Almost and Pseudo Almost Periodic Solutions in Shifts Delta(+/-) for Nonlinear Dynamic Equations on Time Scales with Applications

Lili Wang, Pingli Xie, Yuwei Wang

***Abstract*—In this paper, we first study two nonlinear dynamic equations on time scales. By means of the theory of dynamic equations on time scales and the properties of the shift operators $\delta_{\pm}$ as well as the Schauder's fixed point theorem, some existence theorems of almost periodic solution in shifts $\delta_{\pm}$ and pseudo (v-pseudo) almost periodic solution in shifts $\delta_{\pm}$ of the equations are established. Secondly, based on the obtained results, we bring two ecosystems under investigation on some specific time scales to obtain more general results.**

***Index Terms*—Almost periodicity; Pseudo almost periodicity; Nonlinear dynamic equation; Delay; Time scale.**

I. Introduction

In this paper, we study the following nonlinear dynamic equations on time scales

$$y^{\Delta}(x)=D(y(x))y(x)+\xi(x,y(x),y(\delta_{-}(\tau,x))),x\in\mathbb{T}, \tag{1}$$

and

$$y^{\Delta}(x)=D(x,y(x))y(x)+\xi(x,y(x),y(\delta_{-}(\tau,x))),x\in\mathbb{T}, \tag{2}$$

where $\mathbb{T}$ is a time scale; $D_{n\times n}$ and $\xi_{n\times 1}$ are continuous functions. The equations (1) and (2) can be used to describe many phenomena in physics, ecology and other fields under different time scales.

It is well known that periodic dynamic systems in nature may exhibit almost periodicity and pseudo almost periodicity due to the influence of external or human factors; see, for example [1-11]. In recent years, by means of the shift operators $\delta_{\pm}$, the concepts and properties of periodic and almost (pseudo almost) periodic function in shifts $\delta_{\pm}$ on time scales have been defined and studied in [12-15] and [16-18], respectively. The theory of periodic and almost (pseudo almost) periodic dynamic equations in shifts $\delta_{\pm}$ on time scales have been rapidly developed and applied. The existence and uniqueness theorems of almost (pseudo almost) periodic solution in shifts $\delta_{\pm}$ of the linear dynamic equation on time scales

$$y^{\Delta}(x)=D(x)y(x)+\xi(x),x\in\mathbb{T}, \tag{3}$$

has been studied in [16,17,18]. However, there is no result on the existence of almost periodic and pseudo almost periodic solutions for equations (1) and (2).

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On the basis of the above works, to further construct the theory of almost periodicity and pseudo almost periodicity on time scales, the main work of this paper is to explore the existence theorems of almost periodic solution in shifts $\delta_{\pm}$ and pseudo (v-pseudo) almost periodic solution in shifts $\delta_{\pm}$ of (1) and (2).

Furthermore, we applying the obtained results to study two ecosystems, i.e. a Schoener's competition system and a delayed dynamic equation, the corresponding examples on some specific time scales are given to illustrate the usefulness of our main results.

II. Preliminaries

The theory of time scales and its applications on dynamic equations, see [19].

Lemma 1. ([19]) *If $\alpha\in\mathcal{R}$, then*

(1) $e_0(x,z)\equiv 1, e_{\alpha}(x,x)\equiv 1;$

(2) $e_{\alpha}(\sigma(x),z)=(1+\mu(x)\alpha(x))e_{\alpha}(x,z);$

(3) $e_{\alpha}(x,z)=\dfrac{1}{e_{\alpha}(z,x)}=e_{\ominus\alpha}(z,x);$

(4) $e_{\alpha}(x,z)e_{\alpha}(z,r)=e_{\alpha}(x,r);$

(5) $(e_{\ominus\alpha}(x,z))^{\Delta}=(\ominus\alpha)(x)e_{\ominus\alpha}(x,z);$

(6) $\left(\dfrac{1}{e_{\alpha}(\cdot,z)}\right)^{\Delta}=-\dfrac{\alpha(x)}{e^{\sigma}_{\alpha}(\cdot,z)}.$

A comprehensive review on periodicity and almost (pseudo almost) periodicity in shifts $\delta_{\pm}$ on time scales, see [12-18].

Consider the linear equation

$$y^{\Delta}(x)=D(\varphi(x))y(x),x\in\mathbb{T}, \tag{4}$$

where $\varphi(x)$ is a bounded continuous function.

Definition 1. ([20, 21]) *Suppose that $\Psi(x)$ is the fundamental solution matrix of (4), if there exist a projection P and positive constants β and α such that*

$$|\Psi(x)P\Psi^{-1}(\sigma(z))|\le\beta e_{\ominus\alpha}(x,\sigma(z)),\quad z,x\in\mathbb{T},x\ge\sigma(z),$$
$$|\Psi(x)(I-P)\Psi^{-1}(\sigma(z))|\le\beta e_{\ominus\alpha}(\sigma(z),x),\quad z,x\in\mathbb{T},x\le\sigma(z),$$

then (4) satisfies exponential dichotomy on $\mathbb{T}$, $|\cdot|$ is the Euclidean norm.

In the following sections, suppose that $\mathbb{B}$ is a Banach space, $BC(\mathbb{T},\mathbb{B})$ is a set of all $\mathbb{B}$-valued bounded continuous functions.

III. Existence results of solutions

We first consider the existence of almost periodic solutions in shifts $\delta_\pm$ of (1).

Define the sets:

$$APS(\mathbb{T},\mathbb{B}) = \{\psi : \mathbb{T} \to \mathbb{B}, \psi \text{ is almost periodic in shifts } \delta_\pm\};$$
$$APS^\Delta(\mathbb{T},\mathbb{B}) = \{\psi : \mathbb{T} \to \mathbb{B}, \psi \text{ is } \Delta\text{−almost periodic in shifts } \delta_\pm\}.$$

Theorem 1. *Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, (4) satisfies exponential dichotomy, and $D(y) \in APS(\mathbb{R}^n, \mathbb{R}^{n\times n})$, $\xi \in APS^\Delta(\mathbb{T},\mathbb{R}^n)$, then (1) exists a solution $y(x) \in APS(\mathbb{T},\mathbb{R}^n)$.*

Proof: Let $S = \{g(x)|g(x) \in APS(\mathbb{T},\mathbb{R}^n)$ and $g(x)$ is continuous$\}$, then $(S, \|\cdot\|)$ is a Banach space with the norm $\|g(x)\| = \sup\limits_{x\in\mathbb{T}} |g(x)|$. Since

$$\sup_{(x,y_1,y_2)\in\mathbb{T}\times\mathbb{R}^{2n}} |\xi(x, y_1, y_2)| < +\infty,$$

then one can choose a constant $M_0 > 0$ such that

$$\frac{1}{M_0} \sup_{(x,y_1,y_2)\in\mathbb{T}\times\mathbb{R}^{2n}} |\xi(x, y_1, y_2)| < \frac{1}{\beta\alpha_1},$$

where $\alpha_1 = \frac{1}{\inf(\ominus\alpha)} + \frac{1}{\alpha}$.

Take $S_0 \subseteq S$, S_0 is a closed convex subset of S, and

$$S_0 = \{\varphi(x)|\varphi(x) \in S, \|\varphi\| \le M_0\}.$$

For any $\varphi(x) \in S_0$, consider the inhomogenous linear equation

$$y^\Delta(x) = D(\varphi(x))y(x) + \xi(x, \varphi(x), \varphi(\delta_-(\tau, x))). \tag{5}$$

Since $\varphi(x) \in S_0$, then $\varphi(x)$ is bounded, and then the corresponding homogeneous equation of (5) satisfies exponential dichotomy, that is, (4) satisfies exponential dichotomy.

Moreover, $\varphi(x) \in APS(\mathbb{T},\mathbb{R}^n)$, then by the Definition 2.2 in [17], for arbitrarily $\varepsilon > 0$, there exists at least a $p \in [x, \delta_+^{l(\varepsilon)}(x)]([\delta_-^{l(\varepsilon)}(x), x])$, such that

$$|\varphi(\delta_\pm^p(x)) - \varphi(x)| < \varepsilon, \forall x \in \mathbb{T}.$$

Let $x' = \delta_-^\tau(x)$, since $x \in \mathbb{T}$, then $x' \in \mathbb{T}$, and

$$|\varphi(\delta_\pm^p(\delta_-^\tau(x))) - \varphi(\delta_-^\tau(x))| = |\varphi(\delta_\pm^p(x')) - \varphi(x')| < \varepsilon,$$

that is, $\varphi(\delta_-(\tau, x)) \in APS(\mathbb{T},\mathbb{R}^n)$.

By the theorems in [16,17], (5) exists exactly one solution $y_\varphi(x) \in APS(\mathbb{T},\mathbb{R}^n)$, and

$$\begin{aligned} & y_\varphi(x) \\ = & \int_{-\infty}^{x} \Psi_\varphi(x)P\Psi_\varphi^{-1}(\sigma(z))\xi(z, \varphi(z), \varphi(\delta_-(\tau, z)))\Delta z \\ & - \int_{x}^{+\infty} \Psi_\varphi(x)(I - P)\Psi_\varphi^{-1}(\sigma(z)) \\ & \times\xi(z, \varphi(z), \varphi(\delta_-(\tau, z)))\Delta z, \end{aligned} \tag{6}$$

then

$$\begin{aligned} & \|y_\varphi(x)\| \\ = & \Big\| \int_{-\infty}^{x} \Psi_\varphi(x)P\Psi_\varphi^{-1}(\sigma(z))\xi(z, \varphi(z), \varphi(\delta_-(\tau, z)))\Delta z \\ & - \int_{x}^{+\infty} \Psi_\varphi(x)(I - P)\Psi_\varphi^{-1}(\sigma(z)) \\ & \times\xi(z, \varphi(z), \varphi(\delta_-(\tau, z)))\Delta z \Big\| \\ \le & \left(\int_{-\infty}^{x} \beta e_{\ominus\alpha}(x, \sigma(z))\Delta s + \int_{x}^{+\infty} \beta e_{\ominus\alpha}(\sigma(z), x)\Delta z \right) \\ & \times \sup_{x\in\mathbb{T}} |\xi(x, \varphi(x), \varphi(\delta_-(\tau, x)))| \\ \le & \beta\alpha_1 \sup_{x\in\mathbb{T}} |\xi(x, \varphi(x), \varphi(\delta_-(\tau, x)))| \\ \le & M_0, \end{aligned}$$

that is, $y_\varphi(x) \in S_0$.

Define

$$\Phi : S_0 \to S_0, \Phi\varphi = y_\varphi. \tag{7}$$

Next we show that Φ is a compact continuous mapping.

We first prove that Φ is a compact mapping. Consider the sequence $\{\varphi_n(x)\} \subseteq S_0$, then $|\Phi\varphi_n(x)| \le M_0$. By (7), $\Phi\varphi_n(x) = y_{\varphi_n}(x), n = 1, 2, \cdots$, and

$$y_{\varphi_n}^\Delta(x) = D(\varphi_n(x))y_{\varphi_n}(x) + \xi(x, \varphi_n(x), \varphi_n(\delta_-(\tau, x))), \tag{8}$$

then

$$|y_{\varphi_n}^\Delta(x)| \le \left(M + \frac{1}{\beta\alpha_1}\right)M_0, \tag{9}$$

where $M = \sup\limits_{|\varphi_n(x)|\le M_0} |D(\varphi_n(x))|$.

It follows from (9) that $\{y_{\varphi_n}^\Delta(x)\}$ is uniformly bounded, then $\{y_{\varphi_n}(x)\}$ is uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, there exists a subsequence $\{y_{\varphi_{n_k}}(x)\}$ of $\{y_{\varphi_n}(x)\}$ such that $\{y_{\varphi_{n_k}}(x)\}$ converges uniformly on any compact set of $\mathbb{T}$. Since $\{y_{\varphi_{n_k}}(x)\} = \{\Phi\varphi_{n_k}(x)\}$, then $\{\Phi\varphi_{n_k}(x)\}$ converges uniformly on $\mathbb{T}$, that is, Φ is a compact mapping.

Now we prove that Φ is a continuous mapping. For $\{\varphi_n(x)\} \subseteq S_0$, we only need to prove that when $\{\varphi_n(x)\}$ converges uniformly to $\varphi(x)$, there is $\Phi\{\varphi_n(x)\}$ converges uniformly to $\Phi\varphi(x)$.

It is known from (8) that when $\{\varphi_n(x)\}$ converges uniformly to $\varphi(x)$ on $\mathbb{T}$, $\{y_{\varphi_n}(x)\}$ that is $\Phi\{\varphi_n(x)\}$ converges uniformly to the solution y_φ of the equation (5) on $\mathbb{T}$, and $y_\varphi \in APS(\mathbb{T},\mathbb{R}^n)$. On the other hand, (4) satisfies exponential dichotomy, that is, the corresponding homogeneous of the equation (5) satisfies exponential dichotomy, then (5) exists a unique solution $y_\varphi \in APS(\mathbb{T},\mathbb{R}^n)$. By (7), then $\Phi\varphi_n \to \Phi\varphi$. Thus, Φ is a continuous mapping.

By using the Schauder's fixed point theorem, Φ has a fixed point, there exists $\varphi \in S_0$ such that $\Phi\varphi = \varphi$, that is, (1) has an almost periodic solution in shifts $\delta_\pm$. This completes the proof. ■

Next, we consider the existence of pseudo almost periodic solutions in shifts $\delta_\pm$ of (1).

Define the set

$$PAPS(\mathbb{T},\mathbb{B}) = \{\psi : \mathbb{T} \to \mathbb{B}, \psi \text{ is pseudo almost periodic in shifts } \delta_\pm\};$$

$$PAPS_0(\mathbb{T},\mathbb{B}) = \{\psi(x) \in BC(\mathbb{T},\mathbb{B}) : \lim_{d\to+\infty} \frac{1}{(\delta_+^d(x_0) - \delta_-^d(x_0))} \int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)} |\psi(x)|\Delta x = 0\}.$$

Theorem 2. *Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, (4) satisfies exponential dichotomy, $D(y) \in APS(\mathbb{R}^n, \mathbb{R}^{n\times n})$, $\xi = \xi_1 + \xi_2$, and $\xi_1 \in APS^{\Delta}(\mathbb{T},\mathbb{R}^n)$, $\xi_2 \in PAPS_0(\mathbb{T},\mathbb{R}^n)$, then (1) exists a solution $y(x) \in PAPS(\mathbb{T},\mathbb{R}^n)$.*

Proof: Under the conditions of Theorem 2, similarly to the proof of Theorem 3.1 in [18], (5) has a unique solution $y_\varphi(x) \in PAPS(\mathbb{T},\mathbb{R}^n)$ as (6); then similarly to the proof of Theorem 1, (1) exists a solution $y(x) \in PAPS(\mathbb{T},\mathbb{R}^n)$. This completes the proof. ■

Let $\mathbb{U}$ is a set of functions (weight) $v : \mathbb{T} \to (0,+\infty)$, and

$$u(d,v) = \int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)} v(x)\Delta x,$$
$$\mathbb{U}_\infty = \{v \in \mathbb{U} : \lim_{d\to+\infty} u(d,v) = +\infty\},$$
$$\mathbb{U}_B = \{v \in \mathbb{U}_\infty : v \text{ is bounded and } \inf_{x\in\mathbb{T}} v(x) > 0\},$$

then $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$.

For $v \in \mathbb{U}_\infty$, set

$$PAPS(\mathbb{T},\mathbb{B},v) = \{\psi : \mathbb{T} \to \mathbb{B}, \psi \text{ is } v\text{-pseudo almost periodic in shifts } \delta_\pm\};$$

$$PAPS_0(\mathbb{T},\mathbb{B},v) = \{\psi(x) \in BC(\mathbb{T},\mathbb{B}) : \lim_{d\to+\infty} \frac{1}{u(d,v)} \int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)} |\psi(x)|v(x)\Delta x = 0\}.$$

Theorem 3. *Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, (4) satisfies exponential dichotomy, $D(y) \in APS(\mathbb{R}^n, \mathbb{R}^{n\times n})$, $\xi = \xi_1 + \xi_2$, and $\xi_1 \in APS^{\Delta}(\mathbb{T},\mathbb{R}^n)$, $\xi_2 \in PAPS_0(\mathbb{T},\mathbb{R}^n, v)$, then (1) exists a solution $y(x) \in PAPS(\mathbb{T},\mathbb{R}^n, v)$.*

Remark 1. *Theorem 3 can be proved in a similar way as the proof of Theorem 2.*

Remark 2. *If the linear equation*

$$y^{\Delta}(x) = D(x,\varphi(x))y(x), x \in \mathbb{T}, \tag{10}$$

satisfies exponential dichotomy, $\varphi(x)$ is a bounded continuous function. Then the conclusions of Theorems 1, 2 and 3 also hold for equation (2). In fact, it is only necessary to replace $D(\varphi(x))$ with $D(x,\varphi(x))$ and repeat the above proofs processes.

IV. Applications

In this section, denote $\zeta^u = \sup_{x\in[x_0,+\infty)_\mathbb{T}} |\zeta(x)|$, and $\zeta^l = \inf_{x\in[x_0,+\infty)_\mathbb{T}} |\zeta(x)|$, $\mathbb{R}^+ = (0,+\infty)$.

Example 1. Consider the Schoener's competition system

$$\begin{cases} y_1^{\Delta}(x) = \frac{r_1(x)}{\exp\{y_1(\delta_-(\tau_1,x))\}+b_1(x)} \\ \quad -a_{11}(x)\exp\{y_1(x)\} \\ \quad -a_{12}(x)\exp\{y_2(x)\} - c_1(x), \\ y_2^{\Delta}(x) = \frac{r_2(x)}{\exp\{y_2(\delta_-(\tau_2,x))\}+b_2(x)} \\ \quad -a_{21}(x)\exp\{y_1(x)\} \\ \quad -a_{22}(x)\exp\{y_2(x)\} - c_2(x), \end{cases} \tag{11}$$

with the initial conditions

$$y_1(x_0) = y_{10}, y_2(x_0) = y_{20}, y_{10} > 0, y_{20} > 0, x_0 \in \mathbb{T}.$$

Suppose that the coefficients of (11) are positive Δ-almost periodic functions in shifts $\delta_\pm$, and

(H_1) $\frac{r_1^u - a_{11}^l - a_{12}^l - c_1^l}{(1+b_1^l)a_{11}^l} > 1$, $\frac{r_2^u - a_{21}^l - a_{22}^l - c_2^l}{(1+b_2^l)a_{22}^l} > 1$;

(H_2) $\frac{r_1^l}{(e^{M_1}+b_1^u)a_{11}^u} - \frac{a_{12}^u e^{M_2} + c_1^u}{a_{11}^u} > 1$, $\frac{r_2^l}{(e^{M_2}+b_2^u)a_{22}^u} - \frac{a_{21}^u e^{M_1} + c_2^u}{a_{22}^u} > 1$.

Applying the inequalities in [22,23], similarly to the proofs of Theorem 9 and Lemma 10, one can obtain the following lemmas.

Lemma 2. *Suppose that (H_1) and (H_2) hold, then (11) is permanent, and*

$$m_1 \le \liminf_{t\to+\infty} y_1(x) \le \limsup_{t\to+\infty} y_1(x) \le M_1, \tag{12}$$
$$m_2 \le \liminf_{t\to+\infty} y_2(x) \le \limsup_{t\to+\infty} y_2(x) \le M_2, \tag{13}$$

where

$$M_1 = \frac{r_1^u - a_{11}^l - a_{12}^l - c_1^l}{(1+b_1^l)a_{11}^l} - 1,$$
$$M_2 = \frac{r_2^u - a_{21}^l - a_{22}^l - c_2^l}{(1+b_2^l)a_{22}^l} - 1,$$
$$m_1 = \ln\left(\frac{r_1^l}{(e^{M_1}+b_1^u)a_{11}^u} - \frac{a_{12}^u e^{M_2} + c_1^u}{a_{11}^u}\right),$$
$$m_2 = \ln\left(\frac{r_2^l}{(e^{M_2}+b_2^u)a_{22}^u} - \frac{a_{21}^u e^{M_1} + c_2^u}{a_{22}^u}\right).$$

Let $S(\mathbb{T})$ is the set of all solutions of (11).

Lemma 3. $S(\mathbb{T}) \neq \emptyset$.

Remark 3. *Lemma 3 implies that $S(\mathbb{T})$ is a positive invariant set of (11).*

Theorem 4. *Suppose that (H_1) and (H_2) hold, and*

$$-a_{11}^l + a_{21}^u \le -\alpha < 0, \tag{14}$$
$$a_{12}^l - a_{22}^u \le -\alpha < 0, \tag{15}$$

where $\alpha > 0$ is a constant, then (11) exists a positive solution $(y_1, y_2) \in APS(\mathbb{T}, \mathbb{R}^{2+})$.

Proof: From (11), we can get

$$\begin{aligned} y_1^{\Delta}(x) &= \frac{r_1(x)}{\exp\{y_1(\delta_-(\tau_1,x))\}+b_1(x)} \\ &\quad -a_{11}(x)\frac{\exp\{y_1(x)\}}{y_1(x)}y_1(x) \\ &\quad -a_{12}(x)\frac{\exp\{y_2(x)\}}{y_2(x)}y_2(x) - c_1(x), \\ y_2^{\Delta}(x) &= \frac{r_2(x)}{\exp\{y_2(\delta_-(\tau_2,x))\}+b_2(x)} \\ &\quad -a_{21}(x)\frac{\exp\{y_1(x)\}}{y_1(x)}y_1(x) \\ &\quad -a_{22}(x)\frac{\exp\{y_2(x)\}}{y_2(x)}y_2(x) - c_2(x), \end{aligned}$$

that is

$$\begin{aligned}&\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}^{\Delta}\\=&\begin{pmatrix}-a_{11}(x)\frac{\exp\{y_1(x)\}}{y_1(x)} & -a_{12}(x)\frac{\exp\{y_2(x)\}}{y_2(x)}\\-a_{21}(x)\frac{\exp\{y_1(x)\}}{y_1(x)} & -a_{22}(x)\frac{\exp\{y_2(x)\}}{y_2(x)}\end{pmatrix}\\&\times\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}\\&+\begin{pmatrix}\frac{r_1(x)}{\exp\{y_1(\delta_-(\tau_1,x))\}+b_1(x)}-c_1(x)\\\frac{r_2(x)}{\exp\{y_2(\delta_-(\tau_2,x))\}+b_2(x)}-c_2(x)\end{pmatrix}.\end{aligned}\qquad(16)$$

From (14) and (15),

$$\begin{aligned}&\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}^{\Delta}\\=&\begin{pmatrix}-a_{11}(x)\frac{\exp\{y_1(x)\}}{y_1(x)} & -a_{12}(x)\frac{\exp\{y_2(x)\}}{y_2(x)}\\-a_{21}(x)\frac{\exp\{y_1(x)\}}{y_1(x)} & -a_{22}(x)\frac{\exp\{y_2(x)\}}{y_2(x)}\end{pmatrix}\\&\times\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}\end{aligned}$$

satisfies exponential dichotomy. According to Theorem 1, (16) exists a positive solution $(y_1, y_2) \in APS(\mathbb{T}, \mathbb{R}^{2+})$, that is, (11) exists a positive solution $(y_1, y_2) \in APS(\mathbb{T}, \mathbb{R}^{2+})$. This completes the proof. ■

Now, we give a numerical example. Let

$$\mathbb{T} = \overline{\bigcup_{\ell\in\mathbb{Z}}[2\ell, 2\ell+1]},$$

then

$$\mu(x) = \begin{cases} 0, x \in \bigcup_{\ell\in\mathbb{Z}}[2\ell, 2\ell+1), \\ 1, x \in \bigcup_{\ell\in\mathbb{Z}}\{2\ell+1\}. \end{cases}$$

Take $x_0 = 0$, $\delta_-^z(x) = x - z$, $\tau_1 = \tau_2 = 2$, and

$$\begin{aligned}r_1(x) &= 1.8 - 0.1\cos(\sqrt{2}x),\\r_2(x) &= 1 + 0.1\sin(\sqrt{3}x),\\b_1(x) &= 5 + 0.2\cos(x), b_2(x) = 6 + 0.1\sin(x),\\c_1(x) &= 0.001, c_2(x) = 0.005,\\a_{11}(x) &= 0.2 + 0.01\cos(x),\\a_{12}(x) &= 0.003 - 0.001\sin(x),\\a_{21}(x) &= 0.004 - 0.001\sin(x),\\a_{22}(x) &= 0.1 + 0.01\sin(x).\end{aligned}$$

By a direct computation, (H_1) and (H_2) hold, and

$$M_1 = 0.3308, M_2 = 0.4171, m_1 = 0.1836, m_2 = 0.0929,$$

and

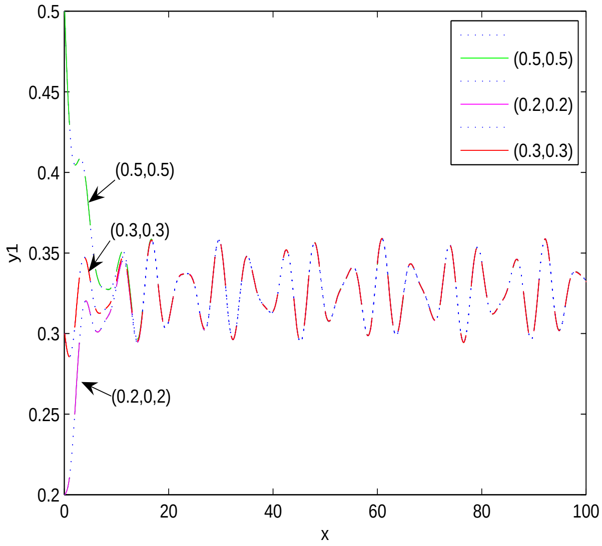
$$\begin{aligned}-a_{11}^l + a_{21}^u &= -0.185 < 0,\\a_{12}^l - a_{22}^u &= -0.986 < 0.\end{aligned}$$

Furthermore, it follows from (16) that,

$$f(x) = \begin{pmatrix}\frac{r_1(x)}{\exp\{y_1(\delta_-(\tau_1,x))\}+b_1(x)}-c_1(x)\\\frac{r_2(x)}{\exp\{y_2(\delta_-(\tau_2,x))\}+b_2(x)}-c_2(x)\end{pmatrix},$$

then $f(x) \in APS^{\Delta}(\mathbb{T}, \mathbb{R}^{2+})$, and $|f(x)| < +\infty$.

According to Theorem 1, (11) exists a positive solution $(y_1, y_2) \in APS(\mathbb{T}, \mathbb{R}^{2+})$. Dynamic simulations of (11), see Figure 1.



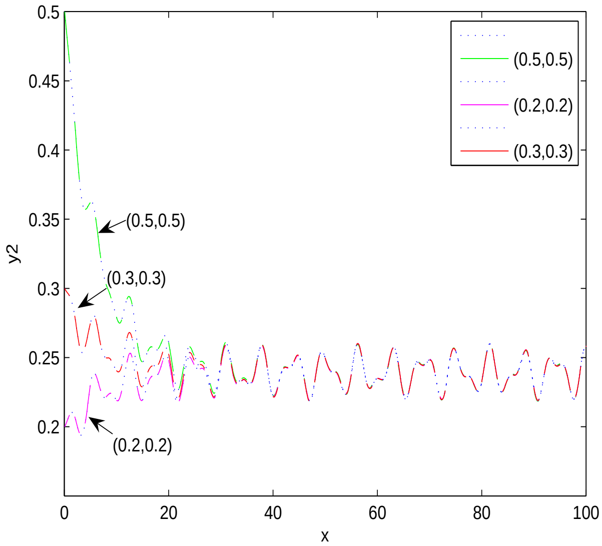


Fig. 1. Numerical solutions of (11) (Example 1) with the initial values $(y_1(0), y_2(0)) = \{(0.2, 0.2), (0.3, 0.3), (0.5, 0.5)\}$.

Example 2. Consider the following delayed dynamic equation

$$\begin{aligned}&y^{\Delta}(x)\\=&-a(y(x))y(x) + b(x)\int_{x_0}^{+\infty}k(z)\eta(y(\delta_-^z(x)))\Delta z\\&+c(x), x_0, x \in \mathbb{T}.\end{aligned}\qquad(17)$$

Suppose that $a(y) \in APS(\mathbb{R}, \mathbb{R}^+)$, $b(x) \in APS^{\Delta}(\mathbb{T}, \mathbb{R}^+)$, $c(x) = c_1(x) + c_2(x)$, and $c_1(x) \in APS^{\Delta}(\mathbb{T}, \mathbb{R}^+)$, $c_2(x) \in PAPS_0(\mathbb{T}, \mathbb{R}^+)$, and

(H_3) $a^l > 0$, and $1 - \mu(x)a^u > 0, \forall x \in [x_0, +\infty)_{\mathbb{T}}$;

(H_4) $\eta \in C(\mathbb{R}, [0, +\infty))$;

(H_5) $\delta_+^{\Delta\zeta}(\cdot, \zeta)$ is bounded, and $0 < \delta_+^{\Delta\zeta}(\cdot, \zeta) \le r$, where $r > 0$ is a constant.

Lemma 4. $\int_{x_0}^{+\infty} k(z)\eta(\phi(\delta_-^z(x)))\Delta z \in PAPS(\mathbb{T}, \mathbb{R}^+)$.

Proof: Let $\phi(x) \in PAPS(\mathbb{T}, \mathbb{R}^+)$, and

$$\eta(\phi(x)) = \eta_1(x) + \eta_2(x),$$

where $\eta_1(x) \in APS(\mathbb{T},\mathbb{R}^+), \eta_2(x) \in PAPS_0(\mathbb{T},\mathbb{R}^+)$, then

$$|\eta_1(\delta_\pm^p(x)) - \eta_1(x)| < \frac{\varepsilon}{1+\int_{x_0}^{+\infty}|k(z)|\Delta z}, \forall x \in \mathbb{T},$$

$$\lim_{d\to+\infty}\frac{1}{(\delta_+^d(x_0)-\delta_-^d(x_0))}\int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)}|\eta_2(x)|\Delta x = 0.$$

It follows that,

$$\begin{aligned}
&\Bigg|\int_{x_0}^{+\infty} k(z)\eta_1(\delta_\pm^p(\delta_-^z(x)))\Delta z \\
&\qquad - \int_{x_0}^{+\infty} k(z)\eta_1(\delta_-^z(x))\Delta z\Bigg| \\
\le\ & \int_{x_0}^{+\infty}|k(z)||\eta_1(\delta_\pm^p(\delta_-^z(x))) - \eta_1(\delta_-^z(x))|\Delta z \\
<\ & \int_{x_0}^{+\infty}|k(z)|\Delta z\frac{\varepsilon}{1+\int_{x_0}^{+\infty}|k(z)|\Delta z} \\
<\ & \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{d\to+\infty}\frac{1}{(\delta_+^d(x_0)-\delta_-^d(x_0))} \\
&\quad \int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)}\left|\int_{x_0}^{+\infty}k(z)\eta_2(\delta_-^z(x))\Delta z\right|\Delta x \\
\le\ & \lim_{d\to+\infty}\frac{1}{(\delta_+^d(x_0)-\delta_-^d(x_0))} \\
&\quad \int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)}\int_{x_0}^{+\infty}|k(z)||\eta_2(\delta_-^z(x))|\Delta z\Delta x \\
=\ & \lim_{d\to+\infty}\frac{1}{(\delta_+^d(x_0)-\delta_-^d(x_0))} \\
&\quad \int_{x_0}^{+\infty}|k(z)|\int_{\delta_-^d(x_0)}^{\delta_+^d(x_0)}|\eta_2(\delta_-^z(x))|\Delta x\Delta z \\
=\ & \lim_{d\to+\infty}\frac{1}{(\delta_+^d(x_0)-\delta_-^d(x_0))} \\
&\quad \int_{x_0}^{+\infty}|k(z)|\int_{\delta_-^{d+z}(x_0)}^{\delta_+^{d-z}(x_0)}|\eta_2(\zeta)|\delta_+^{\Delta_\zeta}(z,\zeta)\Delta\zeta\Delta z \\
\le\ & \lim_{d\to+\infty}\frac{r}{(\delta_+^d(x_0)-\delta_-^d(x_0))} \\
&\quad \int_{x_0}^{+\infty}|k(z)|\int_{\delta_-^{d+z}(x_0)}^{\delta_+^{d+z}(x_0)}|\eta_2(\zeta)|\Delta\zeta\Delta z \\
=\ & 0,
\end{aligned}$$

that is,

$$\int_{x_0}^{+\infty}k(z)\eta_1(\delta_-^z(x))\Delta z \in APS(\mathbb{T},\mathbb{R}^+),$$
$$\int_{x_0}^{+\infty}k(z)\eta_2(\delta_-^z(x))\Delta z \in PAPS_0(\mathbb{T},\mathbb{R}^+),$$

and then,

$$\int_{x_0}^{+\infty}k(z)\eta(\phi(\delta_-^z(x)))\Delta z \in PAPS(\mathbb{T},\mathbb{R}^+).$$

This completes the proof. ■

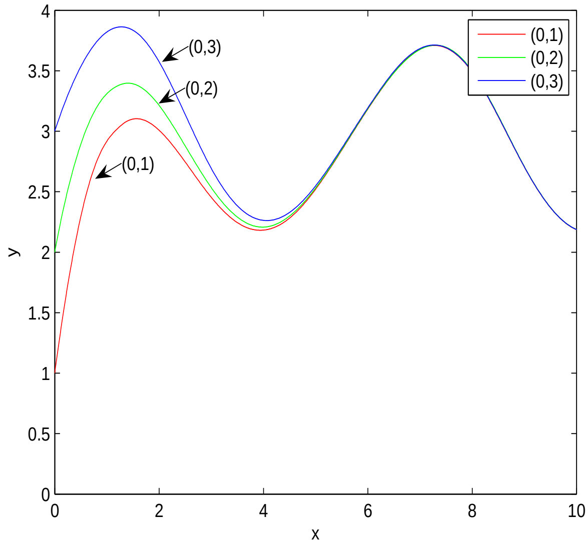
According to Theorem 2, we can obtain the following theorem.

Theorem 5. *Suppose that $(H_3)-(H_5)$ hold, then (17) exists a positive solution $y \in PAPS(\mathbb{T},\mathbb{R}^+)$.*

Now, we give a numerical example. Let $\mathbb{T} = \mathbb{R}$, then $\mu(x) = 0$. Take $x_0 = 0$, $\delta_-^z(x) = x - z$, and

$$\begin{aligned}
a(y) &= \frac{1}{2} - \frac{1}{4}\cos(y), \\
b(x) &= \frac{1}{3} + \frac{1}{6}\sin(x), \\
c(x) &= (2+\cos(x)) + \frac{1}{2+x^2}, \\
k(z) &= e^{-z}, \eta(\theta) = \frac{1}{10}(|\theta+1| - |\theta-1|).
\end{aligned}$$

According to Theorem 5, (17) exists a positive solution $y \in PAPS(\mathbb{T},\mathbb{R}^+)$. Dynamic simulations of (17), see Figure 2.



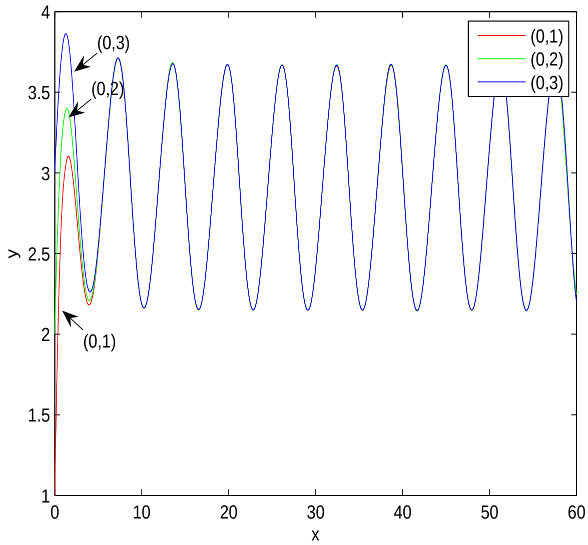


Fig. 2. Numerical solutions of (17) (Example 2) with the initial values $y(0) = \{1, 2, 3\}$.

V. Conclusion

This paper not only provides a new technique for studying the existence of solutions of dynamic equations on time scales, but also gives the existence theorems of almost periodic and pseudo almost periodic solutions in shifts $\delta_{\pm}$ of two kinds of nonlinear dynamic equations.

The study of ecosystem defined on specific time scale (see Example 1) shows that the results of this paper have practical significance in improving the theory of almost periodic and pseudo almost periodic on time scales, which provides a theoretical basis for the study of nonlinear dynamic equations on more general time scales, especially the time scales are not closed under addition.

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