(s-\tau(s)), s)\\
&V_{xx}(x_m(s), x_m(s-\tau(s)), s)\\
&h(x_m(s), x_m(s-\tau(s)), s)ds]\\
&\leq 0.
\end{aligned}$$

Then,

$$\begin{aligned}
&\widehat{\mathbb{E}}[V(x_m(t\wedge\eta_m), x_m(t\wedge\eta_m-\tau(t\wedge\eta_m)), t\wedge\eta_m)]\\
&\leq \widehat{\mathbb{E}}[V(\xi(0), x_m(-\tau(0)), 0)]\\
&+\widehat{\mathbb{E}}[\int_0^{t\wedge\eta_m} \mathcal{L}_m V(x_m(s), x_m(s-\tau(s)), s)ds]\\
&\leq \widehat{\mathbb{E}}[V(\xi(0), x_m(-\tau(0)), 0)]\\
&+ \int_0^{t\wedge\eta_m} \mathbb{E}[V(x_m(s), x_m(s-\tau(s)), s)]ds.
\end{aligned}$$

According to the Gronwall's inequality, we get

$$\begin{aligned}
&\widehat{\mathbb{E}}[V(x_m(t\wedge\eta_m), x_m(t\wedge\eta_m-\tau(t\wedge\eta_m)), t\wedge\eta_m)]\\
&\leq \widehat{\mathbb{E}}[V(\xi(0), x_m(-\tau(0)), 0)]e^{t\wedge\eta_m}.
\end{aligned}$$

Furthermore, as

$$\begin{aligned}
&\mathbb{P}\{\eta_m \leq t\} \inf_{|x|,|y|\geq m, t\geq 0} V(x,y,t)\\
&\leq \int_{\eta_m\leq t}\\
&V(x_m(t\wedge\eta_m), x_m(t\wedge\eta_m-\tau(t\wedge\eta_m)), t\wedge\eta_m)dP\\
&\leq \widehat{\mathbb{E}}V(x_m(t\wedge\eta_m), x_m(t\wedge\eta_m-\tau(t\wedge\eta_m)), t\wedge\eta_m),
\end{aligned}$$

we have

$$\mathbb{P}\{\eta_m \leq t\} \leq \frac{\widehat{\mathbb{E}}[V(\xi(0), x_m(-\tau(0)), 0)]e^{t\wedge\eta_m}}{\inf_{|x|,|y|\geq m, t\geq 0} V(x,y,t)}. \quad (9)$$

When $t \to \infty$,

$$\mathbb{P}\{\eta \leq t\} = 0. \quad (10)$$

Thus,

$$\mathbb{P}\{\eta = \infty\} = 1. \quad (11)$$

■

Theorem 2: For $\forall(x,y,t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+)$, If $\exists V(x,y,t) > 0$, $a_1, a_2, \lambda > 0$ satisfy

$$a_1|x|^p \leq V(x,y,t) \leq a_2|x|^p, \quad (12)$$

$$\mathcal{L}V(x,y,t) \leq -\lambda V(x,y,t), \quad (13)$$

the system (1) is almost sure exponentially stable.

Proof: By using G-Itô formula, for $t > 0$, $\lambda > 0$, we get

$$\begin{aligned}
&e^{\lambda t}V(x(t), x(t-\tau(t)), t) - V(x_0, x(-\tau(0), 0)\\
&= \int_0^t [\lambda e^{\lambda s} V(x(s), x(s-\tau(s)), s)\\
&+e^{\lambda s}V_s(x(s), x(s-\tau(s)), s)\\
&+e^{\lambda s}V_x(x(s), x(s-\tau(s)), s)f(x(s), x(s-\tau(s)), s)\\
&+G(2e^{\lambda s}V_x(x(s), x(s-\tau(s)), s)\\
&g(x(s), x(s-\tau(s)), s)\\
&+e^{\lambda s}h^T(x(s), x(s-\tau(s)), s)\\
&V_{xx}(x(s), x(s-\tau(s)), s)\\
&h(x(s), x(s-\tau(s)), s))]ds\\
&+ \int_0^t e^{\lambda s}V_x(x(s), x(s-\tau(s)), s)\\
&h(x(s), x(s-\tau(s)), s)dB(s)\\
&+ \int_0^t [e^{\lambda s}V_x(x(s), x(s-\tau(s)), s)\\
&g(x(s), x(s-\tau(s)), s)\\
&+\frac{1}{2}e^{\lambda s}h^T(x(s), x(s-\tau(s)), s)\\
&V_{xx}(x(s), x(s-\tau(s)), s)\\
&h(x(s), x(s-\tau(s)), s)]d<B>(s)\\
&-G(2e^{\lambda s}V_x(x(s), x(s-\tau(s)), s)\\
&g(x(s), x(s-\tau(s)), s)\\
&+e^{\lambda s}h^T(x(s), x(s-\tau(s)), s)V_{xx}(x(s), x(s-\tau(s)), s)\\
&h(x(s), x(s-\tau(s)), s))]ds.
\end{aligned}$$

Thence,

$$\begin{aligned}
&e^{\lambda t}V(x(t),x(t-\tau(t)),t)\\
&=V(x_0,x(-\tau(0),0)+\int_0^t e^{\lambda s}\\
&[\lambda V(x(s),x(s-\tau(s)),s)\\
&+\mathcal{L}V(x(s),x(s-\tau(s)),s)]ds\\
&+\int_0^t e^{\lambda s}V_x(x(s),x(s-\tau(s)),s)\\
&h(x(s),x(s-\tau(s)),s)dB(s)\\
&+\int_0^t [e^{\lambda s}V_x(x(s),x(s-\tau(s)),s)\\
&g(x(s),x(s-\tau(s)),s)\\
&+\frac{1}{2}e^{\lambda s}h^T(x(s),x(s-\tau(s)),s)\\
&V_{xx}(x(s),x(s-\tau(s)),s)\\
&h(x(s),x(s-\tau(s)),s)]d<B>(s)\\
&-G(2e^{\lambda s}V_x(x(s),x(s-\tau(s)),s)\\
&g(x(s),x(s-\tau(s)),s)\\
&+e^{\lambda s}h^T(x(s),x(s-\tau(s)),s)\\
&V_{xx}(x(s),x(s-\tau(s)),s)\\
&h(x(s),x(s-\tau(s)),s))]ds.
\end{aligned}$$

According to ([16]), it can be checked that

$$\begin{aligned}
&\widehat{\mathbb{E}}[\int_0^t e^{\lambda s}V_x(x(s),x(s-\tau(s)),s)\\
&h(x(s),x(s-\tau(s)),s)dB(s)]\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\widehat{\mathbb{E}}[\int_0^t [e^{\lambda s}V_x(x(s),x(s-\tau(s)),s)\\
&g(x(s),x(s-\tau(s)),s)\\
&+\frac{1}{2}e^{\lambda s}h^T(x(s),x(s-\tau(s)),s)\\
&V_{xx}(x(s),x(s-\tau(s)),s)\\
&h(x(s),x(s-\tau(s)),s)]d<B>(s)\\
&-G(2e^{\lambda s}V_x(x(s),x(s-\tau(s)),s)\\
&g(x(s),x(s-\tau(s)),s)\\
&+e^{\lambda s}h^T(x(s),x(s-\tau(s)),s)\\
&V_{xx}(x(s),x(s-\tau(s)),s)\\
&h(x(s),x(s-\tau(s)),s))]\\
&\le 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\widehat{\mathbb{E}}[e^{\lambda t}V(x(t),x(t-\tau(t)),t)]\\
&\le \widehat{\mathbb{E}}[V(x_0,x(-\tau(0),0)]\\
&+\mathbb{E}\int_0^t e^{\lambda s}[\lambda V(x(s),x(s-\tau(s)),s)\\
&+\mathcal{L}V(x(s),x(s-\tau(s)),s)]ds\\
&\le a_2|x_0|^p.
\end{aligned} \tag{14}$$

Then,

$$\widehat{\mathbb{E}}[V(x(t),x(t-\tau(t)),t)]\le e^{-\lambda t}a_2|x_0|^p. \tag{15}$$

Hence,

$$a_1\widehat{\mathbb{E}}|x(t)|^p\le e^{-\lambda t}a_2|x_0|^p, \tag{16}$$

which implies that

$$\widehat{\mathbb{E}}|x(t)|^p\le \frac{a_2}{a_1}e^{-\lambda t}|x_0|^p. \tag{17}$$

Therefore, the system (1) is pth moment exponentially stable.

The proof is complete. ■

Corollary 1: Under the conditions in Theorem 2, the system (1) is almost sure exponentially stable.

Proof: According to Theorem 2, when $p=2$, we obtain that

$$\mathbb{E}|x(t)|^2\le \frac{a_2}{a_1}e^{-\lambda t}|x_0|^2. \tag{18}$$

By applying the elementary inequality

$$|a+b+c|^2\le 3(|a|^2+|b|^2+|c|^2),$$

we have

$$\begin{aligned}
&|x(t)|^2=|x_0+\int_0^t f(x(s),x(s-\tau(s)),s)ds\\
&+\int_0^t g(x(s),x(s-\tau(s)),s)d<B>(s)\\
&+\int_0^t h(x(s),x(s-\tau(s)),s)dB(s)|^2\\
&\le 4|x_0|^2+4|\int_0^t f(x(s),x(s-\tau(s)),s)ds|^2\\
&+4|\int_0^t g(x(s),x(s-\tau(s)),s)d<B>(s)|^2\\
&+4|\int_0^t h(x(s),x(s-\tau(s)),s)dB(s)|^2.
\end{aligned}$$

For $\forall q>0$ satisfies $q^2L_K^2(1+5\overline{\sigma}^4 q)<\frac{1}{4}$ and positive integer κ_0, let $\kappa=\kappa_0,\kappa_0+1,\kappa_0+2,\cdots$. By using the

Hölder inequality, we get

$$
\begin{aligned}
&\widehat{\mathbb{E}}[\sup_{\kappa q\leq t\leq(\kappa+1)q}|x(t)|^2]\leq 4\widehat{\mathbb{E}}[|x(\kappa q)|^2]\\
&+4\widehat{\mathbb{E}}(\int_{\kappa q}^{(\kappa+1)q}|f(x(s),s,r(s))|ds)^2\\
&+4\widehat{\mathbb{E}}|\int_0^t g(x(s),x(s-\tau(s)),s)d<B>(s)|^2\\
&+4\widehat{\mathbb{E}}|\int_0^t h(x(s),x(s-\tau(s)),s))dB(s)|^2\\
&\leq 4\widehat{\mathbb{E}}[|x(\kappa q)|^2]\\
&+4\widehat{\mathbb{E}}(q\sup_{\kappa q\leq s\leq(\kappa+1)q}|f(x(s),x(s-\tau(s)),s)|ds)^2\\
&+4\widehat{\mathbb{E}}[\sup_{\kappa q\leq s\leq(\kappa+1)q}|\int_{\kappa q}^{(\kappa+1)q}\\
&g(x(s),x(s-\tau(s)),s)d<B>(s)|^2]\\
&+4\widehat{\mathbb{E}}[\sup_{\kappa q\leq s\leq(\kappa+1)q}|\int_{\kappa q}^{(\kappa+1)q}\\
&h(x(s),x(s-\tau(s)),s)dB(s)|^2]\\
&\leq 4\widehat{\mathbb{E}}[|x(\kappa q)|^2]+4q^2L_K^2\widehat{\mathbb{E}}[\sup_{\kappa q\leq s\leq(\kappa+1)q}|x(s)|^2]\\
&+16\overline{\sigma}^4q^3L_K^2\widehat{\mathbb{E}}[\sup_{\kappa q\leq s\leq(\kappa+1)q}|x(s)|^2]\\
&+4\overline{\sigma}^4q^3L_K^2\widehat{\mathbb{E}}[\sup_{\kappa q\leq s\leq(\kappa+1)q}|x(s)|^2]\\
&\leq 4\frac{a_2}{a_1}|x_0|^2e^{-\lambda\kappa q}\\
&+4q^2L_K^2(1+5\overline{\sigma}^4q)\widehat{\mathbb{E}}[\sup_{\kappa q\leq s\leq(\kappa+1)q}|x(s)|^2].
\end{aligned}
$$

Then,

$$\widehat{\mathbb{E}}[\sup_{\kappa q\leq t\leq(\kappa+1)q}|x(t)|^2]\leq\frac{4\frac{a_2}{a_1}|x_0|^2e^{-\lambda\kappa q}}{1-4q^2L_K^2(1+5\overline{\sigma}^4q)}. \quad (19)$$

Thus,

$$
\begin{aligned}
&\mathbb{P}(\omega:\sup_{\kappa q\leq t\leq(\kappa+1)q}|x(t)|>e^{\frac{-\lambda\kappa q}{2}})\\
&\leq\frac{\widehat{\mathbb{E}}[\sup_{\kappa q\leq t\leq(\kappa+1)q}|x(t)|^2]}{e^{-\lambda\kappa q}}\\
&\leq\frac{4\frac{a_2}{a_1}|x_0|^2}{1-4q^2L_K^2(1+5\overline{\sigma}^4q)}.
\end{aligned}
$$

From the Borel-Cantelli lemma, we get

$$\sup_{\kappa q\leq t\leq(\kappa+1)q}|x(t)|\leq e^{\frac{-\lambda\kappa q}{2}}. \quad (20)$$

Therefore, for $\kappa q\leq t\leq(\kappa+1)q$,

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t)|)<\frac{-\lambda}{2}<0. \quad (21)$$

The proof is complete. ∎

IV. EXAMPLE

Let G-Brownian motion $B(t)\sim N(0,[\underline{\sigma}^2,\overline{\sigma}^2])$.

The nonlinear stochastic delayed systems driven by G-Brownian motion is as follows:

$$
\begin{aligned}
dx(t)\quad &=f(x(t),x(t-\tau(t)),t)dt\\
&+g(x(t),x(t-\tau(t)),t)d<B>(t)\\
&+h(x(t),x(t-\tau(t)),t)dB(t),
\end{aligned}
$$

where

$$f(x(t),x(t-\tau(t)),t)=-6x(t)+x(t-\tau(t)),$$

$$g(x(t),x(t-\tau(t)),t)=\frac{1}{2}x(t)$$

$$h(x(t),x(t-\tau(t)),t)=x(t),$$

$$\tau(t)=1+0.2\sin(t).$$

Hence, $\tau=1.2$. Let $V(x,y,t)=x^2$. According to G-Itô formula, we have

$$\mathcal{L}V(x,y,t)\leq-10x^2+4\overline{\sigma}^2x^2.$$

Let $\overline{\sigma}^2=1$, we obtain that

$$\mathcal{L}V(x,t,1)\leq-6x^2=-6V(x,y,t).$$

Then, the system are pth moment exponential stability and almost surely exponential stability.

V. CONCLUSION

In this article, we have analyzed the pth moment exponential stability and almost sure exponential stability for nonlinear stochastic delayed systems driven by G-Brownian motion. Compared with the previous literature, our results and methods are different from them. We have proved the existence of global unique solution and provided sufficient conditions for the stability based on G-Itô formula, G-Lyapunov function, Gronwall's inequality and Borel-Cantelli lemma. We will consider the stability for delay fractional system in the future.

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