Determination of the Unknown Boundary Conditions of the Laplace Equation via Regularization B-spline Wavelet approach

Xinming Zhang, Kaiqi Wang

***ABSTRACT*—The unknown boundary condition identification problems of the 2-D Laplace equation are considered in this paper. Based on the good characteristics of the B-spline wavelet and Tikhonov regularization method (TRM), a new regularization B-spline wavelet method (RBPWM) is proposed. The novel algorithm could be regarded as one kind of wavelet mesh-free, non-iterative numerical scheme that converts the boundary condition identification problem into a large-scale algebraic equation system that can be solved in a single step. However, the coefficient matrix of the algebraic equation system is ill-conditioned, which will lead to an unstable solution for the case of higher-level noise. The Tikhonov regularization method (TRM) is used to achieve a steady numerical solution to this problem. The current work of this paper has studied four examples with different simulated noise levels for different boundary conditions. The efficiency and accuracy of the presented algorithm are verified with the numerical simulation.**

***Index Terms*—unknown boundary condition identification; Tikhonov regularization method; regularization B-spline wavelet method; noise levels**

I. INTRODUCTION

As well as known, the unknown boundary condition identification problems of the 2-D Laplace equation can be described as follows:

Governing equation (GE):

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad in \quad \Omega\subset\mathbb{R}^2 \tag{1}$$

Partial boundary conditions (PBC):

$$u=g_1 \quad on \quad \Gamma_i\subset\Gamma \ , \ \frac{\partial u}{\partial n}=g_2 \quad on \quad \Gamma_i\subset\Gamma \tag{2}$$

where Γ is the whole boundary, n is the unit normal vector, g_1 and g_2 represent the relevant Dirichlet boundary conditions (DBC) and Neumann boundary conditions(NBC), respectively. Generally speaking, Γ_i is a segment of the entire boundary Γ . Finding the relevant details on the rest boundary or regaining the boundary's shape is our goal.

The 2-D Laplace equation unknown boundary condition identification problems (LEUBCIP) have important applications in scientific research and engineering. The solution to boundary condition identification problems is, however, a very ill-posed problem, meaning that even a slight change in the given data could cause a significant inaccuracy in the solution. Meanwhile, due to the technical and physical limits, some noise is inevitable in the known data. Therefore, how to solve the above problem with noise data efficiently has attracted more and more attention. This problem can be solved numerically using a variety of methods, including QRM [1,2], BGM[3,4], CGM[5], HFM [6-8] , MCTM[9], BKM[10] , PFM[11], BEM[12,13], MFS[14-16], and RBCM [17-19]. Although all these methods can obtain better numerical results, how to improve the computation accuracy and anti-noise property is still a challenging problem.

Due to its exceptional properties, such as local support and vanishing moment features, wavelet methods [20] have recently been employed to solve the numerical solution of all types of differential equations [21–24]. Solving the boundary condition identification problems of the Laplace equation with wavelet method, the coefficient matrix of the linear algebraic equation systems will be relatively tiny compared to the other approaches, so it can be deleted without significantly influencing the solution if we discretize the differential equation based on the wavelet base functions. Additionally, the mother wavelet functions and scaling functions can be utilized to obtain the basis of wavelets, making it simple to apply computationally on a computer. Furthermore, the B-spline wavelets method among the wavelet families may result in greater accuracy for approximating smooth functions [25]. Because of its explicit formulations and finite support property, it is also simple to implement on a computer. Based on the beneficial characteristics of the B-spline wavelet, a regularization B-spline wavelet method (RBSWM) for resolving LEUBCIP with noisy data is proposed in this study. Tikhonov regularization method is used to solve the ill-conditioned coefficient matrix. The regularization technique's efficiency in resolving ill-posed problems has been proved in a number of publications in the literature [26–29].

The remainder of the study is organized as follows: RBSWM is detailly explained in Section 2 along with an introduction of several B-spline wavelet definitions. A few numerical examples are given in Section 3 to show how

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accurate the suggested method is. In Section 4, a succinct conclusion is provided.

II. REGULARIZATION B-SPLINE WAVELET METHOD (RBSWM)

A. B-Spline Wavelet Method (BSWM)

Definition 1. For $j \in \mathbb{Z}$, let $S^{(j)} := \left\{\varsigma_l^{(j)}\right\}_{l=-p+1}^{2^j+p-1}$ as the knot sequence on the interval $[\alpha,\beta]$ with

$$S^{(j)} = \begin{cases} \varsigma_{-p+1}^{(j)} = \varsigma_{-p+2}^{(j)} \cdots = \varsigma_0^{(j)} = \alpha \\ \varsigma_l^{(j)} = \alpha + \dfrac{l(\beta-\alpha)}{2^j}, \quad l = 1,2,\cdots,2^j-1 \\ \varsigma_{2^j}^{(j)} = \varsigma_{2^j+1}^{(j)} = \cdots = \varsigma_{2^j+p-1}^{(j)} = \beta \end{cases}$$

The p order B-spline functions on $[\alpha,\beta]$ are

$$B_{p,l,S^{(j)}}(x) = \frac{x-\varsigma_l}{\varsigma_{l+p-1}-\varsigma_l} B_{p-1,l,S^{(j)}}(x) + \frac{\varsigma_{l+p}-x}{\varsigma_{l+p}-\varsigma_{l+1}} B_{p-1,l+1,S^{(j)}}(x), \quad l = -p+1,\cdots,2^j-1$$

and

$$B_{1,l,S^{(j)}}(x) = \begin{cases} 1 & x \in [\varsigma_l, \varsigma_{l+1}) \\ 0 & otherwise \end{cases}, \quad l = 0,1,\cdots,2^j-1$$

By confirming that, on $[\alpha,\beta]$, at least one wavelet function is complete, the lowest level $j = j_0$ is established by $2^{j_0} \ge 2p-1$.

In the event where $[\alpha,\beta] \equiv [0,1]$, we obtain

$$S^{(j)} = \begin{cases} \varsigma_{-p+1}^{(j)} = \varsigma_{-p+2}^{(j)} \cdots = \varsigma_0^{(j)} = 0 \\ \varsigma_l^{(j)} = \dfrac{l}{2^j}, \quad l = 1,2,\cdots,2^j-1 \\ \varsigma_{2^j}^{(j)} = \varsigma_{2^j+1}^{(j)} = \cdots = \varsigma_{2^j+p-1}^{(j)} = 1 \end{cases}$$

Definition 2. The p order B-spline wavelet scaling functions (BSWSF) can be defined as follows for $j \ge j_0$

$$\xi_{p,k}^{(j)}(x) = \begin{cases} B_{p,k,S^{(j)}}(2^{j-j_0}x) & l = -p+1,\cdots,-1 \\ B_{p,2^j-p-l,S^{(j)}}(1-2^{j-j_0}x) & l = 2^j-p+1,\cdots,2^j-1 \\ B_{p,0,S^{(j)}}(2^{j-j_0}x - 2^{-j_0}l) & l = 0,\cdots,2^j-1 \end{cases}$$

From multi-resolution analysis [30], restricted to a bounded interval, say $[\alpha,\beta]$, one can get by [31]

$$V_0^{[\alpha,\beta]} \subset V_1^{[\alpha,\beta]} \subset \cdots$$

with $\operatorname{close}_{L^2(\alpha,\beta)}\left(\bigcup_{j\in\mathbb{Z}} V_j^{[\alpha,\beta]}\right) = L^2(\alpha,\beta)$

Thus, the orthogonal projection of $g(x)$ into $V_j^{[\alpha,\beta]}$ [32] could be utilized to define an approximation of a function $g(x) \in L^2(\alpha,\beta)$ at a resolution of 2^{-j}, that is

$$(P_j g)(x) = \sum_{l\in\mathbb{Z}} \langle g, \theta_{j,l} \rangle \theta_{j,l}(x) = \sum_{l\in\mathbb{Z}} c_l \theta_{j,l}(x) \in V_j^{[\alpha,\beta]} \tag{3}$$

where $\theta_{j,l}(x) = 2^{\frac{j}{2}} \theta(2^j x - l)$ $(j,l \in \mathbb{Z})$, θ is referred to as scaling function of the MRA. The inner product of f and $\theta_{j,l}$ is denoted as $\langle f, \theta_{j,l} \rangle$.

The BSWSF in $V_j^{[\alpha,\beta]}$ can define a function $g(x)$ in accordance with Equation (3), namely

$$g(x) \approx \sum_{l=-p+1}^{2^j-1} c_l \xi_{p,l}^{(j)}(x) \tag{4}$$

where $\xi_{p,l}^{(j)}(x)$ are the BSWSF, c_k are the undetermined coefficients in matrix form. We can rewrite Equation (4) as follows:

$$g(x) \approx \sum_{l=-p+1}^{2^j-1} c_l \xi_{p,l}^{(j)}(x) = \mathbf{C}^{\mathrm{T}} \mathbf{\Psi}(x) \tag{5}$$

with $\mathbf{C}$ and $\mathbf{\Psi}$ are $(2^j+p-1)\times 1$ matrices, respectively

$$\mathbf{C} = [c_{-p+1},\cdots,c_{2^j-1}]^{\mathrm{T}} \tag{6}$$

$$\mathbf{\Psi}(x) = [\xi_{p,-p+1}^{(j)}(x),\cdots,\xi_{p,2^j-1}^{(j)}(x)]^{\mathrm{T}} \tag{7}$$

The following Equation (8) can be used to solve the undetermined coefficients c_l

$$c_l = \int_0^1 g(x) \tilde{\xi}_{p,l}^{(j)}(x)dx \quad l = -p+1,\cdots,2^j-1 \tag{8}$$

where $\tilde{\xi}_{p,l}^{(j)}$ is the dual function of $\xi_{p,l}^{(j)}$, that is,

$$\left\langle \tilde{\xi}_{p,l}^{(j)}, \xi_{p,q}^{(j)} \right\rangle = \int_0^1 \tilde{\xi}_{p,l}^{(j)}(x) \xi_{p,q}^{(j)}(x)dx = \delta_{lq}$$

with $\delta_{lq} = \begin{cases} 1, & l = q \\ 0, & l \ne q \end{cases}$

Let $\tilde{\mathbf{\Psi}}(x)$ be the dual function of $\mathbf{\Psi}(x)$, we have

$$\tilde{\mathbf{\Psi}}(x) = [\tilde{\xi}_{p,-p+1}^{(j)}(x),\cdots,\tilde{\xi}_{p,2^j-1}^{(j)}(x)]^{\mathrm{T}} \tag{9}$$

From Equation (7) and Equation(9), we obtain

$$\int_0^1 \mathbf{\Psi}(x)\tilde{\mathbf{\Psi}}^{\mathrm{T}}(x)dx = \mathbf{E} \tag{10}$$

where $\mathbf{E}$ is the $(2^j+p-1)\times(2^j+p-1)$ identity matrix.

Considering Equation (7) and the internal and boundary scaling functions, one obtains:

$$\int_0^1 \mathbf{\Psi}(x)\mathbf{\Psi}^{\mathrm{T}}(x)dx = \mathbf{M} \tag{11}$$

where $\mathbf{M}$ is a matrix of order 2^j+p-1

From Equations (10) and(11), we obtain:

$$\tilde{\mathbf{\Psi}}(x) = \mathbf{M}^{-1}\mathbf{\Psi}(x) \tag{12}$$

Consider $u(x,y)$ to be a function of x and y, two independent variables, satisfying $0 \le x \le 1$ and $0 \le y \le 1$, respectively. In both dimensions, one-dimensional BSWSF are utilized. Thus, we could approximate $u(x,y)$ by $\bar{u}(x,y)$ as follows:

$$u(x,y) \approx \bar{u}(x,y) = \sum_{i=-p+1}^{2^j-1}\sum_{j=-p+1}^{2^j-1} u_{ij}\xi_{p,i}^{(j)}(y)\xi_{p,j}^{(j)}(x) = \mathbf{\Psi}^{\mathrm{T}}(y)\mathbf{U}\mathbf{\Psi}(x) \tag{13}$$

where $\mathbf{U}$ is a matrix of order 2^j+p-1, which elements u_{ij} can be obtained by

$$u_{ij} = \int_0^1\int_0^1 \bar{u}(x,y)\tilde{\xi}_{p,i}^{(j)}(y)\tilde{\xi}_{p,j}^{(j)}(x)dydx \quad i,j = -p+1,\cdots,2^j-1 \tag{14}$$

Suppose $\mathbf{\Psi}'(x)$ is the derivative of $\mathbf{\Psi}(x)$, in that case

$$\mathbf{\Psi}'(x) = [\xi'^{(j)}_{p,-p+1}(x),\cdots,\xi'^{(j)}_{p,2^j-1}(x)]^{\mathrm{T}} \tag{15}$$

Also, $\mathbf{\Psi}'(x)$ can be rewritten as follows:

$$\mathbf{\Psi}'(x) = \mathbf{G}\mathbf{\Psi}(x) \tag{16}$$

where $\mathbf{G}$ is a $(2^j+p-1)\times(2^j+p-1)$ matrix

$$\mathbf{G} = \int_0^1 \mathbf{\Psi}'(x)\tilde{\mathbf{\Psi}}^{\mathrm{T}}(x)dx \tag{17}$$

Then, using the Equations (12) and(17), we can have

$$\mathbf{G} = \mathbf{M}^{-1}\int_0^1 \mathbf{\Psi}'(x)\mathbf{\Psi}^{\mathrm{T}}(x)dx = \mathbf{M}^{-1}\mathbf{Q} \tag{18}$$

where $\mathbf{Q}$ is a matrix of order 2^j+p-1, which can be obtained by Equations (7) and (15).

Using Equations(13) and(16), and the well-known chain rule, we have

$$\frac{\partial u}{\partial x}=\Psi^{\mathrm{T}}(y)\boldsymbol{U}\mathbf{G}\Psi(x),\quad \frac{\partial u}{\partial y}=\Psi^{\mathrm{T}}(y)\mathbf{G}^{\mathrm{T}}\boldsymbol{U}\Psi(x) \tag{19}$$

$$\frac{\partial^2 u}{\partial x^2}=\Psi^{\mathrm{T}}(y)\boldsymbol{U}\mathbf{G}^2\Psi(x),\quad \frac{\partial^2 u}{\partial y^2}=\Psi^{\mathrm{T}}(y)(\mathbf{G}^{\mathrm{T}})^2\boldsymbol{U}\Psi(x) \tag{20}$$

Therefore, we can have a linear equations system in the form of matrix by substituting Equations (13), (19), and (20) into Equations (1) and (2)

$$\mathbf{BU}=\mathbf{c} \tag{21}$$

where the coefficient matrix $\mathbf{B}$ is composed with the B-spline scaling functions on the interior collocation points and specified boundary , and the vector $\mathbf{c}$ can be created based on either internal collocation points or boundary conditions..

Therefore, as a result of solving Equation(21), we can have the coefficients u_{ij} in Equation(13), indicating that $u(x,y)$ for all distributed points have been achieved in domain and along the boundary. Thus, the solutions of 2-D Laplace equation unknown boundary condition identification problems can be obtained.

B. Regularization solution

Because of measurement error, the provided data, f and g, typically contain noise. Instead of, we consider the following equation

$$\mathbf{BU}=\tilde{\mathbf{c}} \tag{22}$$

where $\tilde{\mathbf{c}}$ denotes the noisy data.

$$\tilde{\mathbf{c}}=\mathbf{c}(1+e\cdot rand) \tag{23}$$

where $rand$ is a number produced at random from the range [-1, 1] and e stands for noise level

In order to solve Equation (22) stably, the TRM has been introduced to find the minimum of the following quadratic functional

$$J_\alpha(\mathbf{U})=\|\mathbf{BU}-\tilde{\mathbf{c}}\|_2^2+\alpha\|\mathbf{U}\|_2^2 \tag{24}$$

where $\alpha>0$ is the regularization parameter(RP). A proper RP must be chosen for the Tikhonov regularization process. In this study, we opt to determine the RP using the L-curve method [33].

III. NUMERICAL SIMULATION

We consider four examples of LEUBCIP, in this section, on a rectangular region $[0,1]\times[0,1]$. The first three problems are defined in a domain $\boldsymbol{\Omega}=\{(x,y)\in\mathbb{R}^2: 0\le x\le 1, 0\le y\le 1\}$ with different boundary conditions. The governing equation (GE) is as follows:

$$\nabla^2 u(x,y)=0 \ \text{ in } \boldsymbol{\Omega}\subset\mathbb{R}^2 \tag{25}$$

with the boundary

$$\boldsymbol{\Gamma}_0=\{(x,y)\in\mathbb{R}^2: y=0, 0\le x\le 1\} \tag{26}$$

$$\boldsymbol{\Gamma}_1=\{(x,y)\in\mathbb{R}^2: x=0, 0\le y\le 1\} \tag{27}$$

$$\boldsymbol{\Gamma}_2=\{(x,y)\in\mathbb{R}^2: y=1, 0\le x\le 1\} \tag{28}$$

$$\boldsymbol{\Gamma}_3=\{(x,y)\in\mathbb{R}^2: x=1, 0\le y\le 1\} \tag{29}$$

For convenience, we utilize several symbols to represent the collocation points satisfying various conditions (see Table I for details).

TABLE I
TYPES OF MATCHING POINTS

Symbol	Matching condition
×	Controlling equation
●	Controlling equation, Dirichlet BC
▲	Controlling equation, Newman BC
■	Controlling equation, Dirichlet BC, Newman BC

The cubic B-spline wavelet scaling functions (i.e. $p=4$) have been used and the level j, j_0 are chosen as $j=j_0=3$. Thus, 11 scaling functions exist in V_3.

A. Problem with over-determined boundary conditions

A 2-D Laplace equation with over-determined boundary conditions is considered firstly. The analytical solution is:

$$u(x,y)=\cos x\cosh y \tag{30}$$

Based on the analytical solution, two kinds of boundary conditions, Dirichlet boundary condition (DBC) and Newman boundary condition (NBC), are prescribed on the boundary $\boldsymbol{\Gamma}_4=\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_3$, respectively.

$$u|_{\Gamma_4}=u_0,\ \frac{\partial u}{\partial n}|_{\Gamma_4}=g \tag{31}$$

We calculate Equation (31) to obtain the following specific boundary conditions (BC)：

$$\boldsymbol{\Gamma}_1\text{:}\ \ u(0,y)=\cosh y,\quad \frac{\partial u(0,y)}{\partial x}=0$$

$$\boldsymbol{\Gamma}_2\text{:}\ \ u(x,1)=\cos x\cosh 1,\quad \frac{\partial u(x,1)}{\partial y}=\cos x\sinh 1$$

$$\boldsymbol{\Gamma}_3\text{:}\ \ u(1,y)=\cos 1\cosh y,\quad \frac{\partial u(1,y)}{\partial x}=-\sin 1\cosh y$$

We need to use RBSWM to find the unknown boundary conditions on the boundary $\boldsymbol{\Gamma}_0=\{(x,y)\in\mathbb{R}^2\text{:}\ y=0, 0\le x\le 1\}$, which exact values can be obtained with $u(x,0)=\cos x$.

To obtain the linear equations system

$$\mathbf{BU}=\mathbf{c} \tag{32}$$

$n1$ points are chosen to satisfy Equation (25), $n2$ points satisfy Equations (27)- (29), and $n1+n2=121$.

By solving the Equation(32), we can obtain the unknown boundary conditions based on the Equation(13). However, the coefficient matrix $\mathbf{B}$ in Equation (32) is usually an ill-conditioned matrix, and here its condition number is $4.0189\text{e}+03$, which will lead to an unstable solution especially for stronger noise. The double boundary collocation method (DBCM) [34] has been devised to generate an over-determined system in order to solve Equation(32). In order to use DBCM, the collocation points on the boundary must fulfill both GE and BC. The new coefficients matrix is thus well-conditioned. The solution of the linear equations system is further stabilized by using TRM. Table II and Fig.1 show the RP for various noise levels.

The collocation points in this study are distributed on an 11×11 grid, and the distribution is depicted in Fig.2. The DBC of Γ_4 is subjected to the imposed simulated noise, which level range from 0 to 0.5. Fig.3 demonstrates the inversion results by using BSWM and RBSWM for various noise levels. BSWM works well for the situation of $e < 0.01$, notably for $e = 0$, as illustrated in Fig.3 (a). However, in the case of $e \geq 0.1$, we can conclude that from Fig.3 (b), RBSWM has more advantages than the method without regularization.

TABLE II
RP SETTING FOR VARIOUS NOISE

Noise	0	0.001	0.01	0.1	0.5
α	0.007081	0.007081	0.007081	0.007082	0.007145

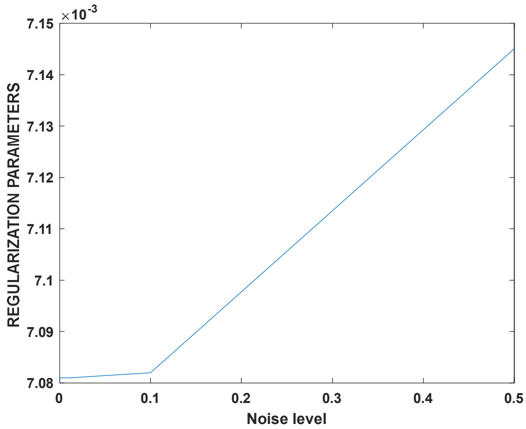


Fig. 1. RP Vs various noise

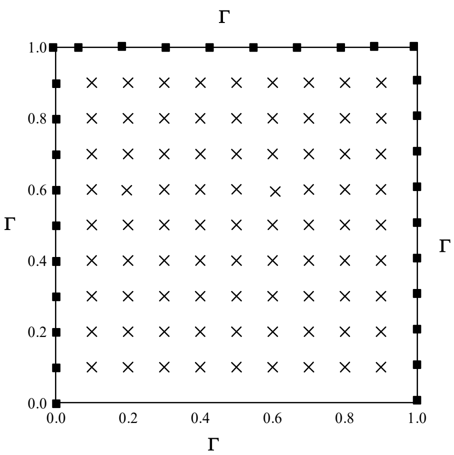
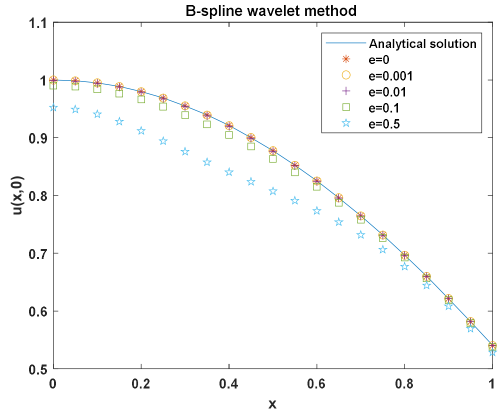
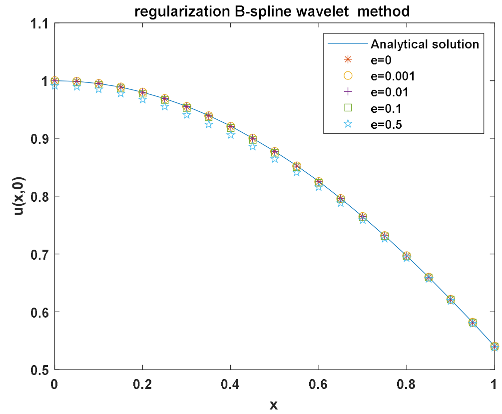


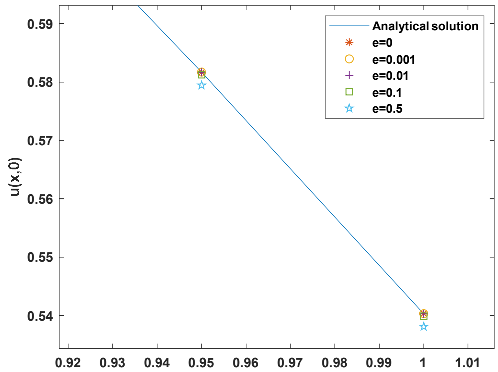
Fig. 2. Distribution map



(a)



(b)



“zoom in” figure of Fig.3(b) for the interval [0.9, 1]

Fig. 3. The inverted $u(x,0)$ for various noise for two methods

The relative root means square error (RRMSE) of both methods with various noise levels is displayed in Fig.4 to further investigate the impact of noise on the inversion results. According to Fig. 4, BSWM performs better for the case of $e < 0.001$, whereas RBSWM provides significantly superior solutions for the case of $e > 0.001$. It demonstrates that in comparison to BSWM, RBSWM can increase the stability of the solution and strengthen the capacity to resist the higher noise levels.

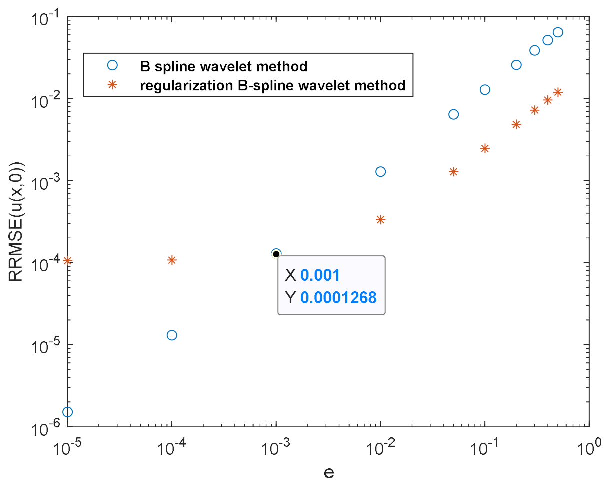


Fig. 4. The RRMSE Vs various noise for two methods

B. Problem with unknown Dirichlet boundary conditions

Consider a 2-D ill-posed boundary condition identification problem on $[0,1]\times[0,1]$, which analytical solution is:

$$u(x,y)=x^2-y^2 \tag{33}$$

The known BCs:

$$\Gamma_1:\quad u(0,y)=-y^2 \tag{34}$$

$$\Gamma_2:\quad u(x,1)=x^2-1 \tag{35}$$

$$\Gamma_3:\quad u(1,y)=1-y^2 \tag{36}$$

The unknown BCs:

$$\Gamma_0:\quad u(x,0)=x^2 \tag{37}$$

Next, we will use BSWM and RBSWM to obtain the unknown BCs on Γ_0. Based on the analytical solution, the potential values on four internal points are given as follows to increase precision of the numerical solution:

$$\begin{aligned} &u(0.2,0.2)=0, \quad u(0.2,0.6)=-0.32, \\ &u(0.6,0.2)=0.32, \quad u(0.6,0.6)=0 \end{aligned} \tag{38}$$

The collocation points are still scattered on a grid of 11×11, as in the previous instance, and the distribution is illustrated in Fig. 5. The internal potential values in Equation (38) and the DBC are subjected to the simulated noise, which level range from 0 to 0.3. In Table III and Fig. 6, the RP for various noise levels are presented.

Figure 7 displays the inversion results $u(x,0)$ for various noise levels using BSWM and RBSWM. From Fig. 7, we can observe how, for various noise levels, the recovered boundary condition $u(x,0)$ fluctuates with x in comparison to the exact solution. Meanwhile, both the outcomes of the two methods for the instance of $e<0.001$ are shown to be in excellent agreement with the exact solution.

However, RBSWM produces significantly better inversion results than BSWM when the noise level reaches 0.1 and 0.3. The solution of BSWM cannot produce a stable effect, but RBSWM does. It is clear that there exists a bigger solution distorted in Fig 7(a).

The RRMSE for four distinct methods, including BSWM, RBSWM, radial basis collocation method, and least squares based radial basis collocation method, is shown in Fig. 8 for various noise levels. In comparison to other methods, it has been found that RBSWM has the best good qualities regarding accuracy and anti-noise.

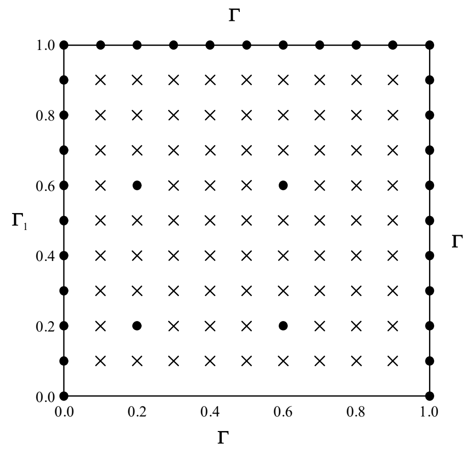


Fig.5. Distribution map for Example 2

TABLE III
RP SETTING FOR VARIOUS NOISE

Noise	0	0.001	0.01	0.1	0.3
α	0.006801	0.006801	0.006801	0.006826	0.006826

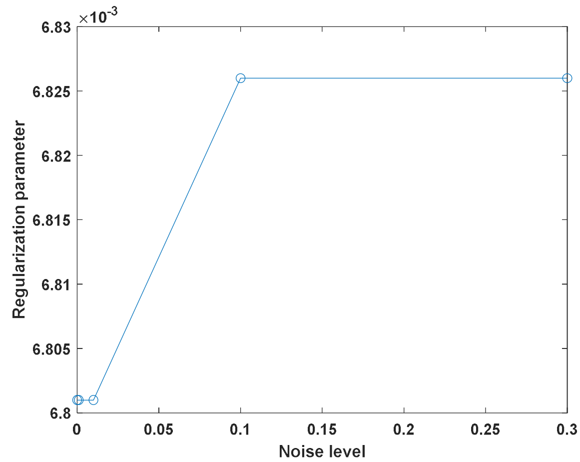
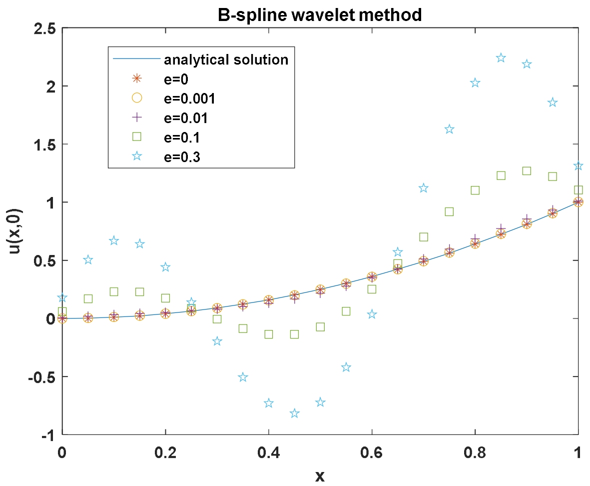
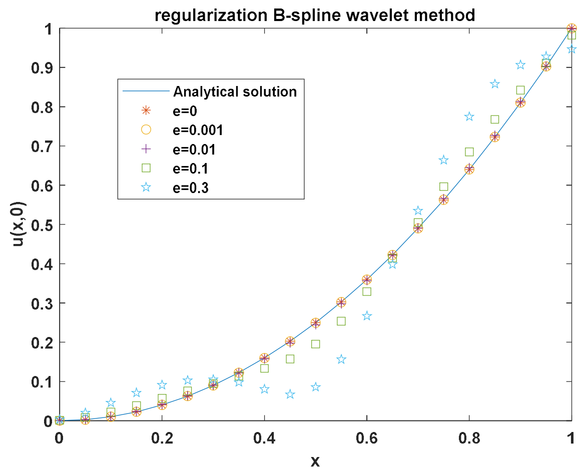


Fig. 6. RP Vs various noise



(a)



(b)

Fig 7. The inverted $u(x,0)$ for various noise

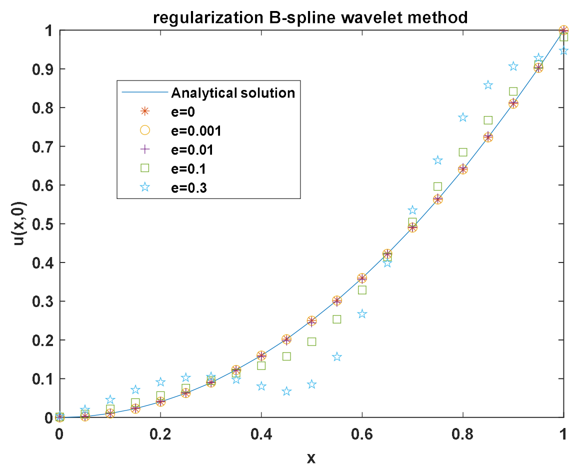


Fig 8. The RRMSE Vs various noise

C. *Problem with the unknown DBC and NBC*

An ill-posed inverse Cauchy problem of heat conduction on $[0,1]\times[0,1]$, which has been studied in [6] , is considered in this part. The exact solution of this problem is as follows:

$$u(x,y)=\sin(x)\sinh(y)+\cos(x)\cosh(y) \tag{39}$$

The known BCs:

$$\Gamma_1:\ u(x,0)=\cos(x) \tag{40}$$

$$\Gamma_2: \frac{\partial u(x,1)}{\partial y}=\sin(x)\cosh(1)+\cos(x)\sinh(1) \tag{41}$$

$$\Gamma_3:\ u(1,y)=\sin(1)\sinh(y)+\cos(1)\cosh(y) \tag{42}$$

$$\Gamma_3: \frac{\partial u(1,y)}{\partial x}=\cos(1)\sinh(y)-\sin(1)\cosh(y) \tag{43}$$

The undetermined BC is on the left-side boundary $\Gamma_0=\{(x,y)\in\mathbb{R}^2: x=0, 0\le y\le 1\}$. We need to find the following two conditions

$$u(0,y)=\cosh(y) \tag{44}$$

$$\frac{\partial u(0,y)}{\partial x}=\sinh(y) \tag{45}$$

As in the previous examples, similar treatments have been taken. The collocation points are still scattered on a grid of 11×11, and the distribution is illustrated in Fig. 9. Based on the exact solution, the internal potential values on four internal points are shown as follows:

$$u(0.2,0.2)=\sin(0.2)\sinh(0.2)+\cos(0.2)\cosh(0.2)$$
$$u(0.2,0.6)=\sin(0.2)\sinh(0.6)+\cos(0.2)\cosh(0.6)$$
$$u(0.6,0.2)=\sin(0.6)\sinh(0.2)+\cos(0.6)\cosh(0.2)$$
$$u(0.6,0.6)=\sin(0.6)\sinh(0.6)+\cos(0.6)\cosh(0.6)$$

In this example, boundary conditions of Γ_3 are subjected to the simulated noise, which level range from 0 to 0.2. Table IV and Fig.10 give the RP setting with various noise for inverted $u(0,y)$ and $\partial u/\partial x(0,y)$.

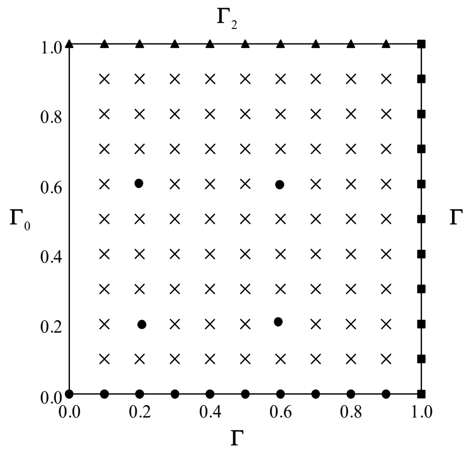


Fig 9. Distribution map for Example 3

TABLE IV
RP SETTING FOR VARIOUS NOISE

Noise	0	0.0001	0.001	0.01	0.1	0.2
α	0.006814	0.006814	0.006814	0.006814	0.006826	0.006826

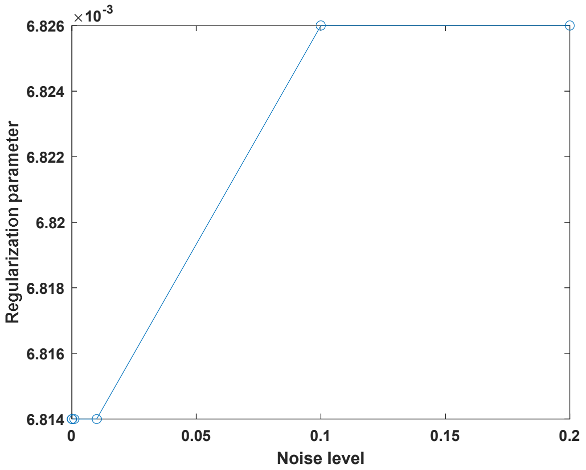
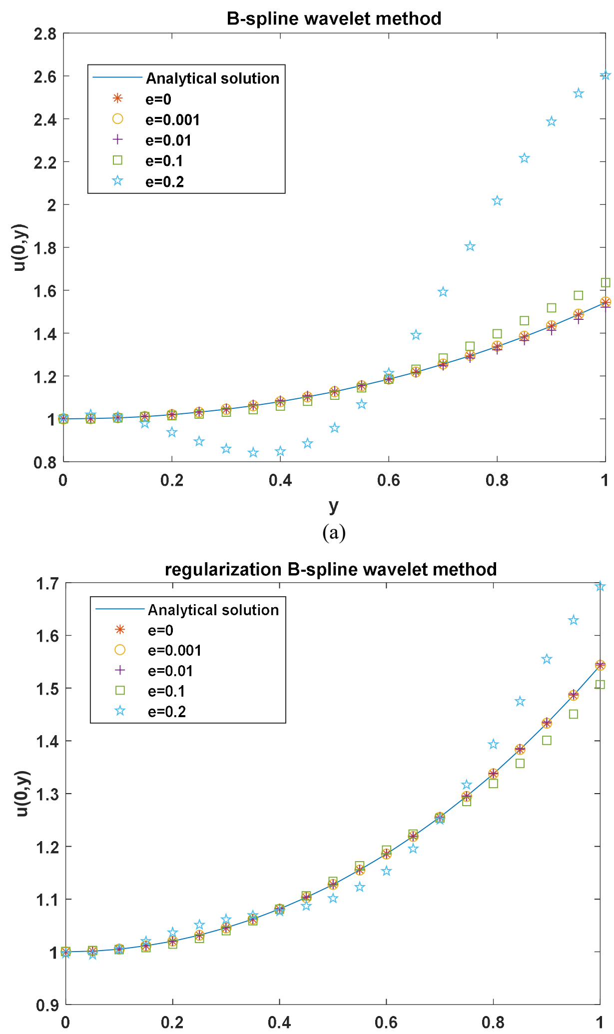


Fig. 10. RP Vs various noise for Example 3



(a)

(b)

Fig 11. The inverted $u(0,y)$ for various noise

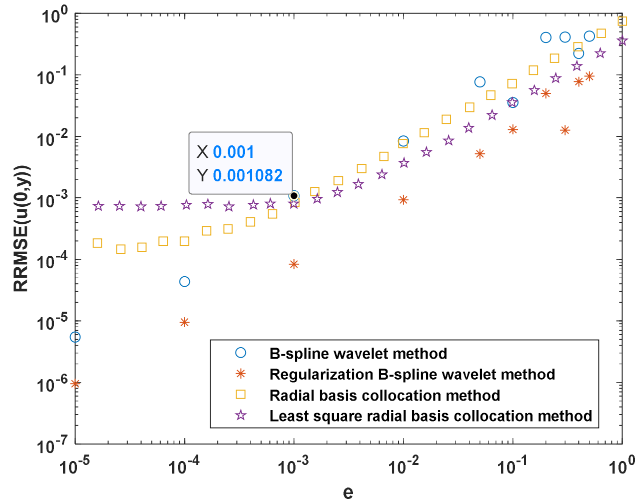
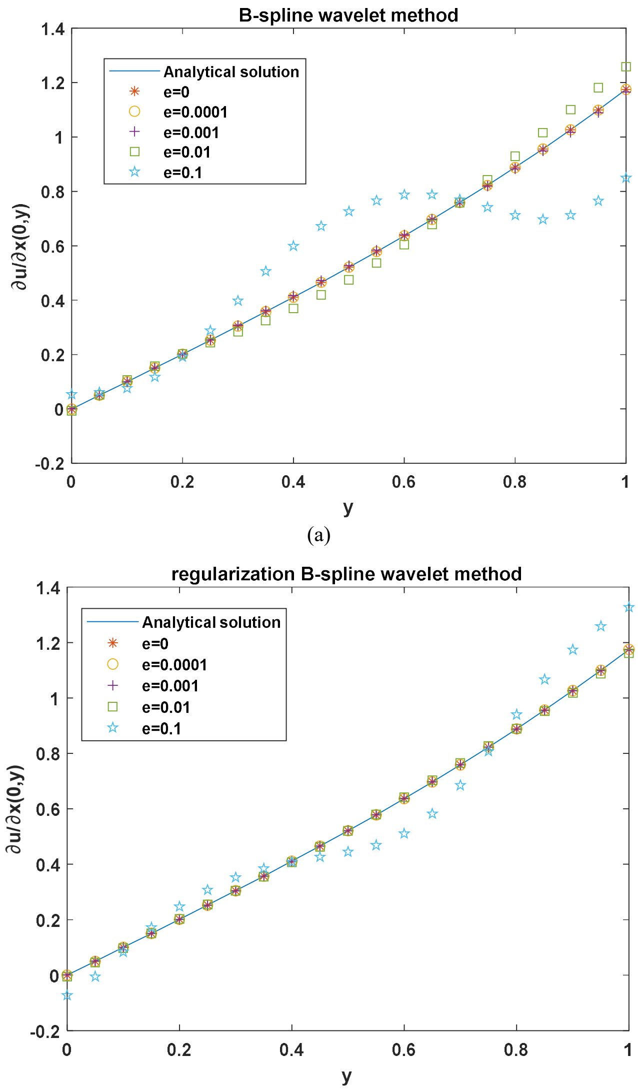


Fig 12. The RRMSE Vs various noise



(a)

(b)

Fig 13. The inverted $\partial u / \partial x(0,y)$ Vs different noise

Fig.11 and Fig.13 show the inversion results of $u_x(0,y)$ and $u(0,y)$ on Γ_0 for various noise levels. For $e<0.001$, the inversion results agree well with the analytical solution. However, when the noise level reaches 10% or 20%, there are obvious solutions distorted in Fig.11 (a) and Fig.13 (a); whereas the recovered results of the RBSWM are much better. Fig. 12 and Fig.14 show the comparison of the relative root means square error (RRMSE) of four different methods at different noise levels. It is seen that, from Fig.12, the RRMSE of the regularization B-spline wavelet method is much smaller than the other three methods for all noise levels; whereas the B-spline wavelet method performs well only for the smaller noise level. For the inverted $u_x(0,y)$ in Fig.14, when the noise level hits 10%, the precision of the suggested algorithm is not as excellent. However, our method's RRMSE is still higher than those of the other methods in the literature. Moreover, the recovered data are compared with the results obtained in literature [32], shown in Fig.15. The results show that we recover the unknown boundary conditions very well for e=0.01, and the regularization B-spline wavelet method has favorable anti-noise property.

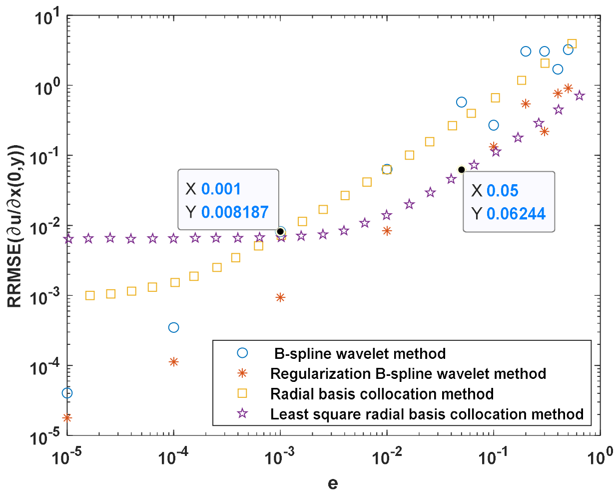
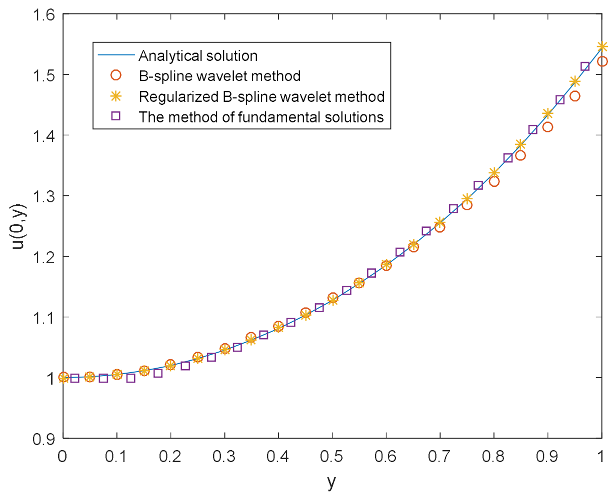
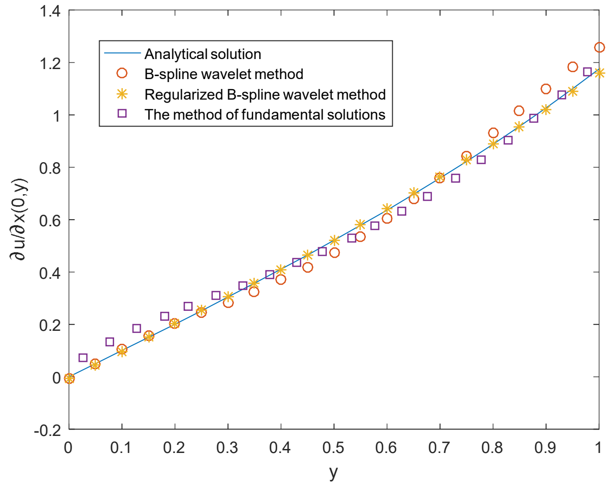


Fig 14. The RRMSE Vs various error



(a)



(b)

Fig 15. The comparison of recovered data for e=0.01

D. *Problem with the unknown convex boundary*

In this subsection, our objective is to recover the shape of part of the boundary. We take $\Omega = \{(x,y) \in R^2 \mid 0 < x < 1, 0 < p(x)\}$ as a solution domain, see Fig.16 for an illustration, where $p(x)$ is an unknown part of boundary to be determined.

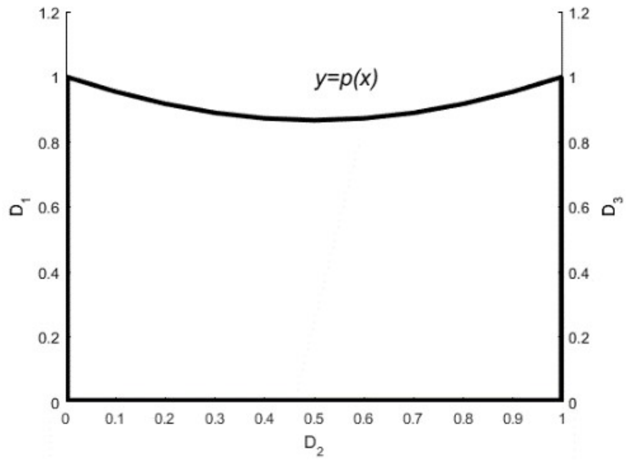


Fig. 16 Solution domain

Assume that this problem have an analytical solution as follows:

$$u(x,y) = y^2 - (x - \frac{1}{2})^2 - \frac{3}{4}$$

and the boundary $p(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{3}{4}}$. The known BCs are:

$$u(1,y) = y^2 - 1 \qquad u_x(1,y) = -1$$

$$u(0,y) = y^2 - 1 \qquad u_x(0,y) = 1$$

$$u(x,0) = -\left(x - \frac{1}{2}\right)^2 - \frac{3}{4} \qquad u_y(x,0) = 0$$

Fig.17 gives the arrangement of effective collocation points in the numerical simulation for this problem.

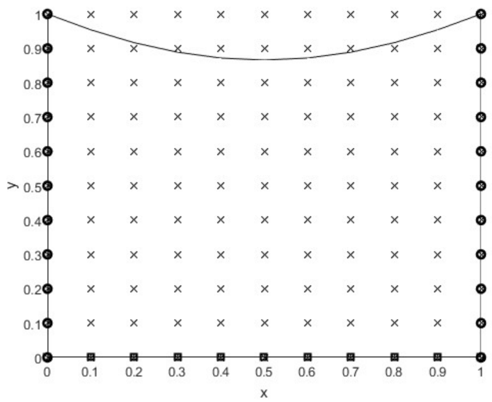
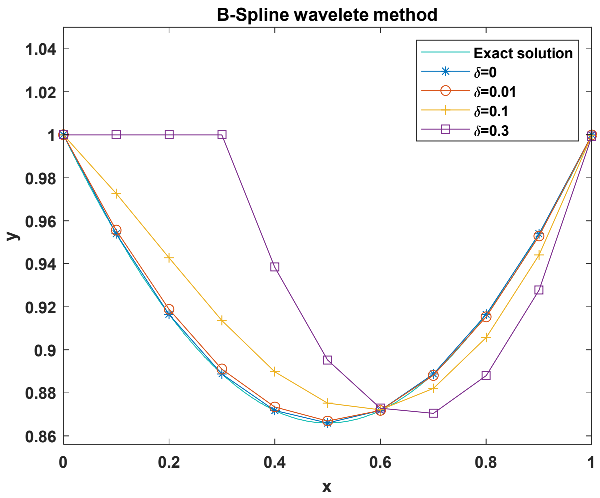
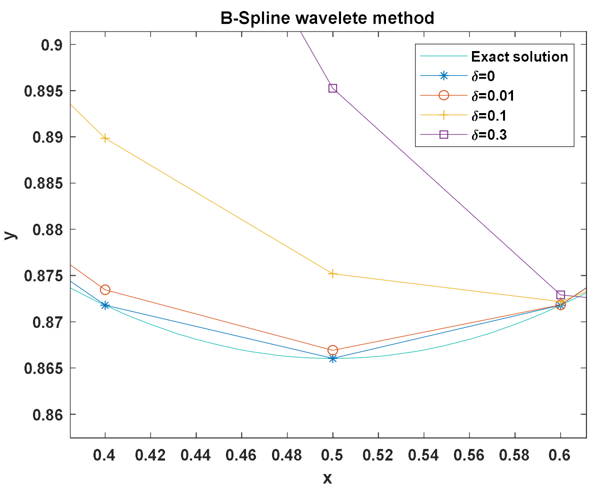


Fig 17. Distribution map for Example 4

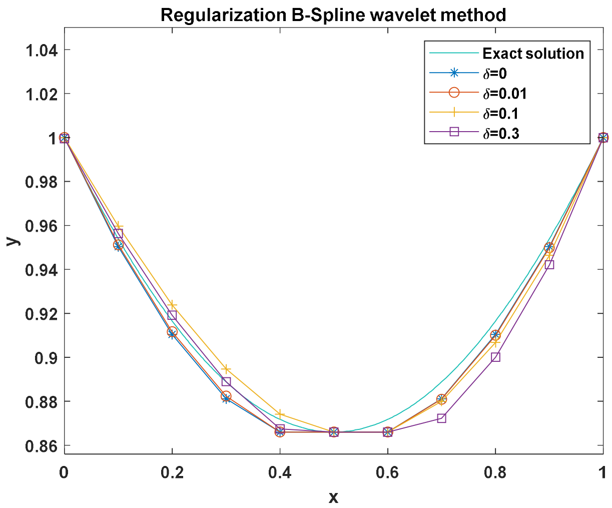
Figure 18 illustrates the numerical results of the BSWM and RBSWM for a range of relative noise levels. As demonstrated, even with a moderately higher noise level, the estimated boundaries with the two methods closely match the precise one. However, the RBSWM greatly outperforms BSWM as the noise level increases.



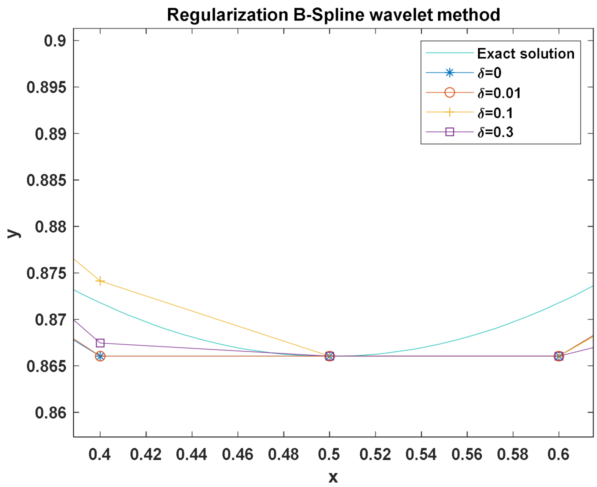
(a)



“zoom in” figure of Fig.18 (a) for interval [0.4, 0.6]



(b)



“zoom in” figure of Fig.18 (b) for interval [0.4, 0.6]

Fig 18. The recovered boundary for different noise

In Table V and Fig. 19, the RRMSE obtained by using two approaches for various noise levels is displayed. In Fig. 20, the RP for various noise are listed. It is observed that the RRMSE for two methods are close to each other for $\delta < 0.05$. However, for the bigger noise level, the advantages of the regularization are obvious. As a result, for this example, our method is feasible and works well.

TABLE V
THE RP AND RRMSE

Noise	α	RBSWM	BSWM
0	0.023081575	0.005542427	7.63513E-09
0.01	0.02312256	0.005271201	0.001459207
0.1	0.02312256	0.006458295	0.015779775
0.3	0.047051798	0.008958761	0.055294667

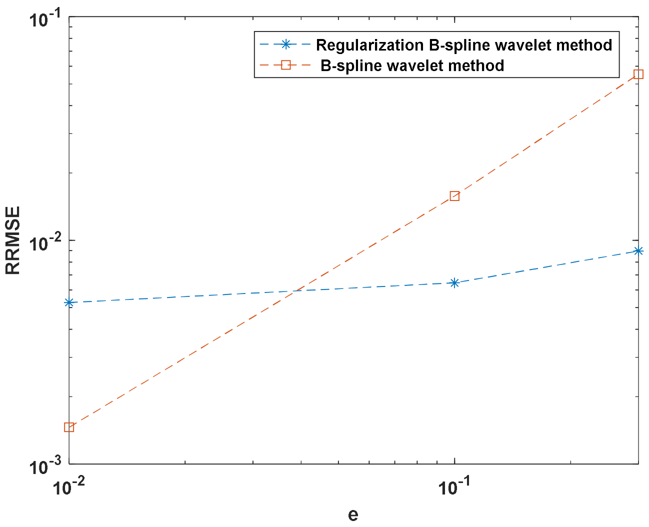


Fig.19 RRMSE for two methods

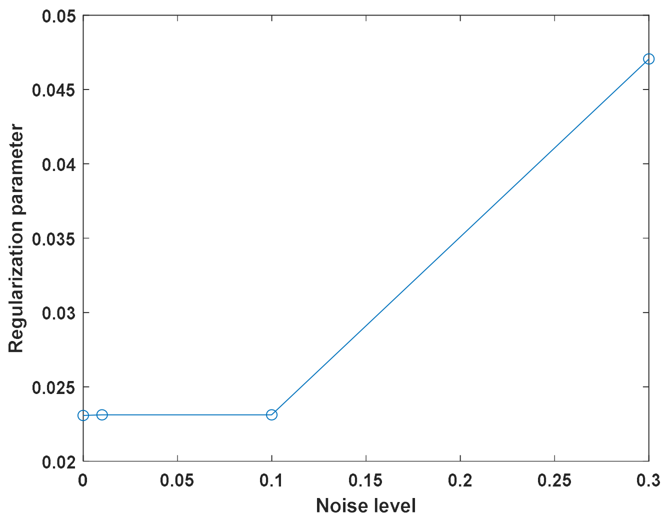


Fig. 20. RP Vs various noise for Example 4

IV. CONCLUSION

In this study, for the ill-posed boundary value identification problems of the 2-D Laplace equation, we propose an effective regularization B-spline wavelet method. Based on the B-spline wavelet scaling functions, the boundary value identification problems has been transformed into a linear algebraic equations system, which contains an ill-conditioned coefficient matrix. To overcome the difficulty, the Tikhonov regularization technology is introduced to make the solution procedure stable. Four numerical examples with different boundary conditions demonstrate that the regularization B-spline wavelet method can solve the ill-posed boundary value identification problems stably in the case with larger noise levels. This paper also show that the regularization B-spline wavelet method performs better for the ill-posed problems with larger noise than the B-spline wavelet method and the other methods.

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