A Computational Study on Unsteady Anisotropic Helmholtz Type Equation of Quadratically Varying Coefficients

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***Abstract*—This paper explores the unsteady Helmholtz type equation with quadratically varying coefficients for anisotropic inhomogeneous media. The paper proposes using a combined Laplace transform and boundary element method to find numerical solutions to problems governed by the equation. The variable coefficients equation is transformed into a constant coefficients equation which is then written in a boundary integral equation involving a time-free fundamental solution. The boundary-only integral equation is used with a standard boundary element method to find the numerical solutions. The results are then transformed numerically using the Stehfest formula to get solutions in the time variable. The paper concludes that the combined Laplace transform and boundary element method is both easy to implement and accurate, as demonstrated by problems related to anisotropic quadratically graded media.**

***Index Terms*—computational study, unsteady anisotropic Helmholtz, quadratically varying coefficients, boundary element method**

I. Introduction

We will consider initial boundary value problems governed by a Helmholtz type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x_j}\right]+\beta^2(\mathbf{x})T(\mathbf{x},t) = \psi(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}\quad i,j=1,2 \tag{1}$$

where the coefficients $[\kappa_{ij}]$ is a symmetric matrix with positive determinant, and summation convention holds for repeated indices so that explicitly equation (1) takes the form

$$\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial T}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial T}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial T}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial T}{\partial x_2}\right)+\beta^2 T=\psi\frac{\partial T}{\partial t}$$

Equation (1) is usually used to model acoustic problems (see for examples [1], [2]).

Over the past ten years, functionally graded materials (FGMs) have been the focus of many studies for a range of applications. FGMs are materials whose properties change according to a mathematical function, making equation (1) applicable to them. FGMs are mainly human-made materials that are designed to have specific properties (see for example [3], [4]), making the solution of equation (1) relevant to their production.

Several studies have been conducted on solving the Helmholtz equation numerically, and these studies are categorized based on the anisotropy of the media and the variability of coefficients (inhomogeneity of the media). For example, some studies considered a constant coefficient isotropic equation for homogeneous media (see [5]–[7]), while others solved an isotropic equation with variable coefficients for inhomogeneous media (see [8]). Recently, studies on problems of inhomogeneous anisotropic media for several types of governing equations had been done (see for examples, [9], [10], [11], [12]). These studies addressed classes of inhomogeneities that differ from the constant-plus-variable inhomogeneity class.

This paper is intended to extend the recently published works in [11] for steady anisotropic Helmholtz type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x_j}\right]+\beta^2(\mathbf{x})T(\mathbf{x},t)=0$$

to unsteady anisotropic Helmholtz type equation with spatially variable coefficients of the form (1).

Equation (1) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. The analysis of this paper is purely formal, the main aim is to construct an effective BEM for a class of equations which falls within the type (1).

II. The initial-boundary value problem

By knowing the coefficients $\kappa_{ij}(\mathbf{x}),\beta^2(\mathbf{x})$ we will seek solutions $T(\mathbf{x},t)$ and its derivatives in a spatial-time space (Ω,t), Ω in R^2 with continuous boundary $\partial\Omega$. On the boundary $\partial\Omega_1$, $T(\mathbf{x},t)$ is given and

$$P(\mathbf{x},t)=\kappa_{ij}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x_i}n_j \tag{2}$$

is specified on $\partial\Omega_2$ where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ represents the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$T(\mathbf{x},0)=0 \tag{3}$$

III. The boundary integral equation

The coefficients κ_{ij},β^2,ψ are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij}g(\mathbf{x}) \tag{4}$$

$$\beta^2(\mathbf{x}) = \overline{\beta}^2 g(\mathbf{x}) \tag{5}$$

$$\psi(\mathbf{x}) = \overline{\psi}g(\mathbf{x}) \tag{6}$$

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where the $\overline{\kappa}_{ij}, \overline{\beta}^2, \overline{\psi}$ are constants and g is a differentiable function of $\mathbf{x}$. Further we assume that the coefficients $\kappa_{ij}(\mathbf{x})$, $\beta^2(\mathbf{x})$ and $\psi(\mathbf{x})$ are quadratically graded by taking $g(\mathbf{x})$ as a quadratic function

$$g(\mathbf{x}) = (c_0 + c_i x_i)^2 \tag{7}$$

where c_0 and c_i are constants. Therefore (7) satisfies

$$\overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0 \tag{8}$$

Use of (4)-(6) in (1) yields

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i} \left(g \frac{\partial T}{\partial x_j} \right) + \overline{\beta}^2 g T = \overline{\psi} g \frac{\partial T}{\partial t} \tag{9}$$

Let

$$T(\mathbf{x}, t) = g^{-1/2}(\mathbf{x}) \sigma(\mathbf{x}, t) \tag{10}$$

therefore substitution of (4) and (10) into (2) gives

$$P(\mathbf{x}, t) = -P_g(\mathbf{x}) \sigma(\mathbf{x}, t) + g^{1/2}(\mathbf{x}) P_\sigma(\mathbf{x}, t) \tag{11}$$

where

$$P_g(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \qquad P_\sigma(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial \sigma}{\partial x_j} n_i$$

Also, (9) may be written in the form

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i} \left[g \frac{\partial \left(g^{-1/2} \sigma \right)}{\partial x_j} \right] + \overline{\beta}^2 g^{1/2} \sigma = \overline{\psi} g \frac{\partial \left(g^{-1/2} \sigma \right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i} \left(g^{1/2} \frac{\partial \sigma}{\partial x_j} + g \sigma \frac{\partial g^{-1/2}}{\partial x_j} \right) + \overline{\beta}^2 g^{1/2} \sigma = \overline{\psi} g^{1/2} \frac{\partial \sigma}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i} = -g^{-1} \frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i} \left(g^{1/2} \frac{\partial \sigma}{\partial x_j} - \sigma \frac{\partial g^{1/2}}{\partial x_j} \right) + \overline{\beta}^2 g^{1/2} \sigma = \overline{\psi} g^{1/2} \frac{\partial \sigma}{\partial t}$$

Rearranging and neglecting the zero terms yield

$$g^{1/2} \overline{\kappa}_{ij} \frac{\partial^2 \sigma}{\partial x_i \partial x_j} - \sigma \overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + \overline{\beta}^2 g^{1/2} \sigma = \overline{\psi} g^{1/2} \frac{\partial \sigma}{\partial t}$$

Equation (8) then implies

$$\overline{\kappa}_{ij} \frac{\partial^2 \sigma}{\partial x_i \partial x_j} + \overline{\beta}^2 \sigma = \overline{\psi} \frac{\partial \sigma}{\partial t} \tag{12}$$

Taking the Laplace transform of (10), (11), (12) and applying the initial condition (3) we obtain

$$\sigma^*(\mathbf{x}, s) = g^{1/2}(\mathbf{x}) T^*(\mathbf{x}, s) \tag{13}$$

$$P_{\sigma^*}(\mathbf{x}, s) = \left[P^*(\mathbf{x}, s) + P_g(\mathbf{x}) \sigma^*(\mathbf{x}, s) \right] g^{-1/2}(\mathbf{x}) \tag{14}$$

$$\overline{\kappa}_{ij} \frac{\partial^2 \sigma^*}{\partial x_i \partial x_j} + \left(\overline{\beta}^2 - s \overline{\psi} \right) \sigma^* = 0 \tag{15}$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (15) is given in the form

$$\begin{aligned} \eta(\mathbf{x}_0) \sigma^*(\mathbf{x}_0, s) = & \int_{\partial\Omega} \left[\Gamma(\mathbf{x}, \mathbf{x}_0) \sigma^*(\mathbf{x}, s) \right. \\ & \left. - \Phi(\mathbf{x}, \mathbf{x}_0) P_{\sigma^*}(\mathbf{x}, s) \right] dS(\mathbf{x}) \end{aligned} \tag{16}$$

where $\mathbf{x}_0 = (a, b)$, $\eta = 0$ if $(a, b) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if $(a, b) \in \Omega$, $\eta = \frac{1}{2}$ if $(a, b) \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a, b). The so-called fundamental solution Φ in (16) is any solution of the equation

$$\overline{\kappa}_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \left(\overline{\beta}^2 - s \overline{\psi} \right) \Phi = \delta(\mathbf{x} - \mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \overline{\kappa}_{ij} \frac{\partial \Phi(\mathbf{x}, \mathbf{x}_0)}{\partial x_j} n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \ln R & \text{if } \overline{\beta}^2 - s\overline{\psi} = 0 \\ \frac{\imath K}{4} H_0^{(2)}(\omega R) & \text{if } \overline{\beta}^2 - s\overline{\psi} > 0 \\ \frac{-K}{2\pi} K_0(\omega R) & \text{if } \overline{\beta}^2 - s\overline{\psi} < 0 \end{cases}$$

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \frac{1}{R} \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \overline{\beta}^2 - s\overline{\psi} = 0 \\ \frac{-\imath K \omega}{4} H_1^{(2)}(\omega R) \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \overline{\beta}^2 - s\overline{\psi} > 0 \\ \frac{K\omega}{2\pi} K_1(\omega R) \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \overline{\beta}^2 - s\overline{\psi} < 0 \end{cases} \tag{17}$$

where

$$\begin{aligned} K &= \dot{\tau} / D \\ \omega &= \sqrt{|\overline{\beta}^2 - s\overline{\psi}| / D} \\ D &= \left[\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\dot{\tau} + \overline{\kappa}_{22} \left(\dot{\tau}^2 + \ddot{\tau}^2 \right) \right] / 2 \\ R &= \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2} \\ \dot{x}_1 &= x_1 + \dot{\tau} x_2 \\ \dot{a} &= a + \dot{\tau} b \\ \dot{x}_2 &= \ddot{\tau} x_2 \\ \dot{b} &= \ddot{\tau} b \end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\tau + \overline{\kappa}_{22}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R / \partial x_j$ needed for the calculation of the Γ in (17) are given by

$$\begin{aligned} \frac{\partial R}{\partial x_1} &= \frac{1}{R} (\dot{x}_1 - \dot{a}) \\ \frac{\partial R}{\partial x_2} &= \dot{\tau} \left[\frac{1}{R} (\dot{x}_1 - \dot{a}) \right] + \ddot{\tau} \left[\frac{1}{R} \left(\dot{x}_2 - \dot{b} \right) \right] \end{aligned}$$

Use of (13) and (14) in (16) yields

$$\begin{aligned} \eta g^{1/2} T^* = & \int_{\partial\Omega} \left[\left(g^{1/2} \Gamma - P_g \Phi \right) T^* \right. \\ & \left. - \left(g^{-1/2} \Phi \right) P^* \right] dS \end{aligned} \tag{18}$$

TABLE I
VALUES OF V_m OF THE STEHFEST FORMULA.

V_m	$N=6$	$N=8$	$N=10$	$N=12$
V_1	1	$-1/3$	$1/12$	$-1/60$
V_2	-49	$145/3$	$-385/12$	$961/60$
V_3	366	-906	1279	-1247
V_4	-858	$16394/3$	$-46871/3$	$82663/3$
V_5	810	$-43130/3$	$505465/6$	$-1579685/6$
V_6	-270	18730	-236957.5	1324138.7
V_7		$-35840/3$	$1127735/3$	$-58375583/15$
V_8		$8960/3$	$-1020215/3$	$21159859/3$
V_9			164062.5	-8005336.5
V_{10}			-32812.5	5552830.5
V_{11}				-2155507.2
V_{12}				359251.2

Equation (18) provides a boundary integral equation that can be solved using a standard boundary element method for determining T^* and its derivatives at all points of Ω.

The Stehfest formula is then used for a numerical Laplace transform inversion to find the solutions and their derivatives in the original time variable. The obtained solutions and their derivatives are for the original variable t, which were previously transformed to the Laplace transform variable s.

The Stehfest formula is

$$\begin{aligned} T(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m T^*(\mathbf{x},s_m) \\ \frac{\partial T(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial T^*(\mathbf{x},s_m)}{\partial x_1} \\ \frac{\partial T(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial T^*(\mathbf{x},s_m)}{\partial x_2} \end{aligned} \quad (19)$$

where

$$\begin{aligned} s_m &= \frac{\ln 2}{t} m \\ V_m &= (-1)^{\frac{N}{2}+m} \times \\ &\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)} \frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!} \end{aligned}$$

IV. NUMERICAL EXAMPLES

To confirm the analysis that was employed in Section III to obtain the boundary integral equation 18, we will examine multiple instances. These instances will either serve as examples of analytical solutions, or they will be problems that do not have straightforward analytical solutions.

The authors have used standard BEM to get numerical results and chosen a unit square as the geometrical domain for all problems in order to keep things simple. A total of 320 elements with equal length, i.e., 80 elements on each side of the unit square, have been used.

The time domain is chosen to be the interval $0 \le t \le 5$. The solutions are computed using a FORTRAN script, and a specific command is included to calculate the amount of time taken to obtain the solutions on the CPU. A simple script is also embedded to compute the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any even number N. Table I shows the values of V_m for several values of N.

For all problems the inhomogeneity function is taken to be

$$g^{1/2}(\mathbf{x}) = 1 - 0.15x_1 - 0.25x_2$$

and the constant anisotropy coefficient $\overline{\kappa}_{ij}$

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.85 \end{bmatrix}$$

We take the constant coefficient $\overline{\beta}^2$

$$\overline{\beta}^2 = 1$$

A. Examples with analytical solutions

1) Problem 1:: Other aspects that will be justified are the convergence (as N increases) and time efficiency for obtaining the numerical solutions. The analytical solutions are assumed to take a separable variables form

$$T(\mathbf{x},t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where $h(\mathbf{x}), f(t)$ are continuous functions. The boundary conditions are assumed to be (see Figure 1)

P is given on side AB
P is given on side BC
T is given on side CD
P is given on side AD

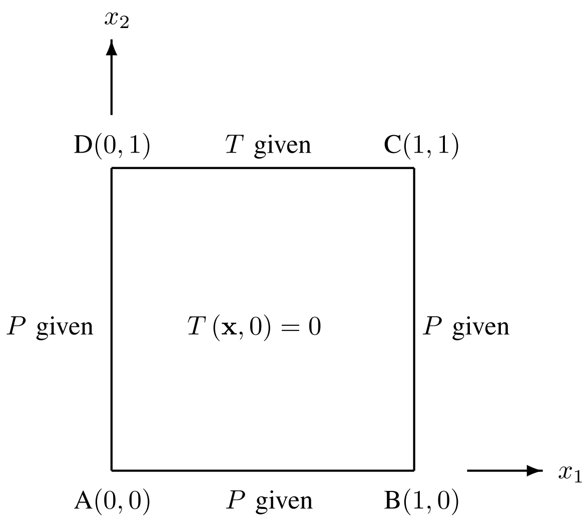

Fig. 1. The boundary conditions for Problem 1.

For each N, numerical solutions for T and the derivatives $T_1 = \partial T/\partial x_1$ and $T_2 = \partial T/\partial x_2$ at 19×19 points inside the space domain which are

$$\begin{aligned} (x_1,x_2) = &\{0.05, 0.1, 0.15, \ldots, 0.9, 0.95\} \times \\ &\{0.05, 0.1, 0.15, \ldots, 0.9, 0.95\} \end{aligned}$$

and 11 time-steps which are

$$t = 0.0005, 0.5, 1, 1.5, \ldots, 4, 4.5, 5$$

are computed. The aggregate relative error E is calculated using the norm

$$E = \left[\frac{\sum_t \sum_{i=1}^{19\times19} \left(\varsigma_{n,i} - \varsigma_{a,i}\right)^2}{\sum_t \sum_{i=1}^{19\times19} T_{a,i}^2}\right]^{\frac{1}{2}}$$

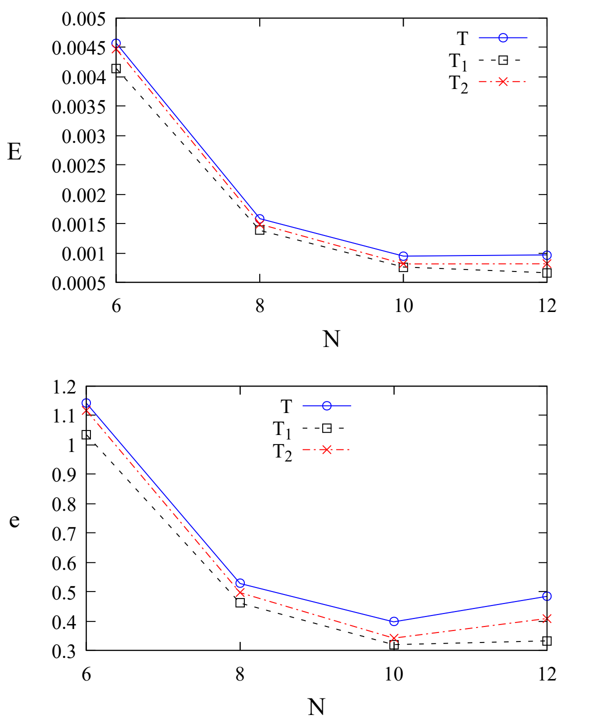

Fig. 2. The global average error E and efficiency number $e = \tau E$ for Case 1 of Problem 1.

where ς_n and ς_a represent respectively the numerical and analytical solutions T or the derivatives $\partial T/\partial x_1$ and $\partial T/\partial x_2$. The elapsed CPU time τ (in seconds) is also computed and the time efficiency number e for obtaining the numerical solutions of error E is defined as

$$e = E\tau$$

This formula explains that the smaller time τ with smaller error E, the more efficient the procedure (smaller e).

Case 1:: We take

$$\begin{aligned} h(\mathbf{x}) &= 1 - 0.3x_1 - 0.7x_2 \\ f(t) &= 1 - \exp(-1.75t) \end{aligned}$$

Thus for $h(\mathbf{x})$ to satisfy (15)

$$\overline{\psi} = \overline{\beta}^2/s = 1/s$$

As shown in Figure 2, for the solutions $T, \partial T/\partial x_2$ the error E and efficiency number e get smaller as N moves up to $N = 10$, and for the solution $\partial T/\partial x_1$ the error E gets smaller as N moves up to $N = 12$ and efficiency number e decreases as N moves up to $N = 10$. Due to round-off errors the accuracy will only increase upto a point N, and then the accuracy will decline (see [13]).

From Table III it is obvious that $N = 10$ is the optimized value of N for solutions $T, \partial T/\partial x_2$ to achieve their smallest error E and efficiency number e. Whereas for the solution $\partial T/\partial x_1$ the optimized values of N to reach its smallest error E and efficiency number e are acheived when $N = 12$ and $N = 10$ respectively.

In addition Table IV shows the numerical and analytical solutions T, $\partial T/\partial x_1$ and $\partial T/\partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$.

TABLE II
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $e = \tau E$ FOR CASE 1 OF PROBLEM 1.

N		6	8	10	12
τ		249.859	332.906	418.594	498.781
T	E	0.00456413	0.00158719	0.00095128	0.00097223
	e	1.140390	0.528384	0.398200	0.484928
$\frac{\partial T}{\partial x_1}$	E	0.00413618	0.00138737	0.00076626	0.00066876
	e	1.033463	0.461865	0.320751	0.333567
$\frac{\partial T}{\partial x_2}$	E	0.00446496	0.00149382	0.00081896	0.00082135
	e	1.115611	0.497304	0.342810	0.409674

TABLE III
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $T, \partial T/\partial x_1, \partial T/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER e FOR CASE 1 OF PROBLEM 1.

	T	$\frac{\partial T}{\partial x_1}$	$\frac{\partial T}{\partial x_2}$
E	$N = 10$	$N = 12$	$N = 10$
e	$N = 10$	$N = 10$	$N = 10$

Case 2:: For the analytical solution we take

$$\begin{aligned} h(\mathbf{x}) &= \cos(1 - 0.3x_1 - 0.7x_2) \\ f(t) &= t/5 \end{aligned}$$

So that in order for $h(\mathbf{x})$ to satisfy (15)

$$\overline{\psi} = 0.4515/s$$

Figure 3 and Tables V and VI show that for solution T the smallest error E and efficiency number e are reached when $N = 12$ and $N = 8$ respectively, whereas for the solutions $\partial T/\partial x_1, \partial T/\partial x_2$ they are reached when $N = 8$. Meanwhile, Table VII shows the numerical and analytical solutions T, $\partial T/\partial x_1$ and $\partial T/\partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$.

Case 3:: We take

$$\begin{aligned} h(\mathbf{x}) &= \exp(-1 + 0.3x_1 + 0.7x_2) \\ f(t) &= 0.16t(5 - t) \end{aligned}$$

Therefore (15) gives

TABLE IV
THE SOLUTIONS T, $\partial T/\partial x_1$ AND $\partial T/\partial x_2$ AT $(x_1, x_2) = (0.5, 0.5)$ FOR CASE 1 OF PROBLEM 1.

t	Analytical	$N = 6$	$N = 8$	$N = 10$	$N = 12$
			T		
0.0005	0.00054	0.00054	0.00054	0.00054	0.00054
1.0	0.51639	0.51525	0.51550	0.51575	0.51582
2.0	0.60612	0.60137	0.60439	0.60528	0.60546
3.0	0.62172	0.61947	0.62117	0.62129	0.62117
4.0	0.62443	0.62398	0.62446	0.62410	0.62385
5.0	0.62490	0.62512	0.62493	0.62444	0.62429
			$\partial T/\partial x_1$		
0.0005	-0.00022	-0.00022	-0.00022	-0.00022	-0.00022
1.0	-0.21301	-0.21272	-0.21283	-0.21293	-0.21297
2.0	-0.25002	-0.24828	-0.24953	-0.24989	-0.24997
3.0	-0.25646	-0.25575	-0.25646	-0.25650	-0.25645
4.0	-0.25757	-0.25762	-0.25781	-0.25766	-0.25757
5.0	-0.25777	-0.25809	-0.25801	-0.25781	-0.25772
			$\partial T/\partial x_2$		
0.0005	-0.00059	-0.00059	-0.00059	-0.00059	-0.00059
1.0	-0.56157	-0.56048	-0.56075	-0.56103	-0.56109
2.0	-0.65916	-0.65417	-0.65745	-0.65841	-0.65860
3.0	-0.67612	-0.67385	-0.67570	-0.67583	-0.67570
4.0	-0.67906	-0.67876	-0.67928	-0.67888	-0.67860
5.0	-0.67958	-0.68000	-0.67979	-0.67926	-0.67911

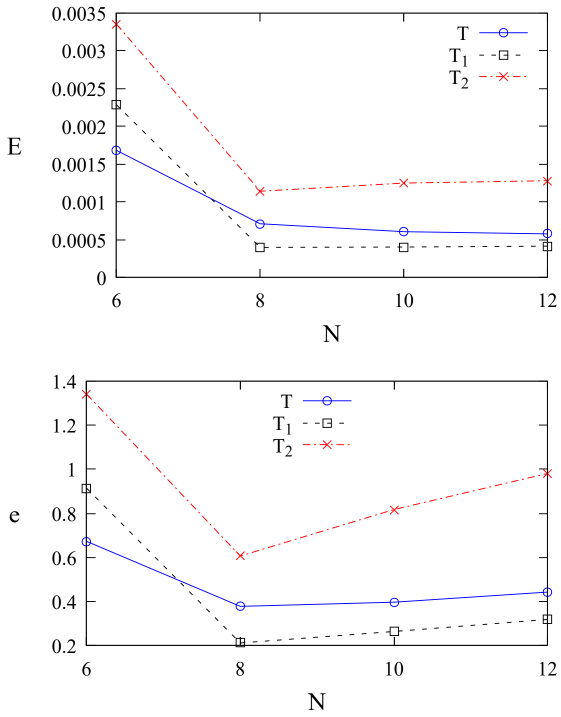

Fig. 3. The global average error E and efficiency number $e = \tau E$ for Case 2 of Problem 1.

TABLE V
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $e = \tau E$ FOR CASE 2 OF PROBLEM 1.

N		6	8	10	12
τ		399.906	530.906	653.391	765.234
T	E	0.00168221	0.00071267	0.00060798	0.00057896
	e	0.672728	0.378362	0.397250	0.443040
$\frac{\partial T}{\partial x_1}$	E	0.00228414	0.00040042	0.00040509	0.00041696
	e	0.913440	0.212583	0.264681	0.319074
$\frac{\partial T}{\partial x_2}$	E	0.00335038	0.00114237	0.00125029	0.00128123
	e	1.339838	0.606494	0.816928	0.980439

$$\overline{\psi} = 1.5485/s$$

Tables VIII and IX show that for solutions $T, \partial T/\partial x_1$ the smallest error E and efficiency number e are achieved when $N = 12$ and $N = 10$ respectively, whereas for the solutions $\partial T/\partial x_2$ they are reached when $N = 10$. Meanwhile, Table X shows the numerical and analytical solutions T, $\partial T/\partial x_1$ and $\partial T/\partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$.

B. A problem without analytical solution

The aim is to show the effect of inhomogeneity and anisotropy of the considered material on the solution T.

1) Problem 2:: The material is supposed to be either inhomogeneous or homogeneous and either anisotropic or isotropic. For a homogeneous material

TABLE VI
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $T, \partial T/\partial x_1, \partial T/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER e FOR CASE 2 OF PROBLEM 1.

	T	$\frac{\partial T}{\partial x_1}$	$\frac{\partial T}{\partial x_2}$
E	$N = 12$	$N = 8$	$N = 8$
e	$N = 8$	$N = 8$	$N = 8$

TABLE VII
THE SOLUTIONS T, $\partial T/\partial x_1$ AND $\partial T/\partial x_2$ AT $(x_1, x_2) = (0.5, 0.5)$ FOR CASE 2 OF PROBLEM 1.

t	Analytical	$N = 6$	$N = 8$	$N = 10$	$N = 12$
			T		
0.0005	0.00011	0.00011	0.00011	0.00011	0.00011
1.0	0.21939	0.21905	0.21955	0.21953	0.21952
2.0	0.43879	0.43810	0.43911	0.43906	0.43904
3.0	0.65818	0.65716	0.65867	0.65859	0.65857
4.0	0.87758	0.87621	0.878229	0.87812	0.87809
5.0	1.09697	1.09527	1.09778	1.09765	1.09761
			$\partial T/\partial x_1$		
0.0005	0.00003	0.00003	0.00003	0.00003	0.00003
1.0	0.07709	0.07692	0.07709	0.07708	0.07708
2.0	0.15418	0.15384	0.15419	0.15417	0.15417
3.0	0.23128	0.23076	0.23129	0.23126	0.23125
4.0	0.30837	0.30768	0.30839	0.30835	0.30834
5.0	0.38546	0.38460	0.38549	0.38544	0.38542
			$\partial T/\partial x_2$		
0.0005	0.00007	0.00007	0.00007	0.00007	0.00007
1.0	0.15246	0.15196	0.15231	0.15229	0.15228
2.0	0.30492	0.30392	0.30462	0.30458	0.30457
3.0	0.45738	0.45589	0.45693	0.45688	0.45686
4.0	0.60984	0.60785	0.60924	0.60917	0.60915
5.0	0.76230	0.75981	0.76156	0.76147	0.76144

TABLE VIII
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $e = \tau E$ FOR CASE 3 OF PROBLEM 1.

N		6	8	10	12
τ		219.906	291.781	361.453	435.750
T	E	0.16845606	0.01074386	0.00066556	0.00055494
	e	37.044541	3.134857	0.240570	0.241814
$\frac{\partial T}{\partial x_1}$	E	0.16858928	0.01092946	0.00059684	0.00056974
	e	37.073837	3.189010	0.215730	0.248265
$\frac{\partial T}{\partial x_2}$	E	0.16871683	0.01109693	0.00029777	0.00030753
	e	37.101886	3.237877	0.107630	0.134007

TABLE IX
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $T, \partial T/\partial x_1, \partial T/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER e FOR CASE 3 OF PROBLEM 1.

	T	$\frac{\partial T}{\partial x_1}$	$\frac{\partial T}{\partial x_2}$
E	$N = 12$	$N = 12$	$N = 10$
e	$N = 10$	$N = 10$	$N = 10$

TABLE X
THE SOLUTIONS T, $\partial T/\partial x_1$ AND $\partial T/\partial x_2$ AT $(x_1, x_2) = (0.5, 0.5)$ FOR CASE 3 OF PROBLEM 1.

t	Analytical	$N = 6$	$N = 8$	$N = 10$	$N = 12$
			T		
0.0005	0.00030	0.00030	0.00030	0.00030	0.00030
1.0	0.48522	0.47706	0.48506	0.48550	0.48548
2.0	0.72783	0.69716	0.72637	0.72827	0.72823
3.0	0.72783	0.66030	0.72394	0.72833	0.72825
4.0	0.48522	0.36647	0.47777	0.48566	0.48556
5.0	0.00000	-0.18430	-0.01215	0.00027	0.00013
			$\partial T/\partial x_1$		
0.0005	0.00014	0.00014	0.00014	0.00014	0.00014
1.0	0.23654	0.23244	0.23634	0.23655	0.23655
2.0	0.35482	0.33969	0.35392	0.35485	0.35483
3.0	0.35482	0.32173	0.35274	0.35487	0.35484
4.0	0.23654	0.17856	0.23279	0.23664	0.23659
5.0	0.00000	-0.08980	-0.00592	0.00013	0.00006
			$\partial T/\partial x_2$		
0.0005	0.00030	0.00030	0.00030	0.000307	0.00030
1.0	0.49129	0.48265	0.49075	0.49119	0.49116
2.0	0.73693	0.70534	0.73489	0.73681	0.73677
3.0	0.73693	0.66804	0.73243	0.73687	0.73680
4.0	0.49129	0.37077	0.48337	0.49136	0.49126
5.0	0.00000	-0.18646	-0.01229	0.00027	0.00014

$$g(\mathbf{x}) = 1$$

and if it is isotropic then

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that there are four cases regarding the material, namely anisotropic inhomogeneous, anisotropic homogeneous, isotropic inhomogeneous and isotropic homogeneous material. We set $\overline{\psi} = 1$ and the boundary conditions are (see Figure 4)

$$\begin{array}{l} P = P(t) \text{ on side AB} \\ P = 0 \text{ on side BC} \\ T = 0 \text{ on side CD} \\ P = 0 \text{ on side AD} \end{array}$$

where $P(t)$ takes four forms

$$\begin{array}{rcl} P(t) = P_1(t) & = & 1 \\ P(t) = P_2(t) & = & 1 - \exp(-1.75t) \\ P(t) = P_3(t) & = & t/5 \\ P(t) = P_4(t) & = & 0.16t(5-t) \end{array}$$

Therefore the system is geometrically symmetric about $x_1 = 0.5$. We use $N = 12$ for all cases of this problem.

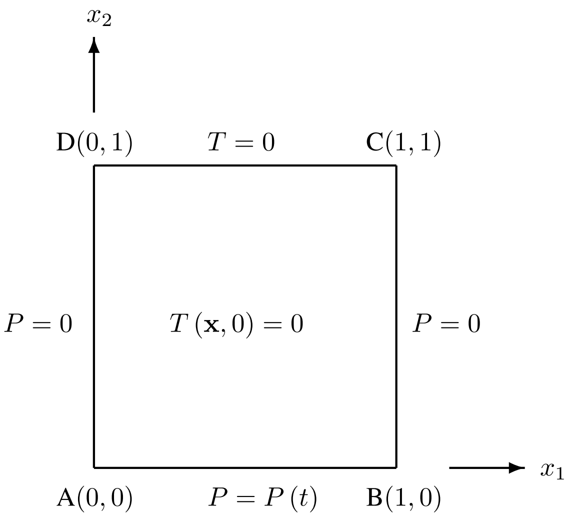

Fig. 4. The boundary conditions for Problem 2.

The results are shown in Figures 5, 6 and 7. Figure 5 depicts solution T at points $(0.2, 0.5), (0.8, 0.5)$ when the material under consideration is an isotropic homogeneous material. It can be seen that the values of T at point $(0.2, 0.5)$ coincide with those at point $(0.8, 0.5)$. This is to be expected as the system is symmetrical about $x_1 = 0.5$ when the material is isotropic homogeneous. However, if the material is anisotropic homogeneous the values of T at point $(0.2, 0.5)$ do not coincide with those at point $(0.8, 0.5)$. See Figure 6. This means anisotropy gives effect on the values of T. Similarly, if the material is isotropic inhomogeneous (see Figure 7) the values of T at point $(0.2, 0.5)$ differ from those at point $(0.8, 0.5)$. This indicates that inhomogeneity also gives effect on the values of T.

In addition, Figures 5, 6 and 7 show that the trends of T values (as the time t changes) follow the time variation of $P(t)$ except for the form of $P(t) = 1$. This is to be expected as $P(t)$, acting as the boundary condition on side AB, is the only time-dependent quantity for the system, and the coefficients $\kappa_{ij}(\mathbf{x}), \beta^2(\mathbf{x}), \psi(\mathbf{x})$ are time independent. Moreover, as shown in Figures 5 and 7, it is also expected that the values of T for the cases of $P_1(t) = 1$ and $P_2(t) = 1 - \exp(-1.75t)$ tend to approach same steady state solution as t increases. Both functions $P_1(t) = 1$ and $P_2(t) = 1 - \exp(-1.75t)$ will converge to 1 as t gets bigger.

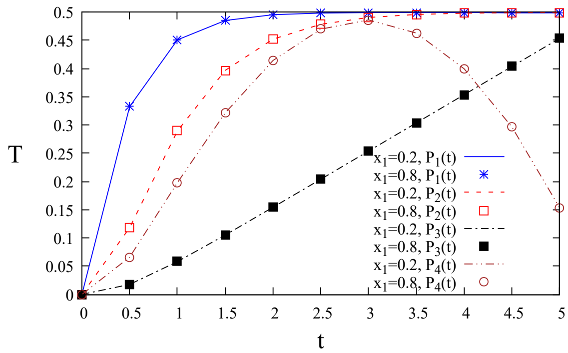

Fig. 5. Solution T at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of isotropic homogeneous material.

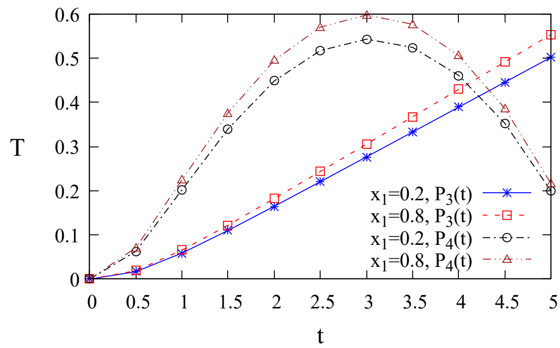

Fig. 6. Solution T at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of anisotropic homogeneous material.

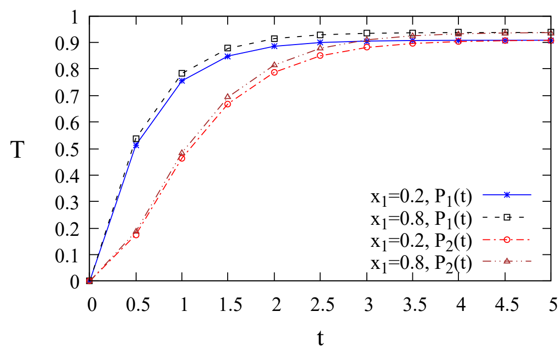

Fig. 7. Solution T at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of isotropic inhomogeneous material.

V. Conclusion

The authors have utilized a combination of Laplace transform and standard boundary element method to solve initial boundary value problems for anisotropic quadratically graded materials governed by the Helmholtz type equation (1). This method does not involve time variable fundamental solution, making it easy to implement and accurate. On the other hand, methods with time variable fundamental solution may produce less accurate solutions due to singular time points and round-off error propagation. In order to use the boundary integral equation, the boundary conditions in the time variable t need to be Laplace transformed, highlighting the importance of accurate numerical Laplace transform inversion. The approach has been applied to quadratically graded materials where the coefficients only depend on the spatial variable with the same inhomogeneity function. The authors suggest the extension of this study to cases where the coefficients depend on different gradation functions varying with the time variable t.

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