Robust Control of the Caputo-Fabrizio Fractional-order Finance System with Double Constraints in Input

Xiaomin Tian, Xingliu Hu, Juan Gu, Jiaqi Ge, Juan Yin

Abstract—**A new definition of fractional derivatives called Caputo-Fabrizio definition is proposed in this paper, and used to develop mathematical models for the finance system. For the purpose of realizing the robust stabilization of fractional-order finance system, the influence of double constraints in input is taken into account. Meanwhile, the details about numerical computation method of Caputo-Fabrizio fractional differential equation are shown. Furthermore, in order to improve the feasibility of fractional-order financial system based on new definition, we also consider the effects of unknown parameters and external perturbations, and their upper bounds are previously unknown. In the final step, simulation results are given to verify that the proposed control strategy is effectively in overcome constraints of inputs, and the controlled system has a good robustness against the external disturbances and system uncertainties.**

Index Terms—**Caputo-Fabrizio definition, finance system, double constraints, fractional derivatives.**

I. Introduction

FRACTIONAL calculus is often used to model many physical problems with memory and nonlocal characteristics that can be difficult to model using integer order derivatives. Fractional calculus has a history of more than three centuries, which can be traced back to the achievements of some researchers, such as, Leibniz, Riemann, Liouville, Grunwald and Letnikov, et al [1]. At present, most of the existing results were based upon the definition of the Riemann-Liouville or the Caputo fractional-order calculus. Such as fractional order sliding mode control method [2], the stabilization or synchronization of fractional-order chaotic system [3-6], fractional-order backstepping control strategy [7-8], etc.

However, there are researchers pointed out the above definitions of fractional-order derivatives have singular kernels at the endpoint of the defined interval, which reduces the accuracy of modeling. Thus, an new definition of fractional order derivative called Caputo-Fabrizio definition has been proposed to overcome this issue. For example, Moore used Caputo-Fabrizio fractional differential equation to model HIV/AIDS disease [9], Baleanu applied Caputo-Fabrizio derivative to research the transmission of COVID-19 [10]. The fundamental distinctions between the Caputo-Fabrizio definition with traditional definition of fractional calculus are the former gives less noise and always has a suitable kernel to accurately describe the memory effect in a real system.

Moreover, when the system is working, the perturbations are inevitable, so, it must be taken into account in the process of designing a robust controller. Further, the nonlinearity in control inputs is often encountered in various systems and may be the cause of instability. Therefore, it is evident that the influence of input nonlinearity must be considered when analyzing and implementing control strategies. Although there are some results have been reported for compensating the input nonlinearities, almost of them are only focus on single nonlinearity, such as dead-zone nonlinearity [11], sector nonlinearity [12], and saturation nonlinearity [13].

Finance system is composed by interest rate, the investment demand and the price exponent, the finance system is a complex nonlinear system and may behave chaotically with the change of system parameters. In 2001, Ma [14,15] presented a macro-financial system, but the model is based on integer order derivative equations, that can not describe the intrinsic memory properties, then reduced the modeling accuracy. At present, much work has been done to research the dynamic performance of fractional-order finance systems [16-18]. Considering the shortcomings of traditional fractional-order derivative equations, in this paper, the Caputo-Fabrizio definition is applied to research the dynamic characteristics of fractional-order financial system.

Furthermore, in most studies on control or stabilization of fractional-order systems, system parameters are known in advance. In fact, the influence of unknown parameters cannot be ignored, as they may disrupt the system's behavior and even lead to unexpected outputs, therefore, it is necessary to design an adaptive controller to solve this problem.

Motivated by the above discussions, studying the stability of fractional-order financial systems with double nonlinearity in input based on Caputo-Fabrizio definition remains a highly challenging and necessary topic. In order to deal with the double restraints in input, the nonlinear inputs can be decomposed into two class of nonlinear inputs. Appropriate estimation laws are provided for the unknown upper bound of external disturbances and system parameters. The fractional-

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order version of Lyapunov stability theory is used to verify the stability of the controlled system.

The remaining part of this article is given as follows: Relevant Caputo-Fabrizio definitions and computation method are introduced in section 2. Main theory results are presented in Section 3. A numerical simulation example is shown in section 4, the simulation results confirmed the effectiveness and feasibility of the proposed method. Finally, conclusions are given in Section 5.

II. Preliminaries

A. Definitions

Definition 1 (see [9,19]). Let $f \in H^1(a,b)$ and $\rho \in (0,1)$. Then the definition of Caputo-Fabrizio fractional derivative is given as

$$ {}^{CF}D_t^{\rho}(f(t)) = \frac{M(\rho)}{1-\rho}\int_a^t f'(x)\exp\Big[-\rho\frac{t-x}{1-\rho}\Big]dx \quad (1)$$

where $M(\rho)$ is a normalization function such that $M(0) = M(1) = 1$. However, if $f \notin H^1(a,b)$, then the derivative is defined as

$$ {}^{CF}D_t^{\rho}(f(t)) = \frac{\rho M(\rho)}{1-\rho}\int_a^t (f(t)-f(x))\exp\Big[-\rho\frac{t-x}{1-\rho}\Big]dx \quad (2)$$

Definition 2 (see [9]). Let $0 < \rho < 1$. The fractional integral of order ρ of a functional $f(t)$ is defined by

$$ {}^{CF}I_t^{\rho}(f(t)) = \frac{2(1-\rho)}{(2-\rho)M(\rho)}f(t) + \frac{2\rho}{(2-\rho)M(\rho)}\int_0^t f(x)dx \quad (3)$$

where $t \geq 0$, $M(\rho) = \frac{2}{2-\rho}$, $0 < \rho < 1$.

Definition 3 (see [20]). Let $0 < \rho < 1$. The fractional Caputo-Fabrizio derivative of order ρ of a function $f(t)$ is given by

$$ {}^{CF}D_t^{\rho}(f(t)) = \frac{1}{1-\rho}\int_a^t f'(x)\exp\Big[-\rho\frac{t-x}{1-\rho}\Big]dx \quad (4)$$

where $t \geq 0$, and its fractional integral is defined as

$$ {}^{CF}I_t^{\rho}(f(t)) = (1-\rho)f(t) + \rho\int_0^t f(x)dx, \quad t \geq 0 \quad (5)$$

In the following part of this paper, we will use D^{ρ} to instead of ${}^{CF}D_t^{\rho}$.

B. Computation Method

Consider a Caputo-Fabrizio fractional-order differential equation [21]:

$$ {}_0^{CF}D_t^{\alpha}x(t) = f(t,x,u), \quad 0 < \alpha \leq 1 \quad (6)$$

where ${}_0^{CF}D_t^{\alpha}(\cdot)$ is Caputo-Fabrizio fractional-order calculus operator defined in ref.[9]. By integrating eq.(6) using the Caputo-Fabrizio fractional integral, we obtain:

$$ {}_0^{CF}I_t^{\alpha}({}_0^{CF}D_t^{\alpha}x(t)) = {}_0^{CF}I_t^{\alpha}(f(t,x,u)) \quad (7)$$

$$ x(t)-x(0) = \frac{1-\alpha}{M(\alpha)}f(t,x,u) + \frac{\alpha}{M(\alpha)}\int_0^t f(z,x(z),u(z))dz \quad (8)$$

Separate time intervals into steps with h intervals; we thus have $t_0 = 0$. $t_{k+1} = t_k + h, k = 0 : n-1$. Now, eq.(8) can be rewritten as

$$ \begin{aligned} x(t_{k+1}) - x(0) = {} & \frac{1-\alpha}{M(\alpha)}f(t_k, x(t_k), u(t_k)) \\ & + \frac{\alpha}{M(\alpha)}\int_0^{t_{k+1}} f(z,x(z),u(z))dz \end{aligned} \quad (9)$$

also, we have

$$ \begin{aligned} x(t_k) - x(0) = {} & \frac{1-\alpha}{M(\alpha)}f(t_{k-1}, x(t_{k-1}), u(t_{k-1})) \\ & + \frac{\alpha}{M(\alpha)}\int_0^{t_k} f(z,x(z),u(z))dz \end{aligned} \quad (10)$$

subtracting eq.(10) from eq.(9) gives

$$ \begin{aligned} x(t_{k+1}) - x(t_k) = {} & \frac{1-\alpha}{M(\alpha)}(f(t_k, x(t_k), u(t_k)) \\ & - f(t_{k-1}, x(t_{k-1}), u(t_{k-1})))) \\ & + \frac{\alpha}{M(\alpha)}\int_{t_k}^{t_{k+1}} f(t,x(t),u(t))dt \end{aligned} \quad (11)$$

For calculating eq.(11), we approximated the integral $\int_{t_k}^{t_{k+1}} f(t,x(t),u(t))dt$ by $\int_{t_k}^{t_{k+1}} K(t)dt$, where $K(t)$ is a second-order Lagrange interpolation polynomial, which can be calculated by the following formula:

$$ K(t) = \sum_{i=0}^{2} f(t_{k-i}, x(t_{k-i}), u(t_{k-i}))L_i(t) \quad (12)$$

where the $L_i(z)$ terms is a Lagrange basis polynomials for each point. Using the above approximation, it can be proven that

$$ \begin{aligned} \int_{t_{k+1}}^{t_k} f(t,x(t),u(t))dv = {} & h\Big[\frac{23}{12}f(t_k, x(t_k), u(t_k)) - \frac{4}{3} \times \\ & f(t_{k-1}, x(t_{k-1}), u(t_{k-1})) + \frac{5}{12} \times \\ & f(t_{k-2}, x(t_{k-2}), u(t_{k-2}))\Big] \end{aligned} \quad (13)$$

where v is defined as $v = \frac{t_{k+1}-t}{h}$. Then, using eq.(13), according to eq. (11), the recursive formula is as follows:

$$ \begin{aligned} x(t_{k+1}) = {} & x(t_k) + \frac{1}{M(\alpha)}\Big[(1-\alpha) + \frac{23}{12}h\alpha\Big]f(t_k, x(t_k), u(t_k)) \\ & - \frac{1}{M(\alpha)}\Big[(1-\alpha) + \frac{4}{3}h\alpha\Big]f(t_{k-1}, x(t_{k-1}), u(t_{k-1})) \\ & + \frac{5h\alpha}{12M(\alpha)}f(t_{k-2}, x(t_{k-2}), u(t_{k-2})) \end{aligned} \quad (14)$$

III. Main Results

The following mathematical model of fractional-order finance system is proposed in [22]:

$$ \begin{aligned} D^{\alpha}x_1 &= x_3 + x_1(x_2 - a) + \Delta f_1(x) + d_1(t) + \phi_1(u) \\ D^{\alpha}x_2 &= 1 - bx_2 - x_1^2 + \Delta f_2(x) + d_2(t) + \phi_2(u) \\ D^{\alpha}x_3 &= -x_1 - cx_3 + \Delta f_3(x) + d_3(t) + \phi_3(u) \end{aligned} \quad (15)$$

where $x^T = (x_1, x_2, x_3)$ is system state vector, the state variables x_1, x_2 and x_3 represent interest rate, investment demand, and price index, respectively; The system parameters a, b and c represent saving amount, cost per investment, and elasticity of commercial markets' demand, respectively.

All parameters a, b and c are nonnegative. $\Delta f_i(x)$ and $d_i(t)$ are unmodeled dynamics and external perturbation, respectively. $\phi_i(u), i = 1, 2, 3$ is nonlinear input with double constraints, which described as follows [23]:

$$\phi(u) = \begin{cases} U_M, & u \geq \omega_M \\ p(u - b_r), & b_r \leq u < \omega_M \\ 0, & b_l < u < b_r \\ p(u - b_l), & \omega_m < u \leq b_l \\ U_m, & u \leq \omega_m \end{cases} \tag{16}$$

where $\omega_M = \frac{U_M}{p} + b_r$, $\omega_m = \frac{U_m}{p} + b_l$, $p \in R$ is linear region slope. A typical nonlinear input with double constraints is drawn in Fig. 1.

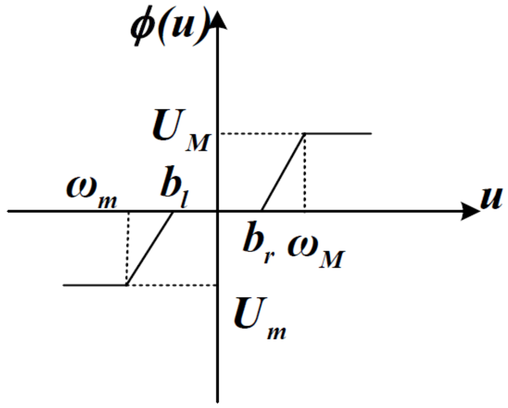


Fig. 1. A Typical Nonlinear Input with Double Constraints

In order to deal with the nonlinear inputs with double constraints, according to the theoretical results proposed in ref.[23], the input can be equivalent to saturated and dead-zone nonlinear inputs by projective decomposition, which shows in Fig. 2.

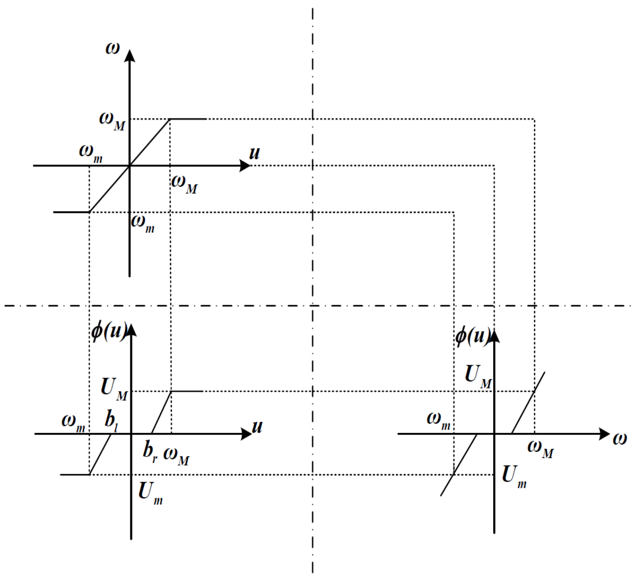


Fig. 2. Projective Decomposition of Input with Double Constraints

According to projective decomposition, the nonlinear input (16) can be divided into:

$$\omega(u) = \begin{cases} \omega_M, & u \geq \omega_M \\ u, & \omega_m \leq u < \omega_M \\ \omega_m, & u < \omega_m \end{cases} \tag{17}$$

and

$$\phi(\omega(u)) = \begin{cases} p[\omega(u) - b_r], & \omega(u) \geq b_r \\ 0, & b_l \leq \omega(u) < b_r \\ p[\omega(u) - b_l], & \omega(u) < b_l \end{cases} \tag{18}$$

Furthermore, the nonlinear input $\phi(\omega(u))$ can be rewritten as

$$\phi(\omega(u)) = p\omega(u) + d''(\omega) \tag{19}$$

where

$$\omega(u) = u + \Delta u \tag{20}$$

$$\Delta u = \begin{cases} \omega_M - u, & u \geq \omega_M \\ 0, & \omega_m \leq u < \omega_M \\ \omega_m - u, & u < \omega_m \end{cases} \tag{21}$$

and

$$d''(\omega) = \begin{cases} -pb_r, & \omega(u) \geq b_r \\ -p\omega(u), & b_l \leq \omega(u) < b_r \\ -pb_l, & \omega(u) < b_l \end{cases} \tag{22}$$

thus, the controlled fractional-order finance system (15) correspondingly transformed into

$$\begin{aligned} D^\alpha x_1 &= x_3 + x_1(x_2 - a) + \Delta f_1(x) + d_1(t) + [p_1(u_1 + \Delta u_1) \\ &\quad + d_1''(\omega)] \\ D^\alpha x_2 &= 1 - bx_2 - x_1^2 + \Delta f_2(x) + d_2(t) + [p_2(u_2 + \Delta u_2) \\ &\quad + d_2''(\omega)] \\ D^\alpha x_3 &= -x_1 - cx_3 + \Delta f_3(x) + d_3(t) + [p_3(u_3 + \Delta u_3) \\ &\quad + d_3''(\omega)] \end{aligned} \tag{23}$$

according to the following standard form of fractional-order nonlinear system

$$\begin{aligned} D^\alpha x_1 &= F_1(x)\theta_1 + f_1(x) + \Delta f_1(x) + d_1(t) \\ D^\alpha x_2 &= F_2(x)\theta_2 + f_2(x) + \Delta f_2(x) + d_2(t) \\ &\vdots \\ D^\alpha x_n &= F_n(x)\theta_n + f_n(x) + \Delta f_n(x) + d_n(t) \end{aligned} \tag{24}$$

Comparing eq.(23) and eq.(24), we know that $n = 3$, $x = (x_1, x_2, x_3)^T$ is state vector. $F_1(x) = -x_1$, $F_2(x) = -x_2$, and $F_3(x) = -x_3$ are linear sections of the controlled system. $\theta_1 = a$, $\theta_2 = b$, and $\theta_3 = c$ are system parameters, we assume them are unknown in advance. $f_1(x) = x_1x_2 + x_3$, $f_2(x) = 1 - x_1^2$, and $f_3(x) = -x_1$ are nonlinear sections of the controlled system. $\Delta f_i(x)$ and $d_i(t)$ are bounded, and the bounds are given as follows

$$|\Delta f_i(x) + d_i(t)| \leq \gamma_i \tag{25}$$

where $i = 1, 2, ..., n$, γ_i is unknown positive real number, the form of unmodeled dynamics $\Delta f_i(x)$ and external perturbations $d_i(t)$ will given in the next section.

Meanwhile, in system (23), the input uncertainties $p_i\Delta u_i + d_i''(\omega)$ are bounded as follows

$$|p_i\Delta u_i + d_i''(\omega)| \leq \delta_i \tag{26}$$

where δ_i is an unknown positive real number.

Our next aim is to design an adaptive robust controller to achieve the stabilization of fractional-order finance system (23) with dual constraint nonlinear inputs. In order to improve the feasibility of the designed controller, we assume that the system parameters and the upper bounds of uncertainties are all unknown beforehand, then apply the fractional-order version of Lyapunov theory to discriminate the controlled system's stability

Theorem 1 Considering the fractional-order finance system (23) with double constraints in inputs, if the controllers are designed as follow

$$\begin{aligned} u_1 &= -\frac{1}{p_1}\Big(|x_1||\hat{a}|+|x_1x_2+x_3|+\hat{\gamma}_1+\hat{\delta}_1+k_1\Big)sgn(x_1) \\ u_2 &= -\frac{1}{p_2}\Big(|x_2||\hat{b}|+|1-x_1^2|+\hat{\gamma}_2+\hat{\delta}_2+k_2\Big)sgn(x_2) \\ u_3 &= -\frac{1}{p_3}\Big(|x_3||\hat{c}|+|x_1|+\hat{\gamma}_3+\hat{\delta}_3+k_3\Big)sgn(x_3) \end{aligned} \tag{27}$$

where k_1, k_2, k_3 are given positive constants, called control gains. The following fractional-order estimation laws of all unknown parameters are given

$$\begin{aligned} D^\alpha\hat{a} &= -x_1^2 \\ D^\alpha\hat{b} &= -x_2^2 \\ D^\alpha\hat{c} &= -x_3^2 \\ D^\alpha\hat{\gamma}_1 &= \lambda_1|x_1| \\ D^\alpha\hat{\gamma}_2 &= \lambda_2|x_2| \\ D^\alpha\hat{\gamma}_3 &= \lambda_3|x_3| \\ D^\alpha\hat{\delta}_1 &= \eta_1|x_1| \\ D^\alpha\hat{\delta}_2 &= \eta_2|x_2| \\ D^\alpha\hat{\delta}_3 &= \eta_3|x_3| \end{aligned} \tag{28}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{\gamma}_i$, $\hat{\delta}_i$ are estimations of a, b, c, γ_i, δ_i, respectively. λ_i and η_i are positive adaptive gains, $i = 1, 2, 3$. Then the controlled system (23) can be stabilized, and all unknown parameters can be fully identified.

Next, we use the theoretical results of [24] to validate the effectiveness of the proposed control strategy.

Proof. Selecting the following positive definite matrix Q as

$$Q = diag\Big(I_6, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\eta_1}, \frac{1}{\eta_2}, \frac{1}{\eta_3}\Big) \tag{29}$$

where I_6 is a 6-dimensional identity matrix. Denote $X^T = (x_1, x_2, x_3, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3, \widetilde{\delta}_1, \widetilde{\delta}_2, \widetilde{\delta}_3)$, in which, $\widetilde{a} = \hat{a} - a$, $\widetilde{b} = \hat{b} - b$, $\widetilde{c} = \hat{c} - c$, $\widetilde{\gamma}_i = \hat{\gamma}_i - \gamma_i$, $\widetilde{\delta}_i = \hat{\delta}_i - \delta_i$ are estimation errors of a, b, c, γ_i, δ_i, respectively. Then a function is constructed to demonstrate the stability of the controlled system, that is

$$J = X^T Q D^\alpha X \tag{30}$$

according to the above denotation, we obtain

$$\begin{aligned} J &= X^T Q D^\alpha X \\ &= x_1 D^\alpha x_1 + x_2 D^\alpha x_2 + x_3 D^\alpha x_3 + \widetilde{a} D^\alpha \widetilde{a} + \widetilde{b} D^\alpha \widetilde{b} \\ &\quad + \widetilde{c} D^\alpha \widetilde{c} + \frac{1}{\lambda_1}\widetilde{\gamma}_1 D^\alpha \widetilde{\gamma}_1 + \frac{1}{\lambda_2}\widetilde{\gamma}_2 D^\alpha \widetilde{\gamma}_2 + \frac{1}{\lambda_3}\widetilde{\gamma}_3 D^\alpha \widetilde{\gamma}_3 \\ &\quad + \frac{1}{\eta_1}\widetilde{\delta}_1 D^\alpha \widetilde{\delta}_1 + \frac{1}{\eta_2}\widetilde{\delta}_2 D^\alpha \widetilde{\delta}_2 + \frac{1}{\eta_3}\widetilde{\delta}_3 D^\alpha \widetilde{\delta}_3 \end{aligned} \tag{31}$$

according to eq.(23), one has

$$\begin{aligned} J &= x_1\Big[x_3 + x_1(x_2 - a) + \Delta f_1(x) + d_1(t) + p_1(u_1 + \Delta u_1) \\ &\quad + d_1^{''}(\omega)\Big] + x_2\Big[1 - bx_2 - x_1^2 + \Delta f_2(x) + d_2(t) \\ &\quad + p_2(u_2 + \Delta u_2) + d_2^{''}(\omega)\Big] + x_3\Big[-x_1 - cx_3 \\ &\quad + \Delta f_3(x) + d_3(t) + p_3(u_3 + \Delta u_3) + d_3^{''}(\omega)\Big] \\ &\quad + \widetilde{a} D^\alpha \widetilde{a} + \widetilde{b} D^\alpha \widetilde{b} + \widetilde{c} D^\alpha \widetilde{c} + \sum_{i=1}^{3} \frac{1}{\lambda_i}\widetilde{\gamma}_i D^\alpha \widetilde{\gamma}_i \\ &\quad + \sum_{i=1}^{3} \frac{1}{\eta_i}\widetilde{\delta}_i D^\alpha \widetilde{\delta}_i \end{aligned} \tag{32}$$

substituting eqs.(25), (26), (28) into (32), it yields

$$\begin{aligned} J &= x_1x_3 + x_1^2x_2 + x_1(\Delta f_1(x) + d_1(t)) + x_1p_1u_1 \\ &\quad + x_1(p_1\Delta u_1 + d_1^{''}(\omega)) + x_2 - x_1^2x_2 + x_2(\Delta f_2(x) \\ &\quad + d_2(t)) + x_2p_2u_2 + x_2(p_2\Delta u_2 + d_2^{''}(\omega)) - x_1x_3 \\ &\quad + x_3(\Delta f_3(x) + d_3(t)) + x_3p_3u_3 + x_3(p_3\Delta u_3 \\ &\quad + d_3^{''}(\omega)) + \hat{a} D^\alpha \hat{a} + \hat{b} D^\alpha \hat{b} + \hat{c} D^\alpha \hat{c} \\ &\quad + \sum_{i=1}^{3} \frac{1}{\lambda_i}\widetilde{\gamma}_i D^\alpha \widetilde{\gamma}_i + \sum_{i=1}^{3} \frac{1}{\eta_i}\widetilde{\delta}_i D^\alpha \widetilde{\delta}_i \\ &\leq |x_1||x_3 + x_1x_2| + \gamma_1|x_1| + x_1p_1u_1 + \delta_1|x_1| \\ &\quad + |x_2||1 - x_1^2| + \gamma_2|x_2| + x_2p_2u_2 + \delta_2|x_2| \\ &\quad + |x_3||x_1| + \gamma_3|x_3| + x_3p_3u_3 + \delta_3|x_3| + |\hat{a}||x_1^2| \\ &\quad + |\hat{b}||x_2^2| + |\hat{c}||x_3^2| + \sum_{i=1}^{3}(\hat{\gamma}_i - \gamma_i)|x_i| \\ &\quad + \sum_{i=1}^{3}(\hat{\delta}_i - \delta_i)|x_i| \\ &= |x_1||x_3 + x_1x_2| + x_1p_1u_1 + |x_2||1 - x_1^2| + x_2p_2u_2 \\ &\quad + |x_3||x_1| + x_3p_3u_3 + |\hat{a}||x_1^2| + |\hat{b}||x_2^2| + |\hat{c}||x_3^2| \\ &\quad + \sum_{i=1}^{3}\hat{\gamma}_i|x_i| + \sum_{i=1}^{3}\hat{\delta}_i|x_i| \end{aligned} \tag{33}$$

bring eq.(27) into (33), we have

$$\begin{aligned} J &\leq |x_1||x_3 + x_1x_2| - |x_1|\big(|x_1||\hat{a}| + |x_1x_2 + x_3| + \hat{\gamma}_1 \\ &\quad + \hat{\delta}_1 + k_1\big) + |x_2||1 - x_1^2| - |x_2|\big(|x_2||\hat{b}| + |1 - x_1^2| \\ &\quad + \hat{\gamma}_2 + \hat{\delta}_2 + k_2\big) + |x_3||x_1| - |x_3|\big(|x_3||\hat{c}| + |x_1| + \hat{\gamma}_3 \\ &\quad + \hat{\delta}_3 + k_3\big) + |\hat{a}||x_1^2| + |\hat{b}||x_2^2| + |\hat{c}||x_3^2| + \sum_{i=1}^{3}\hat{\gamma}_i|x_i| \\ &\quad + \sum_{i=1}^{3}\hat{\delta}_i|x_i| \\ &= -k_1|x_1| - k_2|x_2| - k_3|x_3| \leq k||x|| < 0 \end{aligned} \tag{34}$$

where $k = min\{k_1, k_2, k_3\}$. That is, the controlled fractional-order finance system (23) is asymptotic stability. Therefore, the proof is completed.

IV. Simulation Results

In this section, numerical simulation is presented to demonstrate the correctness and effectiveness of the designed control scheme. Consider the fractional-order system (23), select the system order $\alpha = 0.96$, system parameters $a = 0.3$, $b = 0.1$, $c = 0.1$, $\Delta f_1(x) + d_1(t) = 0.01cos(2x_1) + 0.02sin(2t)$, $\Delta f_2(x) + d_2(t) = 0.02cos(x_2) - 0.015sin(3t)$, $\Delta f_3(x) + d_3(t) = -0.025cos(3x_3) - 0.03sin(4t)$, the adaptive gains $\lambda_1 = \lambda_2 = \lambda_3 = 2$, $\delta_1 = \delta_2 = \delta_3 = 4$, the control gains $k_1 = k_2 = k_3 = 5$. The parameters of nonlinear input are $U_M = 1.8$, $U_m = -1.5$, $b_r = 0.8$, $b_l = -0.5$, $p = 1$. The initial conditions $x(0) = (1, 3, 2)^T$, $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = 0.1$, $\hat{\gamma}_1(0) = \hat{\gamma}_2(0) = \hat{\gamma}_3(0) = 0.2$, $\hat{\delta}_1(0) = \hat{\delta}_2(0) = \hat{\delta}_3(0) = 0.3$.

When the controller is not activated, the system (23) behave chaotically, the strange attractors shown in Fig. 3, and the state trajectories are presented in Fig. 4, it is clearly that the system is unstable.

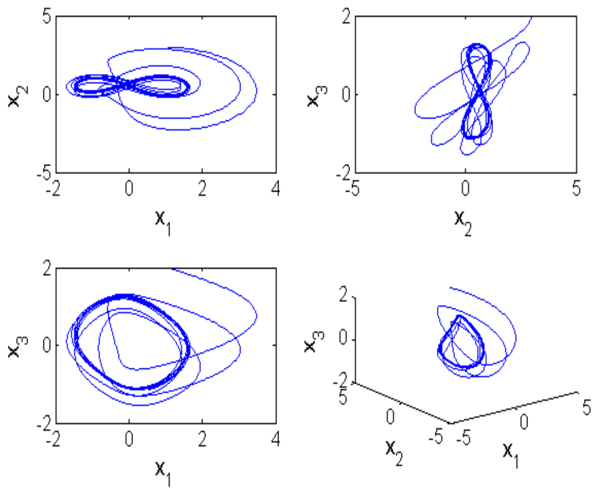
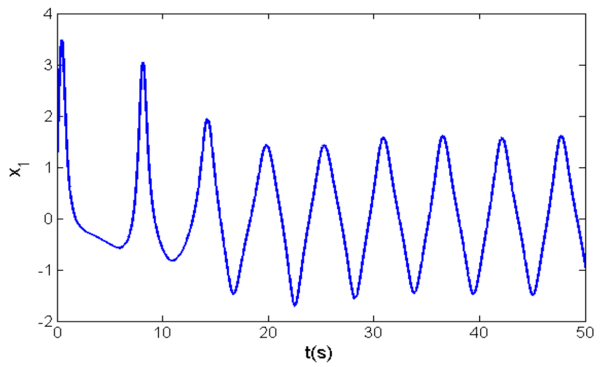
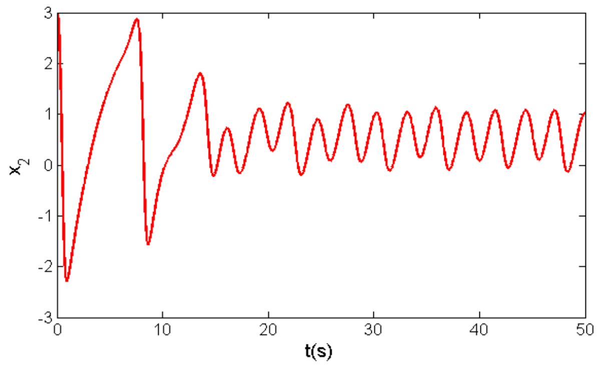


Fig. 3. Strange Attractors of Fractional-order Finance System (23)





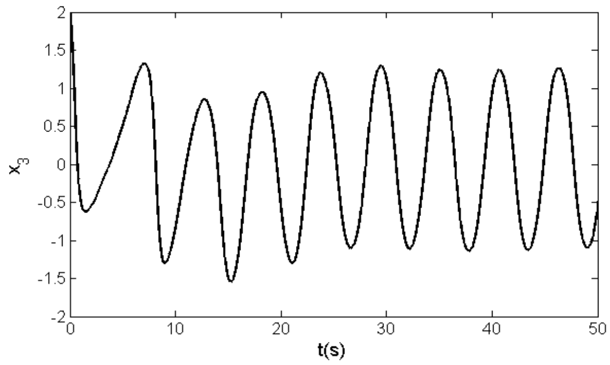


Fig. 4. State Trajectories of Fractional-order Finance System (23) without Controller

When the controller is activated, the time responses of fractional-order finance system (23) with controller are depicted in Fig. 5.

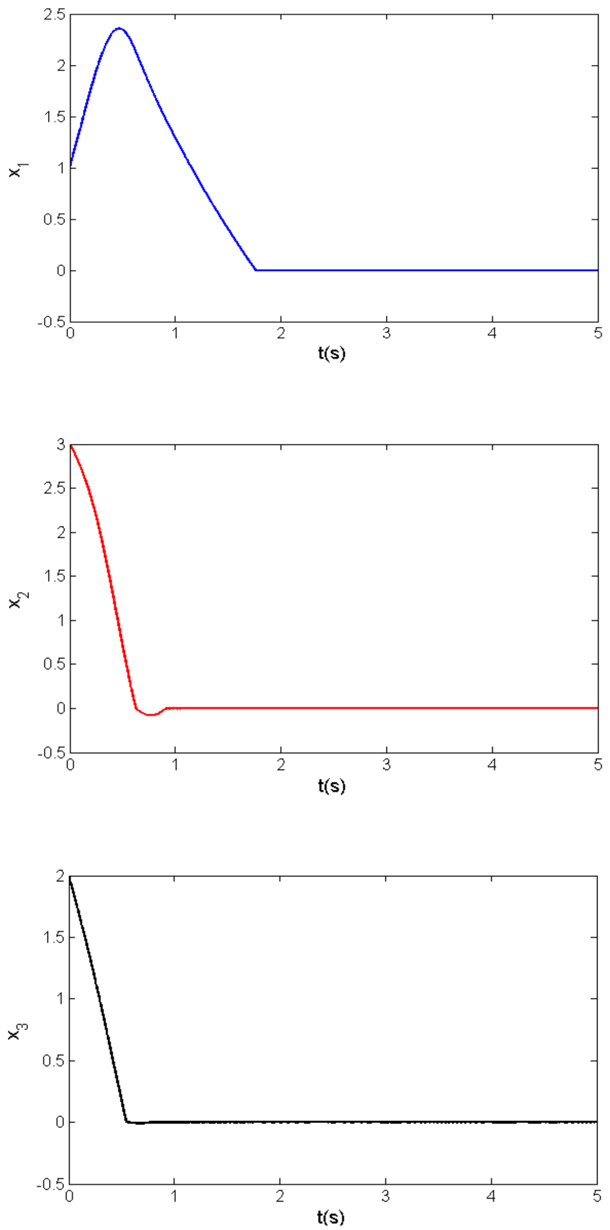


Fig. 5. Time Responses of Fractional-order Finance System (23) with Controller Activated

Meanwhile, all unknown parameters are fully identified, such as system parameters, the bounds of unmodeled dynamics, external disturbances, and input uncertainties, which time evolutions are shown in Fig. 6-8.

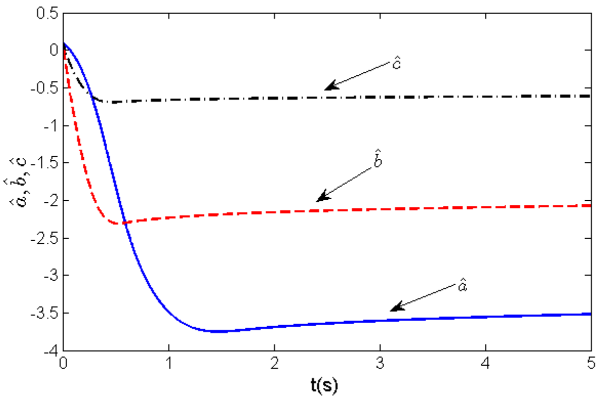


Fig. 6. Time Evolutions of Unknown System Parameters in System (23)

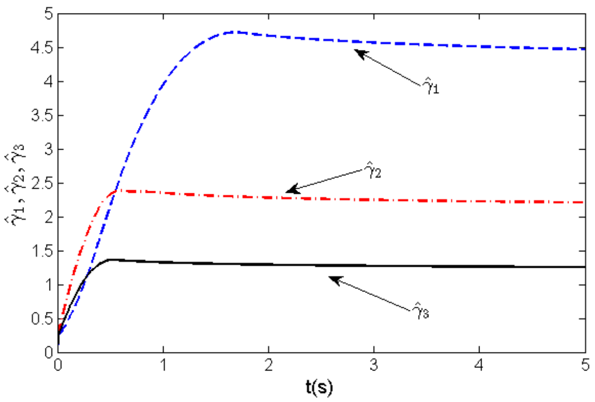


Fig. 7. Time Evolutions of the Bounds of Unmodeled Dynamics and External Disturbances in System (23)

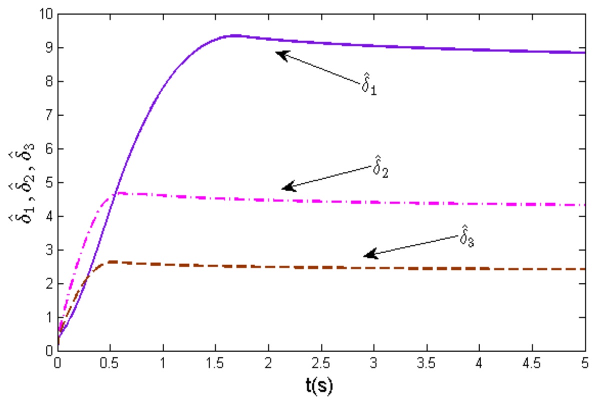


Fig. 8. Time Evolutions of the Bounds of Input Uncertainties in System (23)

The above simulation results confirmed that the proposed control strategy is effective in stabilizing the fractional-order finance system with double constraints in inputs. Under the effect of the designed controller, all unknown parameters can be completely identified.

V. Conclusions

The adaptive stabilization of fractional-order finance system based on Caputo-Fabrizio definition is investigated in this paper. New definition can overcome the problem of singular kernel and improve modeling accuracy. In the processing of design controller, the influence of nonlinear inputs with double constraints is taken into account, and the system parameters, the bounds of uncertainties are all assumed unknown beforehand. For demonstrating the stability of the controlled system, the fractional-order stability discrimination theory is applied, its shown that the designed control scheme is effective to deal with this kind of control problem.

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