On Lambda-Ideal Statistically Convergent in 2-Normed Spaces over Non-Archimedean Fields

R. Sakthipriya and K. Suja*

Abstract—**This paper aims to examine the notion of λ-ideal statistical convergence and λ-ideal statistically Cauchy sequence over non-Archimedean 2-normed spaces. Moreover, we have also studied and proven some significant properties for λ-ideal statistically Cauchy sequence in 2-normed spaces to be λ-ideal statistically Cauchy with respect to $\|.\|_\infty$. And also we show that, λ-ideal statistical convergence in 2-normed spaces is λ-ideal statistical pre-Cauchy in 2-normed spaces over non-Archinedean field. Through this paper, $\mathcal{K}$ symbolizes a complete, non-trivially valued, non-Archinedean field.**

Index Terms—**Ideal, statistical convergence, 2-normed spaces, non-Archimedean field.**

I. Introduction

THE notion of statistical convergence was first established by Steinhaus[23] in 1951 but the expansion of convergence of real sequences to statistical convergence was given by Fast[7] and Schoenberg[22]. Statistical convergence can find its applications in numerous fields of mathematics like measure theory, approximation theory, trigonometric series and summability theory. In the case of real sequences, Fridy[8],[9] obtained the statistical analogue of the Cauchy criterion for convergence. Over the year under different names statistical convergence and the concept of summability method has been examined in numerous fields of mathematics by many others see for instance[15], [24], [25].

A sequence $x = \{x_k\}$ is defined as statistically convergent to '$\mathcal{L}$', if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n; n \in \mathbb{N} : |x_k - \mathcal{L}| \ge \varepsilon\}| = 0$$

$$stat - \lim_{k\to\infty} \{x_k\} = \mathcal{L}.$$

Furthermore, Mursaleen[19] established the study of λ-statistical convergence as a generalization of the statistical convergence and found its connection between statistical convergence and summability theory.

Let $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive integers tending to ∞. So that $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$ where $n \in \mathbb{N}$, where $I_n = \{n - \lambda_n + 1, n\}$.

More particularly, Connor[4] introduced an interesting concept of statistical pre-Cauchy sequences, and it was proved that statistical convergence is always statistically pre-Cauchy. Also, under specific conditions, statistical pre-Cauchy is statistically convergent of a sequence.

The theory of I-convergence was initiated by Kostyrko et al.[16]. Later on, Kostyrko et al.[17] gave some of the basic properties of I-convergence and deals with I-limit points. In recent years, I-statistical convergence by using ideals was introduced by Das et al.[6]. We now define the basic definitions and notions.

A non-empty subset $\mathfrak{I}$ of the subset of $\mathbb{R} \subset \mathbb{N}$ is an ideal in $\mathbb{R}$, if it satisfies the axioms listed below:

(i) $A, B \in \mathfrak{I}$, which implies $A \cup B \in \mathfrak{I}$,
(ii) $A \in \mathfrak{I}, B \in \mathbb{R}, B \subset A$, which implies $B \in \mathfrak{I}$.

An ideal is said to be non-trivial if $\mathbb{N} \notin \mathfrak{I}$ and $\mathfrak{I} \neq \phi$, and if $\{x\} \in \mathfrak{I}$ for each $x \in \mathbb{N}$, then it is said to be an admissible ideal.

A non-empty class $\mathcal{F} \subset X$ is said to be a filter in X, if it satisfies the axioms listed below:

(i) $\Phi \in \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$, which implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subset B$ which implies $B \in \mathcal{F}$.

The study of 2-normed spaces was proposed by Gahler[10] in the 1960s and it has been extended widely in various fields by many researchers[3], [5], [11], [12], [14], [20], [21]. Subsequently, Gurdal and Pehlivan[13] investigated statistical convergence in 2-normed spaces, and proved some significant properties of a real number sequence in 2-normed spaces. Also, studied the concept of a statistically Cauchy sequence in 2-normed spaces and obtained various results related to it. Let X be a linear space with a dimension greater than 1 and $\|.,.\|$ be a non-negative real valued function on $X \times X$ meets the axioms listed below:

(i) $\|x, y\| = 0$ if and only if x and y are not linearly independent vectors,
(ii) $\|x, y\| = \|y, x\|$ for all x, y in X,
(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all α is real,
(iv) $\|x + y, w\| \le \|x, w\| + \|y, w\|$ for all $x, y, w \in X$.

Therefore, $(X, \|.,.\|)$ is defined as linear 2-normed spaces.

[2] Throughout this article, $\mathcal{K}$ denotes a non-Archimedean field that is complete, non-trivially valued, and meets the axioms listed below:

(i) $|x| \ge 0$ and $|x| = 0$ iff $x = 0$.
(ii) $|xy| = |x||y|$.
(iii) $|x + y| \le \max\{|x|, |y|]\}$ for all $x, y \in \mathcal{K}$.

II. Preliminaries

In this section we now propose the primary definitions of this article.

Definition 2.1: [1], [18] Let X be a 'd' dimensional vector space, where $2 \le d < \infty$ over a non-Archimedean valuation $|.|$ with a valued fields $\mathcal{K}$. A non-Archimedean 2-norm is said to be a mapping from $\|.,.\| : X \times X \to [0, \infty)$ if it satisfies the axioms listed below:

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R. Sakthipriya is a Research Scholar in Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603 203, Tamil Nadu, INDIA. (e-mail: sr1398@srmist.edu.in).

K. Suja is an Assistant Professor in Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603 203, Tamil Nadu, INDIA. (corresponding author; phone: +91-9443705466; e-mail: sujak@srmist.edu.in).

(i) $\|x, y\| = 0$ iff x and y are not linearly independent vectors,
(ii) $\|x, y\| = \|y, x\|$,
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for all $\alpha \in \mathcal{K}$,
(iv) $\|x, y + w\| \leq \max\{\|x, w\|, \|y, w\|\}$; for all $x, y, w \in X$.

Therefore $(X, \|.,.\|)$ is defined as non-Archimedean 2-normed spaces.

Definition 2.2: A sequence $\{x_k\}$ of X is λ-$\mathfrak{I}$ statistically convergent to '$\mathcal{L}$' in 2-normed spaces, if for every $\epsilon > 0$ and for every non-zero $w \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; n \in \mathbb{N} : \|x_k - \mathcal{L}, w\| \geq \varepsilon \right\} \in \mathfrak{I} \right| = 0$$

we write,

$$\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k - \mathcal{L}, w\| = 0$$

where '$\mathcal{L}$' is the $\mathfrak{I}$-limit of the sequence $\{x_k\}$.

Definition 2.3: A sequence $\{x_k\}$ of X is λ-$\mathfrak{I}$ statistically Cauchy in 2-normed spaces, if for any $\epsilon > 0$, then there exist $n \in \mathbb{N}$, for every non-zero $w \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; n \in \mathbb{N} : \|x_{k+1} - x_k, w\| \geq \varepsilon \right\} \in \mathfrak{I} \right| = 0.$$

III. Example

Example 3.1: Let $X = \mathbb{R}^2$, is defined with 2-norm by $\|x, y\|$

$$\|x, y\| = max\{\|x_1 y_1, x_2 y_2\|\}$$

where, $x = (x_1, x_2), y = (y_1, y_2)$.

Then $\|.,.\|$ is 2-norm on $\mathbb{R}^2$.

Example 3.2: Let $x = \{x_k\}$ and $\lambda = \{\lambda_n\}$ defined as non-Archimedean valuation to be 2-adic,

$$x_k = \begin{cases} \dfrac{k-1}{k^2+1}, & if\ k\ is\ a\ perfect\ square, \\ 0, & otherwise. \end{cases}$$

The sequence are

$$\{0, 0, 0, 1, 0, 0, 0, 0, \frac{1}{4}, 0, 0, 0, \cdots\}.$$

Then the sequence is λ-ideal statistically convergent to 0. Therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; k \leq n : |x_k - 0| \geq \varepsilon \right\} \in \mathfrak{I} \right| = 0.$$

Thus

$$\mathfrak{I} - stat_\lambda - \lim_{k\to\infty} |x_k - \mathcal{L}| = 0.$$

Example 3.3: Let $x = \{x_k\}$, $\mathfrak{I}$ is an admissible ideal and $\lambda = \{\lambda_n\}$ defined by

$$x_k = \begin{cases} 1, & if\ k = 2n,\ n = 1, 2, \cdots, \\ 0, & if\ k \neq 2n,\ otherwise. \end{cases}$$

Hence,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; k \leq n : \|x_k - 1, w\| \geq \varepsilon \right\} \in \mathfrak{I} \right| \leq \lim_{n\to\infty} \frac{\lambda_n + 1}{2\lambda_n} = 0.$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; k \leq n : \|x_k - 0, w\| \geq \varepsilon \right\} \in \mathfrak{I} \right| \leq \lim_{n\to\infty} \frac{\lambda_n + 1}{2\lambda_n} = 0.$$

That is,

$$\mathfrak{I} - stat_\lambda - \lim_{k\to\infty} \|x_k, w\| = 1$$

and

$$\mathfrak{I} - stat_\lambda - \lim_{k\to\infty} \|x_k, w\| = 0.$$

Thus $x = \{x_k\}$ is λ-ideal statistically convergent to both 1 and 0.

IV. Main Result

This section shows the necessary and sufficient conditions for a sequence to be λ-ideal statistically convergent and λ-ideal statistically Cauchy over non-Archimedean 2-normed spaces.

Theorem 4.1: Let $\{x_k\}$ be a sequence in 2-normed spaces $(X, \|.,.\|)$, if $\mathcal{L}$ and $\mathcal{L}' \in X$, for every $w \in X$,
(i) $\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k - \mathcal{L}, w\| = 0$,
(ii) $\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k - \mathcal{L}', w\| = 0$.

Therefore $\mathcal{L} - \mathcal{L}' = 0$.

Proof: Let $\mathcal{L} - \mathcal{L}' = 0$ (i.e) $\mathcal{L} = \mathcal{L}'$, there exist a non-zero $w \in X$ such that $\mathcal{L} = \mathcal{L}'$ and w is linearly independent.

So, w exists as dimension of X, where $2 \leq d < \infty$, therefore for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|x_k - \mathcal{L}, w\| \geq \varepsilon\} \in \mathfrak{I}| = 0. \quad (1)$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|x_k - \mathcal{L}', w\| \geq \varepsilon\} \in \mathfrak{I}| = 0. \quad (2)$$

Now,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|\mathcal{L} - \mathcal{L}', w\| \geq \varepsilon\} \in \mathfrak{I}|$$

$$= \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|\mathcal{L} - x_k + x_k - \mathcal{L}', w\| \geq \varepsilon\} \in \mathfrak{I}|$$

$$\leq \max \left\{ \begin{array}{l} \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|x_k - \mathcal{L}, w\| \geq \varepsilon\} \in \mathfrak{I}|, \\ \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|x_k - \mathcal{L}', w\| \geq \varepsilon\} \in \mathfrak{I}| \end{array} \right\}$$

$$= 0. \quad \text{(Using (1) and (2))}$$

Thus, $\mathcal{L} - \mathcal{L}^{'} = 0$ (i.e) $\mathcal{L} = \mathcal{L}^{'}$. ■

Theorem 4.2: If $\mathfrak{I}-stat_\lambda \lim_{k\to\infty} \|x_k, w\| = \|x, w\|$ and $\mathfrak{I}-stat_\lambda \lim_{k\to\infty} \|y_k, w\| = \|y, w\|$.
Then

(i) $\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k + y_k, w\| = \|x + y, w\|$,
(ii) $\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|\alpha x_k, w\| = \|\alpha x, w\|$ for all $\alpha \in \mathcal{K}$.

Proof: (i) Let $\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k, w\| = \|x, w\|$ and $\mathfrak{I}-stat_\lambda \lim_{k\to\infty} \|y_k, w\| = \|y, w\|$ for every non-zero $w \in X$. Then

$$A_1(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|x_k - x, w\| \geq \varepsilon\} \in \mathfrak{I}| = 0,$$

$$A_2(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|y_k - y, w\| \geq \varepsilon\} \in \mathfrak{I}| = 0,$$

for all $w \in X$. Let

$$A(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k + y_k) - (x + y), w\| \geq \varepsilon\} \in \mathfrak{I}|.$$

To show that,

$$A(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k + y_k) - (x + y), w\| \geq \varepsilon\} \in \mathfrak{I}| = 0,$$

it is sufficient to prove that $A \subset A_1 \cup A_2$. Let $A_0 \in A$. Then

$$A_0(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_{k_0} + y_{k_0}) - (x + y), w\| \geq \varepsilon\} \in \mathfrak{I}| = 0. \quad (3)$$

Assume that $A_0 \in A_1 \cup A_2$. Then $A_0 \in A_1$ and $A_0 \in A_2$.

This implies,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_{k_0} - x), w\| \geq \varepsilon\} \in \mathfrak{I}| = 0. \quad (4)$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(y_{k_0} - y), w\| \geq \varepsilon\} \in \mathfrak{I}| = 0. \quad (5)$$

We achieve,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_{k_0} + y_{k_0}) - (x + y), w\| \geq \varepsilon\} \in \mathfrak{I}| \\
&\leq \max \left\{ \begin{array}{l} \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_{k_0} - x), w\| \geq \varepsilon\} \in \mathfrak{I}|, \\ \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(y_{k_0} - y), w\| \geq \varepsilon\} \in \mathfrak{I}| \end{array} \right\} \\
&= 0. \qquad \text{(Using (4) and (5))}
\end{aligned}$$

Hence (3) is True. Therefore, $A_0 \in A_1 \cup A_2$ that is $A \subset A_1 \cup A_2$.

(ii) $\mathfrak{I} - stat_\lambda \lim_{n\to\infty} \|\alpha x_k, w\| = \|\alpha x, w\|$ for all $\alpha \in \mathbb{K}$, and $\alpha \neq 0$.

Then

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; n \in \mathbb{N} : \|(x_k - x), w\| \geq \frac{\varepsilon}{|\alpha|} \right\} \in \mathfrak{I} \right|. \quad (6)$$

Now, we shall show that

$$\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|\alpha x_k, w\| = \|\alpha x, w\|$$

for all $\alpha \in \mathcal{K}$.

This is to prove that,

$$\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|\alpha x_k - \alpha x, w\| = 0.$$

This implies,

$$\lim_{n\to\infty} \tfrac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|\alpha x_k - \alpha x, w\| \geq \varepsilon\} \in \mathfrak{I}| = 0.$$

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|\alpha(x_k - x), w\| \geq \varepsilon\} \in \mathfrak{I}| \\
&= \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : |\alpha| \|(x_k - x), w\| \geq \varepsilon\} \in \mathfrak{I}| \\
&= \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : |\alpha| \|(x_k - x), w\| \geq \varepsilon\} \in \mathfrak{I}| \\
&= \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k - x), w\| \geq \frac{\varepsilon}{|\alpha|}\} \in \mathfrak{I}|.
\end{aligned}$$

Now using (6)

$$\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|\alpha x_k - \alpha x, w\| = 0$$

for all $\alpha \in \mathcal{K}$. ■

We suppose $\mathbb{X}$ to give d, where $2 \leq d < \infty$.
Let, $v = \{v_1, v_2, \cdots, v_d\}$ to be a basis for X.
We determine the norm $\|.\|_\infty$ on X by $\|x\|_\infty = max\{\|x, v_i\|\}$ for all $i = 1, 2, \cdots, d$.

Theorem 4.3: A sequence $\{x_k\} \in X$ λ-$\mathfrak{I}$ statistical convergence to $x \in X$ if and only if $\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k - x, v_i\| = 0$ for every $i = 1, 2, \cdots, d$.

Proof: Let $\{x_k\} \in X$ is λ-$\mathfrak{I}$ statistically convergent to $x \in X$.

By the definition of λ-$\mathfrak{I}$ statistically convergent. We have,

$$\mathfrak{I} - stat_\lambda \lim_{k\to\infty} \|x_k - x, w\| = 0$$

where $w = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d\}$ for all $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathcal{K}$.
Then

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, v_i\| = 0$$

for every $i = 1, 2, \cdots, d$.

Assume that

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, w\| = 0 \tag{7}$$

for every $i = 1, 2, \cdots, d$.
To show that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k - x), v_i\| \geq \varepsilon\} \in \Im| = 0. \tag{8}$$

Now, let us consider the 2-norm $\|(x_k - x), w\|$.

Also $v_i = \{v_1, v_2, \cdots, v_d\}$ is a basis of X,

$$\|(x_k - x), w\| = \|x_k - x, \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d\|.$$

Then we have,

$$\|(x_k - x), w\| \leq \max \left\{ \begin{array}{l} \|(x_k - x), \alpha_1 v_1\|, \\ \|(x_k - x), \alpha_2 v_2\|, \cdots, \\ \|(x_k - x), \alpha_d v_d\| \end{array} \right\}$$

$$\leq max\{|\alpha_i| \|(x_k - x), v_i\|\}.$$

By our assumption (8), we have

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, w\| = 0.$$

Thus

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k - x), v_i\| \geq \varepsilon\} \in \Im| = 0$$

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, v_i\| = 0$$

for every $i = 1, 2, \cdots, d$.

Conversely, suppose that,

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, v_i\| = 0$$

for every $i = 1, 2, \cdots, d$.

To show that

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, w\| = 0$$

for all $z \in X$.

Consider,
$\|(x_k - x), w\| = \|x_k - x, \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d\|$.

Then we have,

$$\|(x_k - x), w\| \leq \max \left\{ \begin{array}{l} \|(x_k - x), \alpha_1 v_1\|, \\ \|(x_k - x), \alpha_2 v_2\|, \cdots, \\ \|(x_k - x), \alpha_d v_d\| \end{array} \right\}$$

$$\leq max\{|\alpha_i| \|(x_k - x), v_i\|\}$$

which implies

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k - x), w\| \geq \varepsilon\} \in \Im|.$$

Therefore we have,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n; n \in \mathbb{N} : \|(x_k - x), w\| \geq \varepsilon\} \in \Im| \subseteq \lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; n \in \mathbb{N} : \{\|(x_k - x), v_i\|\} \geq \frac{\varepsilon}{|\alpha_1|} \right\} \in \Im \right|.$$

Hence the right-hand side be a member of ideal, thus the left hand side.

Then

$$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, w\| = 0$$

for all non-zero $w \in X$.

This completes the proof. ■

Theorem 4.4: Any λ-$\Im$ statistically Cauchy sequence $\{x_k\}$ in 2-normed spaces $(X, \|., .\|)$ is λ-$\Im$ statistical convergence iff any λ-$\Im$ statistically Cauchy sequence is λ-$\Im$ statistical convergence with respect to $\|.\|_\infty$.

Proof: Clearly λ-$\Im$ statistical convergence in 2-norm is equivalent to that in $\|.\|_\infty$.

$\Im - stat_\lambda \lim_{k\to\infty} \|x_k - x, w\| = 0$, for all $w \in X$ iff $\Im - stat_\lambda \lim_{k\to\infty} |x_k - x|_\infty = 0$. It suffices to prove that $\{x_k\}$ is λ-$\Im$ statistically Cauchy sequence, with respect to 2-norm iff it is λ-$\Im$ statistically Cauchy with respect to $\|.\|_\infty$.

Let $\{x_k\}$ is λ-$\Im$ statistical Cauchy sequence with respect to 2-normed spaces. Then there exist $n \in \mathbb{N}$. In such a way that for all $k, m \geq \mathbb{N}$. Now we have,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n; n \in \mathbb{N} : \|x_k - x_m, w\| \geq \varepsilon \right\} \in \Im \right|.$$

Consider, $\|x_k - x_m, w\| \geq \varepsilon$.
Now, we have $\|x_k - x_m, v_i\| \geq \varepsilon$ for all $i = 1, 2, \cdots, n$.

Hence, $max\|x_k - x_m, v_i\| \geq \varepsilon$ for all $i = 1, 2, \cdots, n$.

By definition, $\|x_k - x_m\|_\infty \geq \varepsilon$.

Therefore, $\{x_k\}$ λ-$\Im$ statistically Cauchy with respect to $\|.\|_\infty$.

■

Theorem 4.5: Let X be an 2-normed spaces and for $\{\lambda_n\} \in \Delta$. If as $\inf \frac{\lambda_n}{n} > 0$ as $n \geq \infty$.
Then, $stat(X) \subset stat_\lambda(X)$.

Proof: Assume that X is statistically convergent, then

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n; n \in \mathbb{N} : \|x_k - x_m, w\| \geq \varepsilon \right\} \in \Im \right| = 0.$$

Since, $\inf \frac{\lambda_n}{n} > 0$ as $n \geq \infty$

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n;n\in\mathbb{N}:\|x_k-x_m,w\|\ge\varepsilon\}\in\Im|$$
$$\ge\frac{1}{n}|\{k\in I_n;n\in\mathbb{N}: \|x_k-x_m,w\|\ge\varepsilon\}\in\Im|$$
$$\ge\frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k\in I_n;n\in\mathbb{N}: \|x_k-x_m,w\|\ge\varepsilon\}\in\Im|.$$

Consequently, $\{x_k\}\to\mathcal{L}\quad stat(X)$ implies $\{x_k\}\to\mathcal{L}\quad stat_\lambda(X)$.

Thus, $stat(X)\subset stat_\lambda(X)$. ■

Theorem 4.6: Let X be an 2-normed spaces and if $\{\lambda_n\}\in\Delta$ such that $\lim_{n\to\infty}\frac{\lambda_n}{n}=1$. Then, $stat(X)=stat_\lambda(X)$.

Proof: Since, $\lim_{n\to\infty}\frac{\lambda_n}{n}=1$, then for $\varepsilon>0$,

$$\frac{1}{n}|\{k\le n;n\in\mathbb{N}:\|x_k-x_m,w\|\ge\varepsilon\}\in\Im|$$
$$\le max\{\frac{1}{n}|\{k\le n-\lambda_n;n\in\mathbb{N}: \|x_k-x_m,w\|\ge\varepsilon\}\in\Im|\}.$$

$$\frac{1}{n}|\{k\in I_n;n\in\mathbb{N}:\|x_k-x_m,w\|\ge\varepsilon\}\in\Im|$$
$$\le max\{\frac{n-\lambda_n}{n}\frac{1}{n}|\{k\le n-\lambda_n;n\in\mathbb{N}: \|x_k-x_m,w\|\ge\varepsilon\}\in\Im|\}$$
$$\le max\{\frac{n-\lambda_n}{n}\frac{\lambda_n}{n}\frac{1}{n}|\{k\le n-\lambda_n;n\in\mathbb{N}: \|x_k-x_m,w\|\ge\varepsilon\}\in\Im|\}.$$

This is clear that if $\{x_k\}$ is λ-statistically convergent, then $\{x_k\}$ is statistically convergent. That is $stat(X)\supset stat_\lambda(X)$.

Also, since $\lim_{n\to\infty}\frac{\lambda_n}{n}=1$ implies that $\lim_{n\to\infty}\frac{\lambda_n}{n}>0$, by previous theorem we have $stat(X)\subset stat_\lambda(X)$.

Thus, $stat(X)=stat_\lambda(X)$. ■

In this section we have provided essential condition for a sequence to be λ-ideal statistical pre-Cauchy over non-Archimedean 2-normed spaces.

Definition 4.7: A sequence $x=\{x_k\}$ is λ-$\Im$ statistical pre-Cauchy sequence in 2-normed spaces, if for every $\varepsilon>0$ and for $w\in X$,

$$\lim_{n\to\infty}\frac{1}{\lambda_{n^2}}\left|\left\{k\in I_n;n\in\mathbb{N}:\|x_{k+1}-x_k,w\|\ge\varepsilon\right\}\in\Im\right|=0.$$

Theorem 4.8: An λ-$\Im$ statistical convergence in 2-normed spaces is λ-$\Im$ statistical pre-cauchy.

Proof: Let $\{x_k\}$ be $\lambda-\Im$ statistically convergent to '$\mathcal{L}$'. Let $\varepsilon>0$ be given. Therfore

$$A=\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n;n\in\mathbb{N}: \|x_k-\mathcal{L},w\|\ge\varepsilon\right\}\in\Im\right|.$$

Then for $n\in A^c$ where c stands for the complement.

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n;n\in\mathbb{N}:\|x_k-\mathcal{L},w\|\ge\varepsilon\right\}\in\Im\right|.$$
$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n;n\in\mathbb{N}:\|x_k-\mathcal{L},w\|<\varepsilon\right\}\in\Im\right|.$$

Writing

$$B_n=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in I_n;n\in\mathbb{N}:\|x_k-\mathcal{L},w\|<\varepsilon\}\in\Im|$$

we observe that for $k+1,k\in B_n$

$$\|x_{k+1}-x_k,w\|\le\max\left\{\|x_{k+1}-\mathcal{L},w\|,\|x_k-\mathcal{L},w\|\right\}<\varepsilon.$$

Hence

$$B_n\times B_n\subset\{\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k\in I_n;n\in\mathbb{N}: \|x_{k+1}-x_k,w\|<\varepsilon\}\in\Im|\}$$

which implies

$$\left[\frac{|B_n|}{\lambda_n}\right]^2\le\lim_{n\to\infty}\frac{1}{\lambda_{n^2}}\{k\in I_n;n\in\mathbb{N}: \|x_{k+1}-x_k,w\|<\varepsilon\}\in\Im|\}.$$

Hence for all $n\in A^c$,

$$\lim_{n\to\infty}\frac{1}{\lambda_{n^2}}\{k\in I_n;n\in\mathbb{N}: \|x_{k+1}-x_k,w\|<\varepsilon\}\in\Im|\}\ge\left[\frac{|B_n|}{\lambda_n}\right]^2\in A^c.$$

Therefore for all $n\in A$,

$$\lim_{n\to\infty}\frac{1}{\lambda_{n^2}}\{k\in I_n;n\in\mathbb{N}: \|x_{k+1}-x_k,w\|\ge\varepsilon\}\in\Im|\}\in A.$$

■

V. Conclusion

In this article, we have extended the study of λ-ideal statistical convergence in 2-normed spaces and λ-ideal statistically Cauchy sequences in 2-normed spaces over non-Archimedean field and proved some inclusion relations related to it. Also, it has been given that λ-ideal statistical convergence in 2-normed spaces is λ-ideal statistically pre-Cauchy sequences in 2-normed spaces over non-Archimedean field.

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