On the Infinite Growth of Solutions to Second Order Complex Differential Equation

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***Abstract*—By a well known result of G. Gundersen, if the complex differential equation $f'' + A(z)f' + B(z)f = 0$ with the coefficients satisfying $\rho(A) < \rho(B)$, where $\rho(*)$ denotes the growth order of an entire function $*$, then the nontrivial solutions of this equation are of infinite order. Moreover, there exist examples which show that if $\rho(A) = \rho(B)$, then this equation can have a finite order solution. In this paper we discuss the remained case, if $\rho(A) > \rho(B)$ whether the solutions have infinite order. In fact, we prove that if $\rho(A) > \rho(B) > 0$ and $A(z)$ satisfies three conditions respectively, then every nontrivial solution of this equation has infinite growth order. The three conditions are (1) $A(z)$ is a nontrivial solution of $w'' + P(z)w = 0$, where $P(z)$ is a polynomial; (2) $A(z)$ is an entire function of exponential growth; (3) $A(z)$ is a completely regular growth entire function.**

***Index Terms*—entire function, infinite order, regular growth, complex differential equation.**

I. Introduction

Nevanlinna theory is an important tool of value distribution in complex analysis, and it is later used in complex differential equations to study the growth of solutions and the distribution of zeros. In fact, it's the value distribution of solutions to complex differential equations. In this respect, the important literature we cite are [9], [16]. In this paper, we will follow the basic notations and theorems in these literature. Below we first give the definitions of the growth order $\rho(f)$ and the lower order $\mu(f)$ of entire function f, they are the important concepts in this paper.

$$\begin{aligned}\rho(f) &= \limsup_{r\to+\infty} \frac{\log^+ T(r,f)}{\log r} \\ &= \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},\end{aligned}$$

$$\begin{aligned}\mu(f) &= \liminf_{r\to+\infty} \frac{\log^+ T(r,f)}{\log r} \\ &= \liminf_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},\end{aligned}$$

where $T(r, f)$ is the characteristic function and $M(r, f)$ is the maximum modulus of entire function f in a circle $|z| < r$. Moreover,

$$\log^+ x = \max\{\log x, 0\}.$$

The research on the order of the solutions of complex differential equations is always a classic content in the field of complex differential equation. The main purpose of this paper is to study the growth of the solutions of second order linear differential equations

$$f'' + A(z)f' + B(z)f = 0, \quad (1)$$

where the coefficients $A(z)$ and $B(z)$ are entire functions. Because the coefficient functions of this equation are entire, we know that that the solutions to it are also entire functions. In particular, if $B(z)$ is a transcendental entire function, and f_1, f_2 are two independent solutions to this equation, then at least one of these two solutions is of infinite order, see [7]. In this case, there really exists a solution with finite order. For example, $f(z) = e^z$ is a finite order (with $\rho(e^z) = 1$) solution of differential equation

$$f'' + e^{-z}f' - (e^{-z} + 1)f = 0.$$

Then, the researchers think about the conditions under which each of its nontrivial solutions can be guaranteed to be infinite order. Actually, it's a constraint on the coefficient functions. The question becomes what conditions should the coefficients $A(z)$, $B(z)$ satisfy to ensure that every nontrivial solution of equation (1) is of infinite order? Many scholars have made research results in this aspect (for example, see [7], [9], [17]) and the main theorems related to this paper are listed below.

Theorem I.1. *If $A(z)$ and $B(z)$ are nonconstant entire functions, and one of the following additional conditions is satisfied:*

1) *$\rho(A) < \rho(B)$, see [4];*
2) *$A(z)$ is a polynomial and $B(z)$ is transcendental, see [4];*
3) *$\rho(B) < \rho(A) \le \frac{1}{2}$, see [6],*

then every nontrivial solution f of equation (1) *has infinite growth order.*

Although research in this area has yielded many results, several open questions remain unanswered [5], [12]. In this article we continue to discuss the unresolved case of this classic problem. That is, under the condition $\rho(A) > \rho(B)$, what other conditions must $A(z)$ and $B(z)$ satisfy so that the nontrivial solutions to this equation are of infinite order. In addition, we limit the growth order of coefficients function satisfying

$$\max\{\rho(A), \rho(B)\} < \infty$$

when considering problems in this paper, otherwise, the question becomes meaningless.

In the recent literature, a new idea has been introduced to the study of this problem. That is, suppose that the coefficient function $A(z)$ is a nontrivial solution to the following second order differential equation,

$$w'' + P(z)w = 0, \quad (2)$$

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where

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_n \neq 0,$$

another coefficient function $B(z)$ satisfies additional conditions such that the growth order of every nontrivial solution of equation (1) is infinite. The results associated with this method are listed below.

Theorem I.2. *Suppose that $A(z)$ is a nontrivial solution of* (2)*, and $B(z)$ is a transcendental entire function satisfying any one of the following additional hypotheses:*

1) $\rho(B) < 1/2$*, see [15];*
2) $\mu(B) < 1/2$ *and* $\rho(A) \neq \rho(B)$*, see [11];*
3) $\mu(B) < \frac{1}{2} + \frac{1}{2(n+1)}$ *and* $\rho(A) \neq \rho(B)$*, see [14];*

then every nontrivial solution f of equation (1) *has infinite order.*

Our first conclusion is related to the above theorem, we still assume that the function coefficient $A(z)$ satisfies equation (2), and add another condition $0 < \rho(B) < \rho(A)$, then the conclusion of the above theorem I.2 still holds.

Theorem I.3. *Let $A(z)$ be a nontrivial solution of* (2)*. If $B(z)$ is an entire function with $0 < \rho(B) < \rho(A)$, then every nontrivial solution of* (1) *is of infinite order.*

Remark 1. Since $A(z)$ satisfies the equation (2), its growth order is $\rho(A) = \frac{n+2}{2}, n = 0, 1, 2, \cdots$, see [7]. So it's possible that the growth order of $B(z)$ in this theorem is greater than $1/2$, which is obviously an extension of Theorem I.2.

Remark 2. For the case of $\rho(B) < \rho(A)$ and $\rho(A) > \frac{1}{2}$, Gundersen [4] gave an example as following, which shows that there exists a nontrivial solution of (1) with finite order. Let Q be any nonconstant polynomial and $B \not\equiv 0$ be any entire function with $\rho(B) < \deg(Q)$, let f be any antiderivative of e^Q that satisfies $\lambda(f) = \deg(Q)$, where $\lambda(f)$ denote the exponent of convergence of the zero-sequence of f, and set

$$A = -Q' - Bfe^{-Q}.$$

Then $\rho(B) < \rho(A) = \deg(Q)$, and

$$f'' + A(z)f' + B(z)f = 0.$$

We will show that this example does not satisfy the condition of Theorem I.3. In fact $A(z)$ in this example is not a solution of (2). Set $g(z) = e^Q$, then $f = \int g(z)dz$ is a solution of this equation. By calculation, we have

$$\begin{aligned} A'' &= -Q''' - B''fg^{-1} - 2B'(1 - Q'fg^{-1}) \\ &\quad -B[-Q' + fg^{-1}((Q')^2 - Q'')]. \end{aligned}$$

If $A(z)$ satisfies equation (2), substituting A and A'' into (2) we have

$$\begin{aligned} &fg^{-1}\{B'' - 2Q'B' + [(Q')^2 - Q'' + P]B\} \\ &+(2B' - Q'B + Q''' + PQ') = 0. \end{aligned}$$

Since $\rho(fg^{-1}) > \rho(B)$, by [16, Theorem 1.50] we have

$$B'' - 2Q'B' + [(Q')^2 - Q'' + P]B = 0,$$

$$2B' - Q'B + Q''' + PQ' = 0.$$

By the second equality above and Wiman-Valiron theory [9, Chapter 3], we have $\rho(B) = \deg(Q)$, which contradicts

$$\rho(B) < \rho(A) = \deg(Q).$$

This example also shows that the condition on $A(z)$ in Theorem I.3 can not be removed.

Remark 3. Clearly, $A(z) = e^{-z}$ is a solution of (2) for $P(z) = -1$, and the equation

$$f'' + A(z)f' - (e^{-z} + 1)f = 0$$

has a solution $f = e^z$. This shows that $\rho(A)$ can not be equal to $\rho(B)$ in Theorem I.3.

Remark 4. Frei [1] proved that

$$f'' + e^{-z}f' - n^2 f = 0$$

has a nontrivial solution of finite order, where n is a positive integer. This example shows that $\rho(B) > 0$ in Theorem I.3 is sharp.

Notice that the exponential function $e^{p(z)}$, where

$$p(z) = \alpha_n z^n + \cdots + +\alpha_0$$

is a polynomial of degree $n \geq 1$. For this function, we divide the complex plane into $2n$ equal open sectors by the rays

$$\arg z = -\frac{\arg \alpha_n}{n} + (2j - 1)\frac{\pi}{2n},$$

$$j = 0, 1, 2 \cdots, 2n - 1.$$

In each of these sectors, $e^{p(z)}$ either blows up exponentially or decays exponentially to zero. Moreover, these two cases occur in turn for any two adjacent sectors. Inspired by this property, the authors of [8] introduced a more general class of transcendental entire function, called function class $\mathcal{A}$.

Definition I.4. *Suppose that $A(z)$ is a transcendental entire function, its order $\rho(A)$ is equal to $\mu(A)$ and both are finite. Moreover, $\delta_A(\theta)$ is a real-valued function, which is defined on $[0, 2\pi)$ and continuous outside an exceptional set F of finitely many points. Further, let c, d be positive constants. Then for any given $\theta \in [0, 2\pi) \setminus F$, there are a constant τ, and positive constants $R = R(\theta)$ and $M = M(\theta)$ such that when $|z| = r > R$,*

$$\begin{array}{ll} (\mathcal{A}_1) & |A(re^{i\theta})| \geq \exp\{c\delta_A(\theta)r^d\} \ \text{ if } \ \delta_A(\theta) > 0, \\ (\mathcal{A}_2) & |A(re^{i\theta})| \leq Mr^{\tau} \ \text{ if } \ \delta_A(\theta) < 0, \end{array}$$

where $\tau < 2(\rho(A) - 1)$. Especially, for any $\theta \in [0, 2\pi) \setminus F$ with $\delta_A(\theta) > 0$, there exists $l = l(\theta) > 0$ such that $\sup\{R(\widetilde{\theta}), \widetilde{\theta} \in (\theta - l, \theta + l)\} < \infty$. The class $\mathcal{A}$ consists of those functions $A(z)$ that satisfy these above properties.

Since $A(z)$ is transcendental, the first growth case $(\mathcal{A}_1)$ in the above definition must happen. Obviously, exponential function $e^{p(z)}$ belongs to this function class $\mathcal{A}$, and it also contains exponential polynomials. For example,

$$A(z) = a_k(z)e^{kp(z)} + \cdots + a_s e^{sp(z)}$$

is a polynomial of $e^{p(z)}$ with polynomial coefficients, where $s < k$ are positive integers. Moreover, Mittag-Leffler function [2, p.83]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha^{-1}n + 1)}, \quad 0 < \alpha < \infty$$

has the growth property $(\mathcal{A}_1)$ in the sector

$$|\arg z| \le \frac{\pi}{2\alpha}$$

and property $(\mathcal{A}_2)$ in sector

$$\arg z \in (-\pi, \pi] \setminus \Big(-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\Big)$$

for $\alpha > 1/2$.

Let's assume that the coefficient function $A(z)$ belongs to the above function class $\mathcal{A}$, we have a result as follow.

Theorem I.5. *Let $A(z)$ belong to function class $\mathcal{A}$. If $B(z)$ is an entire function with $0 < \rho(B) < \rho(A)$, then every nontrivial solution of* (1) *is of infinite order.*

Finally, we give the concept of completely regular growth. If the function $\rho(r)$ is positive, and when r is sufficiently large, $\rho(r)$ is differentiable with respect to r. Moreover, it satisfies

$$\lim_{r\to\infty} \rho(r) = \rho \in (0, \infty), \qquad \lim_{r\to\infty} \rho'(r) r \log r = 0,$$

then we call $\rho(r)$ is a proximate order, see [2, Section 2, Chapter 2]. In particular, for an entire functions $A(z)$ whose growth order satisfies $0 < \rho(A) < \infty$, it always has a proximate order. For an entire function $A(z)$ with growth order $\rho(A)$, its indicator $h(\theta)$ for proximate order $\rho(r)$ is defined as

$$h(\theta) = \limsup_{r\to\infty} \frac{\log |A(re^{i\theta})|}{r^{\rho(r)}}, \tag{3}$$

where $\rho(r) \to \rho$ as $r \to \infty$. If there exist disks $D(a_k, s_k)$ satisfying

$$\sum_{|a_k|\le r} s_k = o(r) \tag{4}$$

such that

$$\log |A(re^{i\theta})| = h(\theta) r^{\rho(r)} + o(r^{\rho(r)}), \tag{5}$$

where

$$re^{i\theta} \notin \bigcup_k D(a_k, s_k)$$

as $r \to \infty$, uniformly in θ, then $A(z)$ is said to be completely regular growth (in the sense of Levin and Pfluger) [10, p.139-140]. In particular, if the set $I = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$ is of zero Lebesgue linear measure, then $A(z)$ either blows up or decays exponentially in the sectors which are contained in the complex plane excepted some rays. We introduce the property of completely regular growth to the coefficients of equation (1), and give the third main conclusion as follow.

Theorem I.6. *Let $A(z)$ be a completely regular growth entire function with the set $I = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$ of zero Lebesgue linear measure. If $B(z)$ is an entire function with $0 < \rho(B) < \rho(A)$, then every nontrivial solution of* (1) *is of infinite order.*

Below we will only give a detailed proof of theorem 1.3, other theorems can be proved in the same way, just by replacing the property of the coefficient $A(z)$ with the corresponding property in the theorem.

II. Proof of Theorem I.3

In order to prove Theorem I.3, we need an auxiliary theorem, which describes some properties of solutions to the equation (2). Before we state this theorem, let's give you some notation. Let $\alpha < \beta$ satisfy $\beta - \alpha < 2\pi$ and $r > 0$. Denote

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\},$$

$$S(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta, |z| < r\}$$

and $\overline{S}(\alpha, \beta, r)$ denotes the closure of $S(\alpha, \beta, r)$. Let $A(z)$ be an entire function with growth order $\rho(A) \in (0, \infty)$. If for any $\theta \in (\alpha, \beta)$, the formula

$$\lim_{r\to\infty} \frac{\log\log |A(re^{i\theta})|}{\log r} = \rho(A)$$

holds, we say that $A(z)$ blows up exponentially in $S(\alpha, \beta)$. On the contrary, if for any $\theta \in (\alpha, \beta)$, the formula

$$\lim_{r\to\infty} \frac{\log\log |A(re^{i\theta})|^{-1}}{\log r} = \rho(A)$$

holds, we say that $A(z)$ decays to zero exponentially in $S(\alpha, \beta)$. The following lemma plays a crucial role in the proof of the theorem I.3, which comes from the literature [7, Chapter 7.4]. The proof of this lemma used a method called asymptotic integration.

Lemma II.1. *Supopose that A is a nontrivial solution of second order complex differential equation $w'' + P(z)w = 0$, where*

$$P(z) = a_n z^n + \cdots + a_0, a_n \ne 0$$

is a polynomial. Set

$$\theta_j = \frac{2j\pi - \arg(a_n)}{n + 2}$$

and

$$S_j = S(\theta_j, \theta_{j+1}),$$

where

$$j = 0, 1, 2, \cdots, n + 1$$

and

$$\theta_{n+2} = \theta_0 + 2\pi.$$

Then A has the following properties.

1) *In each sector S_j, A either blows up or decays to zero exponentially.*
2) *If, for some j, A decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for A to blow up in many adjacent sectors.*
3) *If A decays to zero in S_j, then A has at most finitely many zeros in any closed sub-sector within*

$$S_{j-1} \cup S_j \cup S_{j+1}.$$

4) *If A blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, A has infinitely many zeros in each sector*

$$S(\theta_j - \varepsilon, \theta_j + \varepsilon),$$

and furthermore, as $r \to \infty$,

$$n(\overline{S}(\theta_j-\varepsilon,\theta_j+\varepsilon,r),0,A) = (1+o(1))\frac{2\sqrt{|a_n|}}{\pi(n+2)}r^{\frac{n+2}{2}},$$

where

$$n(\overline{S}(\theta_j-\varepsilon,\theta_j+\varepsilon,r),0,A)$$

is the number of zeros of A in the region

$$\overline{S}(\theta_j-\varepsilon,\theta_j+\varepsilon,r).$$

The following result which was proved by Gundersen [4, Theorem 3] shows the asymptotic properties of finite order solutions of equation (1).

Lemma II.2. *Let $A(z)$ and $B(z)(\not\equiv 0)$ be two entire functions such that for real constants $\alpha,\beta,\theta_1,\theta_2$, where $\alpha>0,\beta>0$ and $\theta_1<\theta_2$,*

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\beta}\}$$

and

$$|B(z)| \leq \exp\{o(1)|z|^{\beta}\}$$

as $z\to\infty$ in

$$\overline{S}(\theta_1,\theta_2) = \{z : \theta_1 \leq \arg z \leq \theta_2\}.$$

Let $\varepsilon>0$ be a given small constant, and let

$$\overline{S}(\theta_1+\varepsilon,\theta_2-\varepsilon) = \{z : \theta_1+\varepsilon \leq \arg z \leq \theta_2-\varepsilon\}.$$

If f is a nontrivial solution of (1) with $\rho(f)<\infty$, then the following conclusions hold.

1) *There exists a constant $b(\neq 0)$ such that $f(z)\to b$ as $z\to\infty$ in $\overline{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)$. Furthermore,*

$$|f(z)-b| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}$$

as $z\to\infty$ in $\overline{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)$.

2) *For each integer $k\geq 1$,*

$$|f^{(k)}(z)| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}$$

as $z\to\infty$ in $\overline{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)$.

In order to describe the following lemma of the estimation of modulus of the logarithmic derivative of transcendental meromorphic functions, we first give two concepts of the Lebesgue linear measure and the logarithmic measure of a measurable set. The Lebesgue linear measure of a set $E\subset[0,\infty)$ is denoted by $m(E)=\int_E dt$, and the logarithmic measure of a set $E\subset[1,\infty)$ is denoted by $m_l(E)=\int_E \frac{dt}{t}$. The following lemma related to logarithmic derivatives, which is very important in the proof of our theorem, comes from G. Gundersen's article [3].

Lemma II.3. *[3] Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon>0$ be a given real constant, and let k and j be two integers such that $k>j\geq 0$. Then there exists a set $E\subset(1,\infty)$ with $m_l(E)<\infty$, such that for all z satisfying $|z|\notin(E\cup[0,1])$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \tag{6}$$

Lemma II.4. *[13, Corollary 2.3.6] If $g(z)$ is an entire function with $0<\rho(g)<\infty$, then there exists an angular domain $S(\theta_1,\theta_2)$ with $\theta_2-\theta_1\geq\frac{\pi}{\rho(g)}$ such that*

$$\limsup_{r\to\infty}\frac{\log\log|g(re^{i\theta})|}{\log r} = \rho(g) \tag{7}$$

for any $\theta\in(\theta_1,\theta_2)$.

Proof of Theorem I.3: Suppose that there is a nontrivial solution f of (1) with finite order. Applying Lemma II.4 to entire coefficient $B(z)$, then there exists a sector $S_B(\theta_1,\theta_2)$ with

$$\theta_2-\theta_1 \geq \frac{\pi}{\rho(B)}$$

such that

$$\limsup_{r\to\infty}\frac{\log\log|B(re^{i\theta})|}{\log r} = \rho(B) \tag{8}$$

for any $\theta\in S_B$. From Lemma II.1, we denote the union of sectors where A blows up by S_A^+ and the union of sectors where A decays by S_A^-. We split into two cases for the proof.

Case 1. Assume that

$$S_B\cap S_A^- \neq \emptyset.$$

For any

$$z=re^{i\theta}\in S_B\cap S_A^-,$$

we have

$$\lim_{r\to\infty}\frac{\log\log|A(re^{i\theta})|^{-1}}{\log r} = \frac{n+2}{2}. \tag{9}$$

By Lemma II.3, there exists a set $E\subset(1,\infty)$ with finite logarithmic measure such that for all z satisfying

$$|z|=r\notin E\cup[0,1],$$

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{2\rho(f)}, k=1,2. \tag{10}$$

Then for

$$z=re^{i\theta}\in S_B\cap S_A^-$$

satisfying $r\notin E\cup[0,1]$ with $r\to\infty$, we have

$$\begin{aligned} |B(z)| &\leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right| \\ &\leq (1+o(1))r^{2\rho(f)}, \end{aligned} \tag{11}$$

which contradicts (8) since $\rho(B)>0$.

Case 2. Assume that

$$S_B\cap S_A^- = \emptyset,$$

then

$$S_B\cap S_A^+ \neq \emptyset.$$

By Lemma II.1, for any $z=re^{i\theta}\in S_B\cap S_A^+$, we have

$$\lim_{r\to\infty}\frac{\log\log|A(re^{i\theta})|}{\log r} = \rho(A) = \frac{n+2}{2}. \tag{12}$$

Since $\rho(B)>0$, for any given small positive $\varepsilon<\frac{\rho(B)}{4}$, we can take positive number β close to $\rho(A)$ such that

$$\rho(B)-\varepsilon > \rho(A)+\varepsilon-(\beta-\varepsilon),$$

that is $\rho(B) > \rho(A) - \beta + 3\varepsilon$ and

$$\rho(A) - \varepsilon > \beta > \rho(B) + \varepsilon,$$

we have

$$\begin{aligned} |A(z)| &\geq \exp\{|z|^{\rho(A)-\varepsilon}\} \\ &\geq \exp\{(1+o(1))\alpha|z|^{\beta}\}, \end{aligned} \tag{13}$$

$$\begin{aligned} |B(z)| &\leq \exp\{|z|^{\rho(B)+\varepsilon}\} \\ &\leq \exp\{o(1)|z|^{\beta}\} \end{aligned} \tag{14}$$

as $z \to \infty$ in $S_B \cap S_A^+$. Then by Lemma II.2, there exists constant $b \neq 0$ such that

$$|f(z) - b| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\} \tag{15}$$

and

$$|f^{(k)}(z)| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}, k = 1, 2, 3 \cdots \tag{16}$$

as $z \to \infty$ in a smaller sector which is contained in $S_B \cap S_A^+$, denoted by $\widetilde{S}$.

By (8) we can take a sequence $\{r_n\}$ tending to infinite such that

$$\lim_{r_n \to \infty} \frac{\log\log|B(r_n e^{i\theta})|}{\log r_n} = \rho(B) \tag{17}$$

for any $\theta \in S_B$. Then for any given $\varepsilon > 0$ and sufficiently larger r_n, we have

$$|B(r_n e^{i\theta})| \geq \exp\{r_n^{\rho(B)-\varepsilon}\} \tag{18}$$

for any $\theta \in S_B$. By equation (1) we have

$$|Bf| \leq |f''| + |Af'|. \tag{19}$$

Substituting (15), (16) and (18) into (19), we obtain

$$\begin{aligned} & (|b| - o(1)) \exp\{r_n^{\rho(B)-\varepsilon}\} \\ \leq\ & o(1) + \frac{\exp\{r_n^{\rho(A)+\varepsilon}\}}{\exp\{(1+o(1))\alpha r_n^{\beta}\}} \\ \leq\ & o(1) + \exp\{r_n^{\rho(A)-\beta+2\varepsilon}\} \end{aligned} \tag{20}$$

for $r_n e^{i\theta} \in \widetilde{S}$ as $r_n \to \infty$, which contradicts with the chosen β. Thus, the conclusion follows.

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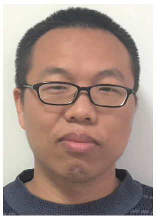

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