Lambda-Statistical Convergence in Paranormed Spaces over Non-Archimedean Fields

N. Saranya and K. Suja*

***Abstract*—The objective of this paper is to investigate the concept of λ-statistical convergence, λ-statistically Cauchy sequences, and ideal statistically pre-Cauchy sequences in non-Archimedean paranormed spaces. Here, $\lambda = (\lambda_n)$ is a non-decreasing sequence that tends to ∞, with the properties $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. The study includes proving significant properties of λ-statistical convergence in paranormed spaces and establishing criteria for λ-statistical convergence and λ-statistically Cauchy sequences. The implications of these concepts are discussed in the framework of non-Archimedean fields. Paranormed spaces are considered to have more general properties than normed spaces. The paper also introduces the concept of ideal statistically pre-Cauchy sequences and proves that ideal statistical convergence implies ideal statistical pre-Cauchy behavior in paranormed spaces over non-Archimedean fields. The field $\mathcal{K}$ is assumed to be complete, non-trivially valued, and non-Archimedean field throughout the article.**

***Index Terms*—Non-Archimedean fields, λ - statistically convergent, ideal statistically pre-Cauchy sequence, paranormed spaces.**

I. Introduction

STATISTICAL convergence was first proposed by Steinhaus[8] in 1951, and further developed by Fridy[10] who established the concepts of statistical limit point and statistical cluster point of a number sequence. Since then,several authors[2], [4], [16] have made generalizations of this notion, among others. Statistical convergence has found applications in various areas of mathematics, including trigonometric series, number theory, and summability theory.

Maddox[9] studied statistical convergence in locally convex Hausdorff topological spaces, while Kolk extended the concept to Banach spaces. Cakalli[6] expanded statistical convergence to topological Hausdorff groups, and more recently, statistical convergence has been investigated in paranormed spaces. Mursaleen et al.[13], [14], [15] generalized the idea of statistical convergence for sequences.

The concept of statistical convergence was also independently introduced by Fast[7], Buck, and Schoenberg for real and complex sequences, and further investigated by Salat[17], Connor[11], Fridy[10], and many others. Alotaibi et al.[1] developed the concept of statistical convergence in a paranormed space. The study of statistical convergence in paranormed spaces involves investigating the behavior of sequences or series of elements in paranormed spaces with respect to statistical convergence. The notion of statistical convergence has also been extended to non-Archimedean fields by Srinivasan, Suja[12] and more recently, to non-Archimedean Kothe sequence spaces by Eunice Jemima and Srinivasan. Non-Archimedean Analysis is the study of analysis over non-Archimedean fields.

Connor et al.[11] introduced the concept of statistically pre-Cauchy sequences as a generalization of Cauchy sequences. Connor showed that statistically convergent sequences are always statistically pre-Cauchy, and under certain conditions, statistical pre-Cauchy condition implies statistical convergence. However, Connor, Fridy, and Klin gave an example showing that statistically pre-Cauchy sequences are not necessarily statistically convergent. Das et.al[3] developed statistically pre Cauchy sequences into ideal statistically pre-Cauchy sequences.

An ideal statistically pre-Cauchy sequence refers to a statistically pre-Cauchy sequence that satisfies certain ideal conditions. The study of ideal statistically pre-Cauchy sequences has applications in various fields, providing insights and tools for analyzing and modeling complex systems. The notion of ideal convergence is a generalization of statistical convergence, initially examined by Kostyrko, Salat, and Wilezynski. Recently, Savas et al. studied the ideal statistical convergence of sequences and obtained some results related to this concept.

In this paper, we will review various notations and definitions that will be utilized throughout the paper to study the concepts of statistical convergence, statistically pre-Cauchy sequences, ideal statistically pre-Cauchy sequences, and ideal convergence.

[5]Throughout this article, $\mathcal{K}$ denotes a non-Archimedean field that is complete, non-trivially valued, and meets the axioms listed below:

(i) $|x| \geq 0$ and $|x| = 0$ iff $x = 0$.
(ii) $|xy| = |x||y|$.
(iii) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathcal{K}$.

II. Preliminaries

Definition 2.1: If X is a non-trivial linear space over $\mathcal{K}$, then by a paranorm on X, we are referring to a map $g : X \to R$ satisfying the following conditions:

(i) $g(0) = 0$.
(ii) $g(x) = g(-x)$ on x, for all $x \in X$
(iii) $g(x + y) \leq \max\{g(x), g(y)\}$, for all $x, y \in X$
(iv) for $\lambda, \lambda_0 \in K$,$x, x_0 \in X$ and $\lambda \to \lambda_0$, $g(x - x_0) \to 0$ imply $g(\lambda x - \lambda_0 x_0) \to 0$.
The space X with a paranorm g is called a paranormed space.

Definition 2.2: A sequence $x = \{x_j\}$ is stated to be statistically convergent to ℓ in (X, g) if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{j \in n : g(x_j - \ell) \geq \epsilon\}| = 0.$$

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This is denoted by $\mathrm{stat}(g) - \lim x = \ell$.

Example 2.3: Let $x = \ell\left(\frac{1}{j}\right) = \{x = \{x_j\} : \sum_j |x_j|^{1/j} < \infty\}$ with the paranorm, $g(x) = \sum_j |x_j|^{1/j}$.

Define a sequence $x = (x_j)$ by

$$\{x_j\} = \begin{cases} j, & if \ \ j = n^2, n \in \mathbb{N}; \\ 0, & otherwise. \end{cases}$$

we see that, $g(x_j) = \begin{cases} (j)^{1/j}, & if \ \ j = n^2, n \in \mathbb{N}; \\ 0, & otherwise. \end{cases}$

and hence, $\lim_j g(x_j) = \begin{cases} 1, & if \ \ j = n^2, n \in \mathbb{N}; \\ 0, & otherwise. \end{cases}$

Thus, $stat(g) - \lim\ x = 0$.

Definition 2.4: A sequence $\{x_j\}$ is known as statistically Cauchy sequence, if there exists an $n \in \mathbb{N}$ such that $\lim_n \frac{1}{n}|\{j \le n : g(x_{j+1} - x_j) \ge \epsilon\}| = 0$. for every $\epsilon > 0$.

Definition 2.5: Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers that approaches ∞, satisfying $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 0$. Let $I_n = [n - \lambda_n + 1, n]$. A sequence $x = \{x_j\}$ is considered to be λ-statistically convergent to ℓ if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : |x_j - \ell| \ge \epsilon\}| = 0.$$

Example 2.6:
Define the sequence $x = \{x_j\}$ by

$$\{x_j\} = \begin{cases} j, & n - [\sqrt{\lambda_n}] + 1 \le j \le n, \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : |x_j - 0| \ge \epsilon\}| = \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n} = 0.$$

Example 2.7: Define the sequence $x = \{x_j\}$ by

$$\{x_j\} = \begin{cases} 1, & n - \sqrt{\lambda_n} + 1 \le j \le n, \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : |x_j - 1| \ge \epsilon\}| = \lim_{n\to\infty} \frac{\sqrt{\lambda_n}}{\lambda_n} = 0.$$

III. λ - STATISTICALLY CONVERGENT

In this section, we prove few basic theorems which comprise the mains results of this paper. These theorems parallel the corresponding theorems in real analysis.

Definition 3.1: Let where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = \{x_j\}$ is known to be λ-*statistically convergent to* ℓ in (X, g) if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(x_j - \ell) \ge \epsilon\}| = 0.$$

This is denoted by $\mathrm{stat}_\lambda(g) - \lim x = \ell$.

Example 3.2: Let $x = \ell\left(\frac{1}{j}\right) = \{x = (x_j) : \sum_j |x_j|^{1/j} < \infty\}$ with the paranorm, $g(x) = \sum_j |x_j|^{1/j}$.

Define a sequence $x = (x_j)$ by
$x_j = \begin{cases} j, & if \ \ n - [\lambda_n] + 1 \le j \le n, n \in \mathbb{N} \\ 0, & otherwise. \end{cases}$

we see that,

$$g(x_j) = \begin{cases} (j)^{1/j}, & if \ \ n - [\lambda_n] + 1 \le j \le n, n \in \mathbb{N} \\ 0, & otherwise. \end{cases}$$

and hence,

$$\lim_j g(x_j) = \begin{cases} 1, & if \ \ n - [\lambda_n] + 1 \le j \le n, n \in \mathbb{N} \\ 0, & otherwise. \end{cases}$$

Thus, $stat_\lambda(g) - \lim\ x = 0$.

Definition 3.3: A sequence $x = \{x_j\}$ is said to be λ-*statistically Cauchy sequence* in (X, g) if for every $\epsilon > 0$, there exist a number I_n such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{m \in I_n : g(x_{j+1} - x_j) \ge \epsilon\}| = 0.$$

Theorem 3.4: If a sequence $x = \{x_j\}$ is λ-statistically convergent in (X, g) then $\mathrm{stat}_\lambda(g)$-limit is unique.

Proof: Let

$$A_1(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(x_j - \ell_1) \ge \epsilon\}| = 0, \quad (1)$$

and

$$A_2(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(x_j - \ell_2) \ge \epsilon\}| = 0. \quad (2)$$

Then by a conjunction of (1) and (2), we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\lambda_n} \quad |\{j \in I_n : g(\ell_1 - \ell_2) \ge \epsilon\}| \\ &= \lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(\ell_1 - x_j + x_j - \ell_2) \ge \epsilon\}| \\ &= \max\left\{\lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(x_j - \ell_1) \ge \epsilon\}|,\right. \\ &\qquad \left.\lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(x_j - \ell_2) \ge \epsilon\}|\right\} \\ &= 0. \end{aligned}$$

Hence $\ell_1 = \ell_2$. ■

Theorem 3.5: If $\mathrm{stat}_\lambda(g) - \lim x_j = \ell_1$ and $\mathrm{stat}_\lambda(g) - \lim y_j = \ell_2$, then for any $\alpha \in k$,

$$\mathrm{stat}_\lambda(g) - \lim(x_j \pm y_j) = \ell_1 \pm \ell_2,\ \mathrm{stat}_\lambda(g) - \lim \alpha x_j = \alpha\ell_1.$$

Proof: Let

$$A_1(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(x_j - \ell_1) \ge \epsilon\}| = 0 \quad (3)$$

and

$$A_2(\epsilon) = \lim_{n\to\infty} \frac{1}{\lambda_n}|\{j \in I_n : g(y_j - \ell_2) \ge \epsilon\}| = 0. \quad (4)$$

Then from (3) and (4), we deduce

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\quad |\{j\in I_n : g((x_j+y_j)-(\ell_1+\ell_2))\geq\epsilon\}|$$
$$=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell_1-y_j-\ell_2)\geq\epsilon\}|$$
$$\leq\max\left\{\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell_1)\geq\epsilon\}|\ ,\right.$$
$$\left.\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(y_j-\ell_2)\geq\epsilon\}|\right\}$$
$$=0.$$

Similarly, we have

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g((x_j-y_j)-(\ell_1-\ell_2))\geq\epsilon\}|=0.$$

Hence, we have $\mathrm{Stat}_\lambda(g)$-$\lim(x_j\pm y_j)=\ell_1\pm\ell_2$.

We then show that

$$\mathrm{Stat}_\lambda(g)\text{-}\lim\alpha x_j=\alpha\ell_1.$$

From (3) and (4), we deduce that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\quad |\{j\in I_n : g((\alpha x_j-\alpha\ell_1))\geq\epsilon\}|$$
$$=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(\alpha(x_j-\ell_1)\geq\epsilon\}|$$
$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\alpha\{j\in I_n : g(x_j-\ell_1)\geq\epsilon\}|$$
$$=|\alpha|\cdot 0=0.$$

Therefore,

$$\mathrm{stat}_\lambda(g)-\lim\alpha\cdot x_j=\alpha\ell_1.$$

■

Theorem 3.6: Let (X,g) be a complete paranormed space. If a sequence $x=\{x_j\}$ in (X,g) is λ-Statistically convergent then it is λ-statistically Cauchy sequence.

Proof: Assume that a sequence $x=\{x_j\}$ in (X,g) is λ-statistically convergent. That is, $\mathrm{stat}_\lambda(g)-\lim x_j=l_1$. Let

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell_1)\geq\epsilon\}|=0$$

and

$$\lim_{n}\frac{1}{\lambda_n}|\{j+1\in I_n : g(x_{j+1}-\ell_1)\geq\epsilon\}|=0.$$

By these two equalities,

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\quad |\{j+1,j\in I_n : g(x_{j+1}-x_j)\geq\epsilon\}|$$
$$=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j+1,j\in I_n : g(x_{j+1}-\ell_1-x_j+\ell_1)\geq\epsilon\}|$$
$$=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j+1,j\in I_n : g((x_{j+1}-\ell_1)+(\ell_1-x_j)\geq\epsilon\}|$$
$$\leq\max\left\{\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j+1\in I_n : g(x_{j+1}-\ell_1)\geq\epsilon\}|\ ,\right.$$
$$\left.\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell_1)\geq\epsilon\}|\right\}$$
$$=0.$$

Hence the sequence $x=\{x_j\}$ is λ-statistically a Cauchy sequence.

■

Theorem 3.7: If $g-\lim x=l$ and $\mathrm{stat}_\lambda(g)-\lim y_j=0$ then $\mathrm{Stat}_\lambda(g)-\lim(x+y)=g-\lim x$.

Proof: Suppose that $g-\lim x=\ell$.
Then $g(x_j-\ell)=0$ as $j\to\infty$. Also, $\mathrm{stat}_\lambda(g)-\lim y_j=0$. That is,

$$\lim_{n\to\infty}\frac{1}{n}|\{j\in I_n : g(y_j-0)\geq\epsilon\}|=0.$$

Let $\mathrm{stat}_\lambda(g)-\lim(x+y)=\ell'$.

By definition,
$stat_\lambda(g)=\lim(x+y)$
$$=\lim_{n\to\infty}\tfrac{1}{\lambda_n}|\{j\in I_n : g((x_j+y_j)-\ell')\geq\epsilon\}|=0$$
Hence

$$\lim_{j\to\infty}g(x_j-\ell')+\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g((y_j-0)\geq\epsilon\}|=0.$$

Hence we have

$$\max\left\{\lim_{j\to\infty}g(x_j-\ell'),\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(y_j-0)\geq\epsilon\}|\right\}=0.$$

Therefore,

$$\max\left\{\lim_{j\to\infty}g(x_j-\ell'),0\right\}=0.$$

Hence $g-\lim x=\ell'$.

But, $g-\lim x=\ell$. Therefore, $\ell=\ell'$. In another word, $\mathrm{Stat}_\lambda(g)-\lim(x+y)=g-\lim x$. ■

Theorem 3.8: A sequence $x=\{x_j\}$ is λ statistically convergent if and only if

$$\lim_{n\to\infty}\frac{1}{n}|\{j\leq n,j'\leq n : |\ x_j-x_{j'(r)}|\geq\epsilon\}|\ =0,$$

where $(x_{j'(r)})$ is a subsequence of $\{x_j\}$ such that $\lim_{j\to\infty}x_{j'(r)}=\ell$.

Proof: Let the sequence $\{x_j\}$ be λ-statistically convergent. We show that the condition

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j-x_{j'(r)}\geq\epsilon\}|=0$$

is satisfied. By the definition of λ-statistical convergence, we have

$$\lim_{n}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|=0.$$

Then

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j-x_{j'(r)}+\ell-\ell)\geq\epsilon\}|$$
$$=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j-\ell)+g(\ell-x_{j'(r)})\geq\epsilon\}|$$
$$=\max\left\{\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|\ ,\right.$$
$$\left.\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j'\in I_n : g(x_{j'(r)}-\ell)\geq\epsilon\}|\right\}$$
$$=\max\left\{0,\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j'\in I_n : g(x_{j'(r)}-\ell)\geq\epsilon\}|\right\}$$

By the assumption, $\lim_{j\to\infty}x_{j'(r)}=\ell$. Hence the sequence is statistically convergent. Hence

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j'\in I_n : g(x_{j'(r)}-\ell)\}|=0.$$

This implies that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j - x_{j'(r)}) \geq \epsilon\}| = 0.$$

Conversly, suppose that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j - x_{j'(r)}) \geq \epsilon\}| = 0.$$

We show that the sequence $\{x_j\}$ is statistically convergent. We have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}\right|\\ &=\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j - x_{j'(r)} + x_{j'(r)} - \ell)\geq\epsilon\}|\\ &\leq\max\left\{\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j,j'\in I_n : g(x_j - x_{j'(r)})\geq\epsilon\}|\,,\right.\\ &\quad\left.\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j'\in I_n : g(x_{j'(r)}-\ell)\geq\epsilon\}|\right\}\\ &\leq\max\left\{0,\lim_{n\to\infty}\frac{1}{\lambda_n}|\{j'\in I_n : g(x_{j'(r)}-\ell)\geq\epsilon\}|\right\}=0.\end{aligned}$$

We have shown that the sequence (x_j) is λ-statistically convergent. ■

Theorem 3.9: Let X be a paranormed space and let $\lambda = (\lambda_n) \in \delta$. Then $stat(x) \subset stat_\lambda(x)$ if and only if $lim\inf\frac{\lambda_n}{n} > 0$

Proof: Suppose that x is statistically convergent, then $\lim\limits_{n\to\infty}\frac{1}{n}|j\in n : g(x_j-\ell)\geq\epsilon| = 0.$
Since $\inf\frac{\lambda_n}{n} > 0$ as $n\to\infty$,

$$\begin{aligned}\frac{1}{n}|\{j\leq n : g(x_j-\ell)\geq\epsilon\}| &\geq \frac{1}{n}|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|\\ &\geq\frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|\end{aligned}$$

It follows that,
$x_j \to \ell(stat(x)) \implies x_j \to \ell(stat_\lambda(x))$
Thus, $stat(x) \subset stat_\lambda(x))$

■

Theorem 3.10: Let x be a paranormed space and if $\lambda_n \in \delta$ such that $lim\frac{\lambda_n}{n} = 1$, then $stat(x) = stat_\lambda(x)$

Proof: Since $lim\frac{\lambda_n}{n} = 1$, then for $\epsilon > 0$,

$$\begin{aligned}&\frac{1}{n}|\{j\leq n : g(x_j-\ell)\geq\epsilon\}|\\ &\leq\max\left\{\frac{1}{n}|\{j\leq n-\lambda_n : g(x_j-\ell)\geq\epsilon\}|,\right.\\ &\qquad\left.\frac{1}{n}|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|\right\}\\ &\leq\max\left\{\frac{n-\lambda_n}{n},\frac{1}{n}|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|\right\}\\ &\leq\max\left\{\frac{(n-\lambda_n)}{n},\frac{\lambda_n}{n}\frac{1}{\lambda_n}\right.\\ &\qquad\left.|\{j\in I_n : g(x_j-\ell)\geq\epsilon\}|\right\}.\end{aligned}$$

This implies that if (x_j) is λ statistically convergent, then $\{x_j\}$ is statistically convergent.
i.e., $stat(X) \supset stat_\lambda(X)$.

Also, since $\lim_n\frac{\lambda_n}{n} = 1$, implies that $\inf\frac{(\lambda_n)}{n} > 0$, by previous theorem we have $stat(X) \subset stat_\lambda(X)$.
Thus, $stat(X) = stat_\lambda(X)$.

■

IV. Ideal statistically pre-Cauchy sequences

The concept of ideal statistically pre-Cauchy sequences provides a generalization of the usual convergence of sequences. In this section, we define the ideal statistically pre-Cauchy sequences and give some inclution relations over non-Archimedean paranormed space.

In non-Archimedean analysis, the concept of ideal statistically pre-Cauchy sequences relates to the convergence behavior of sequences in a non-Archimedean field with respect to a chosen ideal. This notion extends the idea of statistical convergence to pre-Cauchy sequences.

Definition 4.1: A sequence $\{x_j\}$ is said to be $\mathfrak{I}$ - convergent to a number ℓ if for every $\epsilon > 0$
$\{j\in\mathbb{N} : g(x_j-\ell)\geq\epsilon\}\in\mathfrak{I}$.
Symbollicaly, it is denoted as $\mathfrak{I}-\lim x_j = \ell$

Definition 4.2: A sequence $\{x_j\}$ is known as $\mathfrak{I}$ - statistically convergent sequence, whenever $\mathfrak{I}$ is an admissible ideal and that for any $\epsilon > 0$ the set,
$\{n\in\mathbb{N} : \frac{1}{n}|\ \{j\leq n : g(x_j-\ell)\geq\epsilon\}|\ \to 0 \text{ as } n\to\infty\}\in\mathfrak{I}$.
Symbollicaly, it is denoted as $\mathfrak{I}-st-\lim_{m\to\infty}x_j = \ell$

Definition 4.3: A sequence $\{x_j\}$ is known as $\mathfrak{I}$ - statistically cauchy sequence, whenever $\mathfrak{I}$ is an admissible ideal and that for any $\epsilon > 0$ the set,
$\{n\in\mathbb{N} : \frac{1}{n}|\ \{j\leq n : g(x_{j+1}-x_j)\geq\epsilon\}|\ \to 0 \text{ as } n\to\infty\}\in\mathfrak{I}$.

Definition 4.4: A sequence $\{x_j\}$ is said to be $\mathfrak{I}$ - statistically pre-cauchy sequence if for any $\epsilon > 0$
$\{n\in\mathbb{N} : \frac{1}{n^2}|\ \{j\leq n : g(x_{j+1}-x_j)|\ \geq\epsilon\}|\ \to 0 \text{ as } n\to\infty\}\in\mathfrak{I}$.

Theorem 4.5: If $x = \{x_j\}$ is an $\mathfrak{I}$-statistically convergent sequence then it is $\mathfrak{I}$ statistically pre-cauchy.

Proof: Let $\{x_j\}$ be $\mathfrak{I}$ statistically convergent to ℓ
Therefore, $\{n\in\mathbb{N} : \frac{1}{n}|\ \{m\leq n : g(x_j-\ell)\geq\epsilon\}|\ \to 0 \text{ as } n\to\infty\}\in\mathfrak{I}$.

Let us consider,
$A = \{n\in\mathbb{N} : \frac{1}{n}|\ \{m\leq n : g(x_j-\ell)\geq\epsilon\}|\ \to 0 \text{ as } n\to\infty\ \}$
$A^c = \{n\in\mathbb{N} : \frac{1}{n}|\ \{m\leq n : g(x_j-\ell)\geq\epsilon\}|\ \to 1 \text{ as } n\to\infty\ \}$
Where c stands for complement.

Let us define, $B_n = \{j\leq n : g(x_j-\ell) < \epsilon\}$.
Now consider,
$g(x_{j+1}-x_j) = g(x_{j+1}-x_j-\ell+\ell)$

$\Rightarrow g(x_{j+1}-x_j)\leq max\{g(x_{j+1}-\ell), g(x_j-\ell)\}$

we observe that for $j\in B_n$,
$g(x_{j+1}-x_j)\leq max\{g(x_{j+1}-\ell)<\epsilon, g(x_j-\ell)<\epsilon\}<\epsilon$

Hence, $B_nXB_n \subset \{j \leq n : g(x_{j+1} - x_j) < \epsilon\}$

$\Rightarrow \left[\frac{|B_n|}{n}\right]^2 \leq \frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) < \epsilon\}|$

$\Rightarrow \frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) < \epsilon\}| \geq [\frac{|B_n|}{n}]^2 \to 1^2$

$\Rightarrow \frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) < \epsilon\}| \to 1^2 - 1^2 \to 0$

$\Rightarrow \frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) < \epsilon\}| \to 0 \subseteq A$

$\Rightarrow \{\frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) \geq \epsilon\}| \to 0 \text{ as } n \to \infty\} \in \mathfrak{I}$

Hence Proved.

■

Theorem 4.6: Let $x = (x_j)$ be a bounded sequence. Then x is $\mathfrak{I}$ - statistically pre-cauchy iff $\lim_n \frac{1}{n^2} \sum_{j \leq n} g(x_{j+1} - x_j) = 0$.

Proof: First suppose that, $\lim_n \frac{1}{n^2} \sum_{j \leq n} g(x_{j+1} - x_j) = 0$.

$$\begin{aligned}
\lim_n \frac{1}{n^2} \sum_{j \leq n} g(x_{j+1} - x_j) \\
&= \frac{1}{n^2} \sum_{\substack{j \leq n \\ g(x_{j+1}-x_j) \leq \epsilon}} g(x_{j+1} - x_j) \\
&\quad + \frac{1}{n^2} \sum_{\substack{j \leq n \\ g(x_{j+1}-x_j) \geq \epsilon}} g(x_{j+1} - x_j) \\
&\geq \frac{1}{n^2} \sum_{\substack{j \leq n \\ g(x_{j+1}-x_j) \geq \epsilon}} g(x_{j+1} - x_j) \\
&\geq \frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) \geq \epsilon\}| \\
&\geq 0.
\end{aligned}$$

Since $\lim_n \frac{1}{n^2} \sum_{j \leq n} g(x_{j+1} - x_j) = 0$, the set $\{n \in \mathbb{N} : \frac{1}{n}| \ \{j \leq n : g(x_{j+1} - \ell) \geq \epsilon\}| \ \to 0 \text{ as } n \to \infty\} \in \mathfrak{I}$.

Conversly, Suppose that x is $\mathfrak{I}$ - statistically pre-cauchy, then since x is bounded sequence, there exist a $B > 0$ such that $|x_j| \leq B$ for all $k \in \mathbb{N}$

$$\begin{aligned}
&\lim_n \frac{1}{n^2} \sum_{j \leq n} g(x_{j+1} - x_j) \\
&= \frac{1}{n^2} \sum_{\substack{j \leq n \\ g(x_{j+1}-x_j) \leq \epsilon}} g(x_{j+1} - x_j) + \\
&\quad \frac{1}{n^2} \sum_{\substack{j \leq n \\ g(x_{j+1}-x_j) \geq \epsilon}} g(x_{j+1} - x_j) \\
&\leq \epsilon + B(\frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) \geq \epsilon\}|)
\end{aligned}$$

Since x is $\mathfrak{I}$ - pre cauchy, there is an $\mathbb{N}$ such that $\epsilon + B(\frac{1}{n^2}|\{j \leq n : g(x_{j+1} - x_j) \geq \epsilon\}|)$ is less that ϵ for all $j \in \mathbb{N}$.
Hence $\lim_n \frac{1}{n^2} \sum_{j \leq n} g(x_{j+1} - x_j) = 0$. ■

Corollary 4.7: A sequence $x = \{x_j\}$ is $\mathfrak{I}$ - convergent $\Leftrightarrow$ $\mathfrak{I}$ - $\lim_n \frac{1}{n^2} \sum g(x_{j+1} - x_j) = 0$

Proof: Let, $A_1 = \{j \in \mathbb{N} : g(x_{j+1} - x_j) < \epsilon\} \in \mathfrak{I}$ $A_1^c = \{j \in \mathbb{N} : g(x_{j+1} - x_j) \geq \epsilon\} \in \mathfrak{I}$

Then, $\{j \in \mathbb{N} : g(x_{j+1} - x_j) \geq \epsilon\} \subset A_1 \bigcup A_1^C \in \mathfrak{I}$.

Hence, $\mathfrak{I}$ - $\lim_n \frac{1}{n^2} \sum g(x_{j+1} - x_j) = 0$.

The desired result can be obtained by directly applying Theorem 4.5.

■

Corollary 4.8: A sequence $x = \{x_j\}$ is $\mathfrak{I}$ - convergent to ℓ $\Leftrightarrow \mathfrak{I} - \lim_n \frac{1}{n} \sum g(x_j - \ell) = 0$.

V. Conclusion

Known results have been extended from Archimedean fields to non-Archimedean fields. We have investigated the concept of λ-statistical convergence and derived basic properties of λ-statistical convergence in paranormed spaces over non-Archimedean fields. Furthermore, the relationship between ideal statistical convergence and ideal statistical pre-Cauchy property has been discussed in this study.

References

[1] A. Alotaibi, A. M. Alroqi, "Statistical convergence in a paranormed space", Journal of inequalities and applications, 2012, 1-6. DOI:10.1186/1029-242X-2012-39

[2] B. Hazarika, "Generalized statistical convergence in 2-normed spaces", Acta et Commentationes Universitatis Tartuensis de Mathematica, 17(2), (2013), 135-143.

[3] Das, Pratulananda, and Ekrem Savas. "On I-statistically pre-Cauchy sequences," .Taiwanese J. Math. 18(1) (2014), 115-126.

[4] D. Rath, and B. C. Tripathy, "On statistically convergent and statistically Cauchy sequences", Indian Journal of Pure and Applied Mathematics, 25 (1994), 381-381.

[5] G. Bachman, "Introduction to p-adic numbers and valuation theory", Academic Press, 1964

[6] H. Cakalli, "A study on statistical convergence", Funct. Anal. Approx. Comput, 1(2),(2009) 19-24.

[7] H. Fast, "Sur la convergence statistique", In Colloquium mathematicae, 2(3-4), (1951), pp. 241-244.

[8] H. Steinhaus, "Sur la convergence ordinaire et la convergence asymptotique", In Colloq. math, 2(1), (1951), 73-74.

[9] I. J. Maddox, "Statistical convergence in a locally convex space", In Mathematical Proceedings of the Cambridge Philosophical Society, 104(1), (1988), 141-145. Cambridge University Press.

[10] J. A. Fridy, "On statistical convergence", Analysis, 5(4), (1985), 301-314. https://doi.org/10.1524/anly.1985.5.4.301

[11] J. Connor, J. Fridy, J. Kline, "Statistically Pre-Cauchy Sequences", Analysis, 14, (1994), 311-317.

[12] K. Suja, and Srinivasan Vaithinathasamy. "On statistically convergent and statistically cauchy sequences in non-archimedean fields", Journal: Journal of Advances in Mathematics 6(3), (2014).

[13] M. Mursaleen, "λ-statistical convergence", Mathematica slovaca 50(1),(2000), 111-115.

[14] M. Mursaleen, C. Çakan, S. A. Mohiuddine, and Ekrem Savaş. "Generalized statistical convergence and statistical core of double sequences", Acta Mathematica Sinica, English Series 26 (2010), 2131-2144.

[15] Alghamdi, Mohammed A., and Mohammad Mursaleen, "λ-Statistical convergence in paranormed space", Abstract and Applied Analysis, 2013(SI27), (2013), 1-5.

[16] O.H.H. Edely, and Mohammad Mursaleen, ”On statistical A-summability”, Mathematical and Computer Modelling 49(3-4) (2009), 672-680.
[17] T. Šalát, ”On statistically convergent sequences of real numbers”, Mathematica slovaca, 30(2) (1980), 139-150.