Properties of Set Scalarization Function and Its Application to Set Optimization Problems

Xiangyu Kong, Yonghong Zhang, Zhen Wang

Abstract—**This article investigates the optimality conditions for solutions to set optimization problems using set scalarization functions defined by oriented distance functions. Specifically, we begin by examining the sup-inf set scalarization function, which is defined by Hiriart-Urruty's oriented distance function. We then proceed to define the Dini directional derivative of set-valued maps and analyze its properties. Finally, we obtain the optimality conditions for solutions to set optimization problems through the use of the Dini directional derivative.**

Index Terms—**set optimization problem, set scalarization functions, oriented distance functions, Dini directional derivative, set order relations**

I. Introduction

SET optimization problem is a vital problem of decision optimization. In many real-life optimization problems, the objective is a set rather than an individual, which means the research on set optimization problems are extremely important. Last decades, the set optimization problems have been extensively used for solving many issues, such as financial mathematics problems, multi-objective problems, vector variational inequalities, and optimal control problems, which attracted extensive attention from numerous scholars [1-5].

There are two standards of characterization for the solutions of set-valued optimization problems, i.e. set criterion and vectorial criterion. The researches on these two criteria are independent. The vectorial criterion is defined as finding the effective points of objective function image set. Therefore, the set-valued optimization problems with vectorial criterion can be referred to as set-valued vector optimization problems or vector optimization problems with set mapping. Many studies have been done on this type of problem. On the other, Kuroiwa[6] studied the solutions of set-valued optimization problems under set criterion. Thereafter, Kuroiwa et al. [7] proposed six set order relations of set-valued optimization problems. Using the set criterion for set-valued optimization problems is naturally rather than vectorial criterion. Therefore, set-valued optimization problems under set criterion can be defined as set optimization problems. The researches of set optimization problems under different set order relations are significant.

The scalarization function is essential for solving vector optimization problems and set optimization problems. Theoretically, it is an essential tool for the research of optimality conditions. Computationally, it is a vital means for discovering new algorithms. The linear scalarization function was first put forward and widely used. After that, two crucial nonlinear scalarization functions were proposed, which are Gerstewitz ’ s function [8] and Hiriart-Urruty oriented distance function [9]. Gerstewitz ’ s function became an essential tool for researching the vector optimization problems. By using the Hiriart-Urruty oriented distance function, well-posedness of the solution sets, the nonlinear scalarization results and Lagrange multiplier rule of vector optimization problems were obtained in [10-14]. In the recent researches, the Hiriart-Urruty oriented distance function was also studied for set optimization problems. In [15], the optimality condition of four kinds of optimal solutions for constraint set optimization problems based on different set order relations were gained by the oriented distance function. In [16], a sup-inf set scalarization function was proposed, and directional derivative of the set-valued map were defined by using the sup-inf set scalarization function. Moreover, the optimality conditions of solutions to set optimization problems were concluded. In [17-18], the properties of the sup-inf set scalarization function were analyzed, and the concept of minimum for set optimization problems was described by the sup-inf set scalarization function. In [19], six generalized oriented distance functions were discussed, and the optimality conditions of solutions for set optimization problems were given by six set scalarization functions. In [20], Dini directional derivative of the set-valued map was defined by imposing sup-inf set scalarization function, and the optimality condition of solutions for set optimization problems was described using this type of directional derivative. From the above results, it is significant to study different kinds of the sup-inf set scalarization functions and research on optimality conditions of solutions for set optimization problems based on these set scalarization functions.

The rest of this paper is organized as follows. The basic knowledge and preliminaries are introduced in Sect. 2. In Sect. 3, the properties of the sup-inf set scalarization function proposed by Jiménez [19] are studied. In Sect. 4, based on the sup-inf set scalarization function proposed by Jiménez, we define the Dini directional derivative and discuss the properties of this directional derivative. Furthermore, the

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optimality condition of solutions for set optimization problems is gained by Dini directional derivative. Finally, we conclude this paper.

II. Preliminaries

Assume V is a normed vector space. Let B_V be a closed unit sphere in V, and B_V^0 be an open unit sphere in V. Define $\wp_0(V)$ as the entire nonempty subsets of V. Suppose that $C \subseteq V$ is a pointed closed convex cone, and C has the nonempty interior. The cone C generates a partial order on V, which can be defined below. $\forall v_1, v_2 \in V$, $v_1 \leq v_2 \Leftrightarrow v_2 - v_1 \in C$.

Denote the topological dual space of V as V^*, and the topological dual cone of C as C^*, which is given as follows:

$$C^* = \{\varphi \in V^* : \varphi(v) \geq 0, \forall v \in C\}.$$

Define $B_C^* = \{\varphi \in C^* : \|\varphi\| = 1\}$, and the support function of A at φ can be defined as $S(\varphi, A) = \sup_{a\in A} \varphi(a)$, where $A \in \wp_0(V)$ and $\varphi \in V^*$. The upper relation "$\leq^u$" can be denoted as $M \leq^u N \Leftrightarrow M \subseteq N - C$, the weak upper relation "$<<^u$" can be noted as $M <<^u N \Leftrightarrow M \subseteq N - \operatorname{int} C$, and the equivalence relation "$\sim^u$" can be defined by $M \sim^u N \Leftrightarrow M \leq^u N$ and $N \leq^u M$, where $M, N \in \wp_0(V)$.

Denote the topological interior of M as $\operatorname{int} M$; the topological closure of M as clM; the topological boundary of M as ∂M; the convex hull of M as coM; and the complementary set of M as M^c. For a nonempty set $M \subseteq V$, if $M + C \neq V$, it is known as C-proper; if $M + C$ is a convex set, it is known as C-convex; if $M + C$ is a closed set, it is known as C-closed; if for each neighbourhood I of zero in V, there exist some number $t > 0$ that $M \subseteq tI + C$, it is known as C-bounded; and if any cover formed as $\{I_\alpha + C : I_\alpha \text{ are open}\}$ of M exists a finite subcover, it is called C-compact. Note $N(0)$ as the neighborhoods of $0 \in V$.

Remark 1 Obviously, if there is $\beta > 0$ such that βB_V is C-closed, thus $\forall \delta > 0$ that δB_V is C-closed.

Note that $M \in \wp_0(V)$ and $m \in M$. If $(M - m) \bigcap (-C) = \{0\}$, m is a minimal point of M as regards C can be defined as $m \in Min(M)$. If $(M - m) \bigcap (-\operatorname{int} C) = \Phi$, m is a weak minimal point of M as regards C can be defined as $m \in WMin(M)$.

Remark 2[21] Evidently, $Min(M) \subseteq WMin(M)$. Furthermore, if M is nonempty and C-compact, therefore $Min(M) \neq \Phi$.

Let U be a normed vector space and D is a nonempty subset of U. Denote $F : U \to 2^V$ is a set-valued mapping, then the set optimization problem is as follows:

(SOP) $\min_{x\in D} F(x)$.

Definition 1 $x^* \in D$ is referred to as

(1) u-minimal solution of (SOP), if $\forall x \in D$, $F(x) \leq^u F(x^*)$ means that $F(x^*) \leq^u F(x)$;

(2) weak u-minimal solution of (SOP), if $\forall x \in D$, $F(x) <<^u F(x^*)$ means that $F(x^*) <<^u F(x)$.

Note that $E_u(F, D)$ and $W_u(F, D)$ are the u-minimal solution set and weak u-minimal solution set of (SOP), respectively.

Example 1 Let $\mathbb{R}^2$ ordered by $\mathbb{R}^2_+$,Let $U = \{S_x : x \in [0, \infty)\}$ be the family of subsets of $\mathbb{R}^2$ defined by

$$S_x = \begin{cases} \{(0,0)\} & if\ x = 0 \\ [(0,0),(x, -\frac{1}{x})] & if\ x \neq 0 \end{cases}$$

It is easy to check that there are not u-minimal sets of U, however each S_x is a weak u-minimal sets of U.

Definition 2[22] A set-valued mapping $\Psi : U \to 2^V$ is called C-convex on D, where D is a convex subset and nonempty of U, if $\forall u_1, u_2 \in D$ and $\forall \lambda \in [0,1]$,

$$\Psi(\lambda u_1 + (1-\lambda)u_2) \subseteq \lambda\Psi(u_1) + (1-\lambda)\Psi(u_2) - C.$$

Definition 3[23] Note that (U, d) is a metric space. Denote M and N are two nonempty subsets of U. The Hausdorff distance can be refereed to as

$$H(M, N) = \max\{e(M, N), e(N, M)\},$$

where

$$e(M, N) = \sup_{m\in M} d(m, N),\ d(m, N) = \inf_{n\in N} d(m, n).$$

Definition 4[9] For a set $D \subseteq V$, define the oriented distance function $\Delta_D : V \to R \bigcup \{\pm\infty\}$ as

$$\Delta_D(v) = d_D(v) - d_{V\setminus D}(v),$$

with $d_\Phi(v) = +\infty$, where $d_D(v) = \inf_{x\in D} \|v - x\|$.

The elementary properties of the oriented distance function are given as follows.

Lemma 1[24-25] If $D \subseteq V$ is nonempty, and $D \neq V$, we have

(1) Δ_D is a real valued function;

(2) Δ_D is a 1-Lipschitzian function;

(3) $\Delta_D(v) < 0 \Leftrightarrow v \in \operatorname{int} D$;

(4) $\Delta_D(v) = 0 \Leftrightarrow v \in \partial D$;

(5) $\Delta_D(v) > 0 \Leftrightarrow v \in \operatorname{int} D^c$;

(6) if D is closed, thus $D = \{v \in V : \Delta_D(v) \leq 0\}$ holds;

(7) if D is a cone, thus Δ_D is a positively homogeneous function;

(8)if D is convex, thus Δ_D is a convex function;

(9) if D is a closed convex cone, $\forall v, v' \in V$, $v - v' \in D \Rightarrow \Delta_D(v) \leq \Delta_D(v')$;

if D has a nonempty interior, therefore $\forall v, v' \in V$,

$$v - v' \in \operatorname{int} D \Rightarrow \Delta_D(v) < \Delta_D(v').$$

Therefore, we can easily derive the lemmas below.

Lemma 2 Let $\delta \geq 0$. Then

$$d_D(v) \geq \delta \Leftrightarrow (v+\delta B_V^0) \bigcap D = \Phi .$$

Lemma 3 Let $\delta \geq 0$. There exist:

(1) if $\delta B_V + D$ is a closed set, $v \in \delta B_V + D$, then $d_D(v) \leq \delta$;

(2) if $\delta B_V + D$ is a closed set and $d_D(v) \leq \delta$, then $v \in \delta B_V + D$.

Proof (1)Since $v \in \delta B_V + D$ and $d_n \in D$, then $v - d_n \in \delta B_V$, i.e. $\|v - d_n\| < \delta$,so $d_D(v) \leq \delta$.

(2)For $n \in N$, because $d_D(v) \leq \delta$, then $d_D(v) \leq \delta < \delta + \frac{1}{n}$. Therefore if $d_n \in D$, there exists $\|v - d_n\| < \delta + \frac{1}{n}$. Thereby,

$$v \in (\delta + \frac{1}{n})B_V + d_n \subseteq (\delta + \frac{1}{n})B_V + D, \ \forall n \in N .$$

Hence, there is $b_n \in B_V$ that $v - \frac{1}{n} b_n \in \delta B_V + D$. Because $v - \frac{1}{n} b_n \to v$ and $\delta B_V + D$ is closed, $v \in \delta B_V + D$ holds. □

The following corollary is given based on Remark 1 and Lemma 3.

Corollary 1 If D is a cone, and $B_V + D$ is a closed set, then for $\forall \delta > 0$, $d_D(v) \leq \delta \Rightarrow v \in \delta B_V + D$.

Proof since $B_V + D$ is a closed set,by Remark 1, $\forall \delta > 0$,that $B_V + D$ is a closed set, from Lemma 3, $d_D(v) \leq \delta \Rightarrow v \in \delta B_V + D$.

Lemma 4 If $\delta > 0$ and D is C -bounded, then $D \not\subset D - \bigcap_{\beta \in \delta B_V} (\beta + \operatorname{int} C)$.

Proof Assume that

$$D \subset D - \bigcap_{\beta \in \delta B_V} (\beta + \operatorname{int} C) . \quad (1)$$

Thus, $\forall d_1 \in D$, there exists $d_2 \in D$, so that

$$d_1 - d_2 \in - \bigcap_{\beta \in \delta B_V} (\beta + \operatorname{int} C) . \quad (2)$$

$\forall \beta \in \delta B_V$, it is clear that $-\beta \in \delta B_V$. Together with (2) means that $d_1 - d_2 \in \beta - \operatorname{int} C$ and $d_1 - d_2 - \beta \in -\operatorname{int} C$. $\forall \beta \in \delta B_V$, we have $d_1 - d_2 - \delta B_V \subseteq -\operatorname{int} C$. Similarly, for $d_n \in D$, there is $d_{n+1} \in D$ so that $d_n - d_{n+1} - \delta B_V \subseteq -\operatorname{int} C$. This shows that $d_1 - d_{n+1} - n\delta B_V \subseteq -\operatorname{int} C$ and

$$d_1 - d_{n+1} - n\delta B_V - C \subseteq -\operatorname{int} C - C \subseteq -\operatorname{int} C . \quad (3)$$

Because D is C -bounded, there is $\eta > 0$ such that $D \subseteq \eta B_V + C$, so we have $-D \subseteq \eta B_V + C$. Obviously, there is n_0 sufficiently large so that $n_0 \delta > \|d_1\| + \eta$. Observing that $-d_{n_0+1} \in D \subseteq \eta B_V + C$, there exists $-b_0 \in \eta B_V$ and $c_0 \in C$ such that $-d_{n_0+1} = -b_0 + c_0$. Because of $d_1 - b_0 \in n_0 \delta B_V$, $c_0 \in C$ and (3), we have

$$0 = d_1 - d_{n_0+1} - (d_1 - b_0) - c_0 \in d_1 - d_{n_0+1} - n_0 \delta B_V - C \subseteq -\operatorname{int} C$$

which is a contradiction. □

Lemma 5 If $F(x')$ is nonempty C -compact, $x' \in D$, thus $x' \in W_u(F, D)$ iff it does not exist $x \in D$ satisfying $F(x) <<^u F(x')$.

III. The properties of set scalarization function

We discuss the set scalarization function of sup-inf type in this section. Denote M and N as nonempty subsets of V. The following scalarization function is introduced in [19]:

$$h_C(M, N) = \sup_{m \in M} \inf_{n \in N} \Delta_{-C}(m - n) .$$

Lemma 6 [16] If N is C -bounded then $h_C(M, N) > -\infty$; if M is C -bounded then $h_C(M, N) < +\infty$; and if both M and N are bounded then $h_C(M, N)$ is finite.

Proof If N is C -bounded. Therefore $N \subset M' + C$ for some nonempty bounded set $M' \subset V$. Fix $m \in M$. For $\forall n \in N$, there exist $m' \in M'$ and $c \in C$, so that $n = m' + c$. Thus $n \geq m'$. By using Lemma 1, there is $\Delta_{-C}(m - n) \geq \Delta_{-C}(m - m') \geq -\|m - m'\| \geq -\|m\| - \|m'\|$. Then $h_C(M, N) \geq \inf_{m' \in M'} -\|m\| - \|m'\| > -\infty$. The other two cases can be checked similarly. □

From Lemma 3.2 of [16], the following Lemma is given.

Lemma 7 Let M and N are nonempty subsets of U and $v \in V$, respectively.

(1) If N is C -compact, therefore there is $n_0 \in N$ that

$$\Delta_{-C}(x - n_0) = \inf_{n \in N} \Delta_{-C}(x - n) .$$

(2) If M, N is C -compact, therefore there is $m_0 \in M$ that

$$h_C(M, N) = \inf_{n \in N} \Delta_{-C}(m_0 - n) .$$

Proof (1) Assume that N is C -compact, thus N is C -bounded. Suppose that $n \in N$ is given, then $t = \inf_{n \in N} \Delta_{-C}(m - n) > -\infty$ can be obtained from Lemma 6. Contrarily, $\Delta_{-C}(m - \cdot)$ does not reach its infimum on N. Therefore for any $n \in N$, there is a positive scalar $\varepsilon(n)$ depending on n so that $\Delta_{-C}(m - n) > t + \varepsilon(n)$. For $n \in N$, let $U_n = \{v \in V \,|\, \Delta_{-C}(v - n) > t + \varepsilon(n)\}$. Because of $0 \in C$, we get $U_n \subset U_n + C$, and because of $\Delta_{-C}(v + c - n) \geq \Delta_{-C}(v - n) > t + \varepsilon(n)$ for any $v \in U_n$ and $c \in C$, we get $U_n + C \subset U_n$. Then, $U_n = U_n + C$. Furthermore, because Δ_{-C} is Lipschitz, sets U_n are open, $n \in U_n$, so that $N \subset \bigcup_{n \in N} U_n$ holds. The C -compactness of N means that there exist finite vectors $n_1, \cdots, n_i$ so that $n_j \in N$ for all $j = 1, \cdots, i$ and $N \subset \bigcup_{j=1}^{i} (U_{n_j} + C)$. Thus, $N \subset \bigcup_{j=1}^{i} U_{n_j}$ and we have $t = \inf_{n' \in N} \Delta_{-C}(m - n') > t + \inf\{\varepsilon(n_j) \,|\, j = 1, \ldots i\} > t$, which is a contradiction. Therefore, there exists $n_0 \in N$ such that $\Delta_{-C}(x - n_0) = \inf_{n \in N} \Delta_{-C}(x - n)$.

(2) From the properties of the function Δ_{-C}, one can easily obtain that the function $\inf_{n \in N} \Delta_{-C}(\cdot - n)$ is 1 -Lipschitz and

monotone as follows:

$$m_2 \leq_C m_1 \Leftrightarrow \inf_{n\in N} \Delta_{-C}(m_1 - n) \leq \inf_{n\in N} \Delta_{-C}(m_2 - n).$$

Then, by Lemma 6, $\forall m \in M$, we get $\inf_{n\in N} \Delta_{-C}(m-n) > -\infty$ and $t = h_C(M,N) < +\infty$. Assume contrarily that $\inf_{n\in N} \Delta_{-C}(\cdot - n)$ does not reach its maximum on N. Fix $m \in M$. Therefore, there has a positive $\varepsilon(m)$ depending on m so that $\inf_{n\in N} \Delta_{-C}(m-n) < t - \varepsilon(m)$. Set $U_m = \{v \in V \,|\, \inf_{n\in N} \Delta_{-C}(v-n) < t - \varepsilon(m)\}$. One can get $U_m = U_m + C$. In the similar way in the proof of (1), and in view of the properties of the function $\inf_{n\in N} \Delta_{-C}(\cdot - n)$ mentioned above, there exist finite numbers of vectors $m_1, \cdots, m_i$ so that $m_j \in M$ for all $j = 1, \cdots, i$ and $M \subset \bigcup_{j=1}^{i} (U_{m_j} + C) = \bigcup_{j=1}^{i} U_{m_j}$. Therefore, $t = \sup_{m\in M} \inf_{n\in N} \Delta_{-C}(m-n) < t - \inf\{\varepsilon(m_j) \,|\, j = 1, \ldots i\} < t$, a contradiction. Then, there is $m_0 \in M$ so that $h_C(M,N) = \inf_{n\in N} \Delta_{-C}(m_0 - n)$. □

From Theorem 5.1 of [17], the proposition is obtained as follows.

Proposition 1 Assume $\delta \geq 0$,

(1) If $N - C - \delta B_V$ is closed and $M \subseteq N - C - \delta B_V$, then $h_C(M,N) \leq \delta$;

(2) If $N - C - \delta B_V$ is closed and $h_C(M,N) \leq \delta$, then $M \subseteq N - C - \delta B_V$.

From Proposition 1, the following corollaries are given.

Corollary 2

(1) If $M \leq^u N$, then $h_C(M,N) \leq 0$;

(2) If M is C-closed and $h_C(M,N) \leq 0$, then $M \leq^u N$.

Proof (1) Suppose that $M \leq^u N$ iff $M \subseteq N - C$. $\forall m \in M$ there is $n_0 \in N$ that $m - n_0 \in -C$, and then $\Delta_{-C}(m - n_0) \leq 0$. $\inf_{n\in N} \Delta_{-C}(m-n) \leq \Delta_{-C}(m-n_0) \leq 0$ holds. As $n \in N$ is arbitrarily chosen, we have $h_C(M,N) \leq 0$.

(2) Contrarily assume $M \not\leq^u N$ iff $M \not\subset N - C$. Then let $\overline{m} \in M$, so $\overline{m} \notin N - C$. For $n \in N$ there exists $\overline{m} - n \notin -C$, then $\Delta_{-C}(\overline{m} - n) > 0$ for $n \in N$. This means $h(\overline{m}, N) = \inf_{n\in N} \Delta_{-C}(\overline{m} - n) > 0$. Thus, $h_C(M,N) = \sup_{m\in M} h(m,N) = \sup_{m\in M} \inf_{n\in N} \Delta_{-C}(m-n) > 0$, a contradiction. Therefore $M \leq^u N$. □

Corollary 3 Let M, $N \in \wp_0(V)$, N is C-compact and let C be solid. Then $M \ll^u N \Leftrightarrow h_C(M,N) < 0$.

Proof (1) Necessity. $M \ll^u N$ iff $M \subseteq N - \operatorname{int} C$, i.e., if and only if for all $m \in M$, there is $n_0 \in N$ such that $m \in n_0 - \operatorname{int} C$, and therefore $m - n_0 \in -\operatorname{int} C$. By the Lemma 1(3), $\Delta_{-C}(m - n_0) < 0$ thus

$$h(m,N) = \inf_{n\in N} \Delta_{-C}(m-n) < 0,\ \forall m \in M.$$

Thus, as N is C-compact, we have $h_C(M,N) = h(m_0,N)$ for some $m_0 \in M$. Then we have $h_C(M,N) < 0$.

(2) Sufficiency. Suppose that $M \not\ll^u N$, i.e., $M \not\subset N - \operatorname{int} C$. Thus, there is $\overline{m} \in M$ so that $\overline{m} \notin N - \operatorname{int} C$. Then, for $n \in N$, $\overline{m} - n \notin -\operatorname{int} C$ holds. Therefore, by Lemma 1(3) we get $\Delta_{-C}(\overline{m} - n) \geq 0$, for $n \in N$. Then $h(\overline{m}, N) = \inf_{n\in N} \Delta_{-C}(\overline{m} - n) \geq 0$. Consequently, $h_C(M,N) = \sup_{m\in M} h(m,N) = \sup_{m\in M} \inf_{n\in N} \Delta_{-C}(m-n) \geq 0$. Then, $M \ll^u N$. □

Proposition 2 Assume that $\delta \geq 0$.

(1) If $h_C(M,N) < \delta$, then $M \subseteq N - C - \delta B_V^0$;

(2) If M is C-compact, N is C-bounded, $M \subseteq N - C - \delta B_V^0$ holds, then $h_C(M,N) < \delta$.

Proof (1) For any $m \in M$, it follows from $h_C(M,N) < \delta$ that $\inf_{n\in N} \Delta_{-C}(m-n) < \delta$. Thus there is $n_0 \in N$ such that $\Delta_{-C}(m - n_0) < \delta$. Two cases are considered here.

case1. $m - n_0 \in -C$, then $m \in n_0 - C$. For any $\beta \in \delta B_V$, then $-\beta \in \delta B_V$, that is $\beta \in -\delta B_V$, thus $m + \beta \in n_0 - C - \delta B_V$, therefore $m \in n_0 - C - \beta - \delta B_V \subseteq N - C - \delta B_V^0$.

case2. $m - n_0 \notin -C$. Then $\Delta_{-C}(m - n_0) = d_{-C}(m - n_0) < \delta$. By Lemma 2, $m - n_0 + \delta B_V^0 \subseteq -C$, and thus $m \in n_0 - C - \delta B_V^0 \subseteq N - C - \delta B_V^0$.

By the arbitrariness of $m \in M$, we get $M \subseteq N - C - \delta B_V^0$.

(2) From lemma 7(1), there is $m_0 \in M$ such that

$$h_C(M,N) = \inf_{n\in N} \Delta_{-C}(m_0 - n) \tag{4}$$

Because $M \subseteq N - C - \delta B_V^0$, there is $n_0 \in N$ such that $m_0 - n_0 \in -C - \delta B_V^0$. By Lemma 2, there exists $\Delta_{-C}(m_0 - n_0) \leq d_{-C}(m_0 - n_0) < \delta$. By (4) that

$$h_C(M,N) = \inf_{n\in N} \Delta_{-C}(m_0 - n) \leq \Delta_{-C}(m_0 - n_0) < \delta. \quad □$$

Proposition 3 Assume that $\delta > 0$.

(1) If $M \subseteq N - \bigcap_{\beta\in\delta B_V^0} (\beta + C)$, thus $h_C(M,N) \leq -\delta$;

(2) If N is C-compact and $h_C(M,N) \leq -\delta$, thus $M \subseteq N - \bigcap_{\beta\in\delta B_V^0} (\beta + C)$.

Proof (1) For any $m \in M$, because $M \subseteq N - \bigcap_{\beta\in\delta B_V^0} (\beta + C)$, there is $\overline{n} \in N$ that $m - \overline{n} \in -\bigcap_{\beta\in\delta B_V^0} (\beta + C)$. This implies that for any $\beta \in \delta B_V^0$, there is $m - \overline{n} \in -\beta - C$, and thus $m - \overline{n} + \beta \in -C$. By the arbitrariness of $\beta \in \delta B_V^0$, there exists $m - \overline{n} + \delta B_V^0 \subseteq -C$. Therefore, $(m - \overline{n} + \delta B_V^0) \bigcap (V \setminus (-C)) = \Phi$. Since Lemma 2, $d_{V\setminus(-C)}(m - \overline{n}) \geq \delta$. Obviously $m - \overline{n} \in -C$. Then

$$\Delta_{-C}(m - \overline{n}) \leq -d_{V\setminus(-C)}(m - \overline{n}) \leq -\delta,$$

and thus

$$\inf_{n\in N}\Delta_{-C}(m-n)\le\Delta_{-C}(m-\bar{n})\le-\delta,\ \forall m\in M,$$

By the arbitrariness of $m\in M$, there exists

$$h_C(M,N)=\sup_{m\in M}\inf_{n\in N}\Delta_{-C}(m-n)\le-\delta.$$

(2) For any given $m\in M$, it follows from $h_C(M,N)\le-\delta$ that $\inf_{n\in N}\Delta_{-C}(m-n)\le-\delta$. Since N is C-compact, by Lemma 7(1), there is $n_0\in N$ so that

$$\Delta_{-C}(m-n_0)=\inf_{n\in N}\Delta_{-C}(m-n)\le-\delta<0.$$

Therefore $m-n_0\in-C$ and $\Delta_{-C}(m-n_0)=-d_{V\setminus(-C)}(m-n_0)\le-\delta$, and thus

$$d_{V\setminus(-C)}(m-n_0)\ge\delta.$$

From Lemma 2, we get $(m-n_0+\delta B_V^0)\bigcap(V\setminus(-C))=\Phi$, which implies that $m-n_0+\delta B_V^0\subseteq-C$. Thus, for any $\beta\in\delta B_V^0$, we get $m-n_0+\beta\in-C$, and thus $m\in n_0-\beta-C$. Due to the arbitrariness of $\beta\in\delta B_V^0$, we obtain $m-n_0\in-\bigcap_{\beta\in\delta B_V^0}(\beta+C)$. Therefore,

$$m\in n_0-\bigcap_{\beta\in\delta B_V^0}(\beta+C)\subseteq N-\bigcap_{\beta\in\delta B_V^0}(\beta+C).$$

It follows from the arbitrariness of $m\in M$ that $M\subseteq N-\bigcap_{\beta\in\delta B_V^0}(\beta+C)$. □

Proposition 4 Assume that $\delta>0$. Then the statements hold:

(1) If $h_C(M,N)<-\delta$, then $M\subseteq N-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)$;

(2) If M is C-compact, N is C-bounded, V is finite dimensional and $M\subseteq N-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)$, then $h_C(M,N)<-\delta$.

Proof (1) For any $m\in M$, it follows from $h_C(M,N)<-\delta$ that $\inf_{n\in N}\Delta_{-C}(m-n)<-\delta$. Then there is $n_0\in N$ such that $\Delta_{-C}(m-n_0)<-\delta<0$. This implies that

$$\Delta_{-C}(m-n_0)=-d_{V\setminus(-C)}(m-n_0)<-\delta.$$

Then, $d_{V\setminus(-C)}(m-n_0)>\delta$. Therefore, there is $\eta\in R$ such that

$$d_{V\setminus(-C)}(m-n_0)>\eta>\delta.$$

From Lemma 3(1), we get

$$(m-n_0+\eta B_V)\bigcap(V\setminus(-C))=\Phi,$$

which means

$$m-n_0+\eta B_V=m-n_0+\delta B_V+(\eta-\delta)\ B_V\subseteq-C.$$

It follows that $m-n_0+\delta B_V\subseteq-\operatorname{int}C$. For any $\beta\in\delta B_V$, we get $m-n_0+\beta\in-\operatorname{int}C$, and then $m-n_0\in-\beta-\operatorname{int}C$. Due to the arbitrariness of $\beta\in\delta B_V$, we get $m-n_0\in-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)$. Therefore,

$$m\in n_0-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)\subseteq N-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C).$$

By the arbitrariness of $m\in M$, we obtain $M\subseteq N-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)$.

(2) By Lemma 7(2), we obtain that there is $m_0\in M$ such that

$$h_C(M,N)=\inf_{n\in N}\Delta_{-C}(m_0-n). \tag{5}$$

Because $M\subseteq N-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)$, there is $n_0\in N$ such that

$$m_0-n_0\in-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C).$$

This implies that $m_0-n_0+\delta B_V\subseteq-\operatorname{int}C$. Since V is finite dimensional, then $m_0-n_0+\delta B_V$ is compact. Thus there is $\xi>0$ such that $m_0-n_0+(\delta+\xi)B_V\subseteq-C$, which means that

$$(m_0-n_0+(\delta+\xi)B_V^0)\bigcap(V\setminus(-C))=\Phi.$$

By Lemma 2, $d_{V\setminus(-C)}(m_0-n_0)\ge\delta+\xi>\delta$ hold. Therefore,

$$\Delta_{-C}(m_0-n_0)=-d_{V\setminus(-C)}(m_0-n_0)<-\delta.$$

Together with (5) means

$$h_C(M,N)=\inf_{n\in N}\Delta_{-C}(m_0-n)\le\Delta_{-C}(m_0-n_0)<-\delta.\quad □$$

Theorem 1 Suppose that M and N are C-bounded.

(1) If $h_C(M,N)\ge0$, therefore

$$h_C(M,N)=\inf\{t\ge0:M\subseteq N-C-tB_V\};$$

(2) If $h_C(M,N)<0$, therefore

$$h_C(M,N)=\inf\{t<0:M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C)\}.$$

Proof (1) Since M is C-bounded, by Lemma 6, $h_C(M,N)<+\infty$ hold. Thus there exists $\varphi>0$ such that $h_C(M,N)<\varphi$. It follows from Proposition 2(1) that

$$M\subseteq N-C-\varphi B_V^0\subseteq N-C-\varphi B_V,$$

which implies that $\{t\ge0:M\subseteq N-C-tB_V\}\ne\Phi$. Note that $\eta=\{t\ge0:M\subseteq N-C-tB_V\}$. For any $\varepsilon>0$, there is $t\ge0$ so that $M\subseteq N-C-tB_V$ and $t<\eta+\varepsilon$. From Proposition 1(1), $h_C(M,N)\le t<\eta+\varepsilon$ hold. By the arbitrariness of $\varepsilon>0$, we get $h_C(M,N)\le\eta$.

Moreover, assume $h_C(M,N)<\eta$. Thus there is $\beta\in R$ so that

$$h_C(M,N)<\beta<\eta \tag{6}$$

From Proposition 2(1), $M\subseteq N-C-\beta B_V^0\subseteq N-C-\beta B_V$ holds. This means that $\eta\le\beta$, which contradicts (6).

Thus, $h_C(M,N)\ge\eta$ .Therefore

$$h_C(M,N)=\inf\{t\ge0:M\subseteq N-C-tB_V\}.$$

(2) Due to $h_C(M,N)<0$, there is $\delta\in R$ such that $h_C(M,N)<\delta<0$. By Proposition 4(1), there is

$$M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+\operatorname{int}C)\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C),$$

which implies that $\{t<0:M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C)\}\ne\Phi$. Assume that

$$\inf\{t<0:M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C)\}=-\infty. \tag{7}$$

Let N is C-bounded, from Lemma 6, $h_C(M,N)>-\infty$ hold. Thus there is $\xi<0$ so that

$$h_C(M,N)>\xi. \tag{8}$$

From (7), $\inf\{t<0: M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C)\}<\xi$ holds. Therefore, there is $t_0<0$ with $t_0<\xi$ so that

$$M\subseteq N-\bigcap_{\beta\in(-t_0)B_V}(\beta+C)\subseteq N-\bigcap_{\beta\in(-t_0)B_V^0}(\beta+C).$$

By Proposition 3(1), $h_C(M,N)\le t_0<\xi$ hold, which contradicts (8). Then,

$$\eta=\inf\{t<0: M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C)\}>-\infty.$$

For any $\varepsilon>0$, there is $t<0$ with $t<\eta+\varepsilon$ so that

$$M\subseteq N-\bigcap_{\beta\in(-t)B_V}(\beta+C)\subseteq N-\bigcap_{\beta\in(-t)B_V^0}(\beta+C).$$

By Proposition 3(1), we get $h_C(M,N)\le t<\eta+\varepsilon$. Due to the arbitrariness of $\varepsilon>0$, we get $h_C(M,N)\le\eta$.

Moreover, assume that $h_C(M,N)>\eta$. Thus there is $\phi\in R$ so that

$$h_C(M,N)<\phi<\eta. \tag{9}$$

From (9) and Proposition 4(1), we have

$$M\subseteq N-\bigcap_{\beta\in(-\phi)B_V}(\beta+\operatorname{int}C)\subseteq N-\bigcap_{\beta\in(-\phi)B_V}(\beta+C).$$

This implies that $\eta\le\phi$, which contradicts (9). Then, $h_C(M,N)\ge\eta$ holds. Therefore,

$$h_C(M,N)=\inf\{t\ge0: M\subseteq N-C-tB_V\}. \qquad \square$$

Lemma 8

(1) If $\xi>0$ and $\delta>0$, then

$$-\bigcap_{\beta\in\xi\delta B_V}(\beta+C)=-\xi(\bigcap_{\beta\in\delta B_V}(\beta+C)),$$

$$-\bigcap_{\beta\in\xi\delta B_V}(\beta+\operatorname{int}C)=-\xi(\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)).$$

(2)If $\delta_1>0$ and $\delta_2>0$, then

$$-\bigcap_{\beta\in\delta_1B_V}(\beta+C)-\bigcap_{\beta\in\delta_2B_V}(\beta+C)\subseteq-\bigcap_{\beta\in(\delta_1+\delta_2)B_V}(\beta+C).$$

(3) If $\delta_2\ge\delta_1>0$, then

$$-\bigcap_{\beta\in\delta_1B_V}(\beta+C)-\delta_2B_V-C\subseteq-(\delta_2-\delta_1)B_V-C.$$

(4) If $\delta_1>\delta_2\ge0$, then

$$-\bigcap_{\beta\in\delta_1B_V}(\beta+C)-\delta_2B_V-C\subseteq-\bigcap_{\beta\in(\delta_1-\delta_2)B_V}(\beta+C).$$

Proof (1) Noting that $z\in-\bigcap_{\beta\in\xi\delta B_V}(\beta+C)$, Thus for any $\beta\in\xi\delta B_V$, $\frac{\beta}{\xi}\in\delta B_V$, we obtain

$$z\in-\beta-C,\ z\in-\xi(\bigcap_{\beta\in\delta B_V}(\beta+C)),$$

so $-\bigcap_{\beta\in\xi\delta B_V}(\beta+C)\subseteq-\xi(\bigcap_{\beta\in\delta B_V}(\beta+C))$, on the contrary, we obtain $-\xi(\bigcap_{\beta\in\delta B_V}(\beta+C))\subseteq-\bigcap_{\beta\in\xi\delta B_V}(\beta+C)$. therefore $-\bigcap_{\beta\in\xi\delta B_V}(\beta+C)=-\xi(\bigcap_{\beta\in\delta B_V}(\beta+C))$. Similarly, we obtain $-\bigcap_{\beta\in\xi\delta B_V}(\beta+\operatorname{int}C)=-\xi(\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C))$.

(2) Noting that $z_1\in-\bigcap_{\beta\in\delta_1B_V}(\beta+C)$, $z_2\in-\bigcap_{\beta\in\delta_2B_V}(\beta+C)$. Thus for any $\beta_1\in\delta_1B_V$, $\beta_2\in\delta_2B_V$, we obtain

$$z_1\in-\beta_1-C,\ z_2\in-\beta_2-C. \tag{10}$$

For any $v\in-(\delta_2+\delta_1)B_V$, there exist $-\frac{\delta_1}{\delta_1+\delta_2}v\in\delta_1B_V$ and $-\frac{\delta_2}{\delta_1+\delta_2}v\in\delta_2B_V$. It follows from (10) that $z_1\in-\frac{\delta_1}{\delta_1+\delta_2}v-C$ and $z_2\in-\frac{\delta_2}{\delta_1+\delta_2}v-C$, and thus $z_1+z_2\in-v-C\subseteq-(\delta_1+\delta_2)B_V-C$. By the arbitrariness of $v\in-(\delta_2+\delta_1)B_V$, we get $z_1+z_2\subseteq-\bigcap_{\beta\in(\delta_1+\delta_2)B_V}(\beta+C)$, and thus

$$-\bigcap_{\beta\in\delta_1B_V}(\beta+C)-\bigcap_{\beta\in\delta_2B_V}(\beta+C)\subseteq-\bigcap_{\beta\in(\delta_1+\delta_2)B_V}(\beta+C).$$

(3) Noting that $z\in-\bigcap_{\beta\in\delta_1B_V}(\beta+C)$, $v\in\delta_2B_V$ and $c_0\in C$.Thus

$$z\in-\beta-C,\ \forall\beta\in\delta_1B_V. \tag{11}$$

Due to $\frac{\delta_1}{\delta_2}v\in-\delta_1B_V$ and (11), we get $z\in\frac{\delta_1}{\delta_2}v-C$. Therefore

$$\begin{aligned}z-v-c_0\in\frac{\delta_1}{\delta_2}v-v-c_0-C&\subseteq-\frac{(\delta_2-\delta_1)}{\delta_2}v-C\\&\subseteq-(\delta_2-\delta_1)B_V-C\end{aligned},$$

which implies that

$$-\bigcap_{\beta\in\delta_1B_V}(\beta+C)-\delta_2B_V-C\subseteq-(\delta_2-\delta_1)B_V-C.$$

(4) Noting that $z\in-\bigcap_{\beta\in\delta_1B_V}(\beta+C)$, $v\in\delta_2B_V$ and $\bar{c}\in C$. For any $\varphi\in(\delta_1-\delta_2)B_V$, there exits $-v+\varphi\in\delta_1B_V$. By (11) that $z\in v-\varphi-C$. Then,

$$z-v-\bar{c}\in v-\varphi-C-v-\bar{c}\subseteq-\varphi-C.$$

By the arbitrariness of $\varphi\in(\delta_1-\delta_2)B_V$, we get $z-v-\bar{c}\in-\bigcap_{\beta\in(\delta_1-\delta_2)B_V}(\beta+C)$, and thus

$$-\bigcap_{\beta\in\delta_1B_V}(\beta+C)-\delta_2B_V-C\subseteq-\bigcap_{\beta\in(\delta_1-\delta_2)B_V}(\beta+C). \qquad \square$$

Theorem 2 Suppose that M_1, M_2 N_1 and N_2 are C-bounded. Therefore

$$h_C(M_1+M_2,N_1+N_2)\le h_C(M_1,N_1)+h_C(M_2,N_2).$$

Proof By Lemma 6 that $h_C(M_1,N_1)$, $h_C(M_2,N_2)$ and $h_C(M_1+M_2,N_1+N_2)$ are finite. Consider the following four cases.

Case 1. $h_C(M_1,N_1)\ge0$ and $h_C(M_2,N_2)\ge0$. For any $\varepsilon>0$, from Theorem 1(1), there exists $h_C(M_1,N_1)\le t_1\le h_C(M_1,N_1)+\varepsilon$ and $h_C(M_2,N_2)\le t_2\le h_C(M_2,N_2)+\varepsilon$ such that $M_1\subseteq N_1-C-t_1B_V$ and $M_2\subseteq N_2-C-t_2B_V$. Then

$$\begin{aligned}M_1+M_2&\subseteq N_1+N_2-C-C-t_1B_V-t_2B_V\\&\subseteq N_1+N_2-C-(t_1+t_2)B_V\end{aligned}.$$

Due to Proposition 1(1) that the following holds

$$\begin{aligned}h_C(M_1+M_2,N_1+N_2)&\le t_1+t_2\\&\le h_C(M_1,N_1)+h_C(M_2,N_2)+2\varepsilon\end{aligned}.$$

By the arbitrariness of $\varepsilon>0$, we have $h_C(M_1+M_2,N_1+N_2)\le h_C(M_1,N_1)+h_C(M_2,N_2)$.

Case 2. $h_C(M_1,N_1)<0$ and $h_C(M_2,N_2)<0$. For any $\varepsilon>0$ with $h_C(M_1,N_1)+\varepsilon<0$ and $h_C(M_2,N_2)+\varepsilon<0$, From Theorem 1(2), there is $h_C(M_1,N_1)\le t_1<h_C(M_1,N_1)+\varepsilon$ and $h_C(M_2,N_2)\le t_2<h_C(M_2,N_2)+\varepsilon$ such that $M_1\subseteq N_1-\bigcap_{\beta\in(-t_1)B_V}(\beta+C)$ and $M_2\subseteq N_2-\bigcap_{\beta\in(-t_2)B_V}(\beta+C)$. Together with Lemma 8(2) means that

$$M_1+M_2\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1)B_V}(\beta+C)-\bigcap_{\beta\in(-t_2)B_V}(\beta+C)$$
$$\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1-t_2)B_V}(\beta+C)\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1-t_2)B_V^0}(\beta+C)$$

From Proposition 3(1), we have

$$h_C(M_1+M_2,N_1+N_2)\le t_1+t_2$$
$$\le h_C(M_1,N_1)+h_C(M_2,N_2)+2\varepsilon\,.$$

By the arbitrariness of $\varepsilon>0$, we get $h_C(M_1+M_2,N_1+N_2)\le h_C(M_1,N_1)+h_C(M_2,N_2)$.

Case 3. $h_C(M_1,N_1)<0$ and $h_C(M_2,N_2)\ge 0$. For any $\varepsilon>0$ with $h_C(M_1,N_1)+\varepsilon<0$, by Theorem1(2), there exists $h_C(M_1,N_1)\le t_1<h_C(M_1,N_1)+\varepsilon$ such that

$$M_1\subseteq N_1-\bigcap_{\beta\in(-t_1)B_V}(\beta+C)\,.$$

From Theorem 1(1), there is $h_C(M_2,N_2)\le t_2\le h_C(M_2,N_2)+\varepsilon$ such that $M_2\subseteq N_2-C-t_2B_V$. Thus

$$M_1+M_2\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1)B_V}(\beta+C)-t_2B_V-C \qquad (12)$$

If $h_C(M_1,N_1)+h_C(M_2,N_2)\ge 0$, then $t_1+t_2\ge h_C(M_1,N_1)+h_C(M_2,N_2)\ge 0$, and thus $t_2\ge -t_1>0$. By Lemma 8(3), we have

$$-\bigcap_{\beta\in(-t_1)B_V}(\beta+C)-t_2B_V-C\subseteq-(t_1+t_2)B_V-C\,.$$

Together with (12) means that $M_1+M_2\subseteq N_1+N_2-(t_1+t_2)B_V-C$. From Proposition 1(1), $h_C(M_1,N_1)+h_C(M_2,N_2)\le t_1+t_2$ holds.

If $h_C(M_1,N_1)+h_C(M_2,N_2)<0$, in a general way, suppose that $t_1+t_2<0$, then $-t_1>t_2\ge 0$. Due to Lemma 8(4), we obain

$$-\bigcap_{\beta\in(-t_1)B_V}(\beta+C)-t_2B_V-C\subseteq-\bigcap_{\beta\in(-t_1-t_2)B_V}(\beta+C).$$

It follows from (12) that

$$M_1+M_2\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1-t_2)B_V}(\beta+C)$$
$$\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1-t_2)B_V^0}(\beta+C)\,.$$

By Proposition 3(1), we get $h_C(M_1,N_1)+h_C(M_2,N_2)\le t_1+t_2$. Then,

$$h_C(M_1+M_2,N_1+N_2)\le t_1+t_2$$
$$\le h_C(M_1,N_1)+h_C(M_2,N_2)+2\varepsilon\,.$$

By the arbitrariness of $\varepsilon>0$, we have

$$h_C(M_1+M_2,N_1+N_2)\le h_C(M_1,N_1)+h_C(M_2,N_2)\,.$$

Case 4. $h_C(M_1,N_1)\ge 0$ and $h_C(M_2,N_2)<0$. From Theorem 1(1), there is $h_C(M_1,N_1)\le t_1\le h_C(M_1,N_1)+\varepsilon$ such that $M_1\subseteq N_1-C-t_1B_V$.

For any $\varepsilon>0$ with $h_C(M_2,N_2)+\varepsilon<0$, by Theorem1(2), there exists $h_C(M_2,N_2)\le t_2<h_C(M_2,N_2)+\varepsilon$ such that

$$M_2\subseteq N_2-\bigcap_{\beta\in(-t_2)B_V}(\beta+C)\,.$$

Thus

$$M_1+M_2\subseteq N_1+N_2-\bigcap_{\beta\in(-t_2)B_V}(\beta+C)-t_1B_V-C \qquad (13)$$

If $h_C(M_1,N_1)+h_C(M_2,N_2)\ge 0$, then $t_1+t_2\ge h_C(M_1,N_1)+h_C(M_2,N_2)\ge 0$, and thus $t_1\ge -t_2>0$. By Lemma 8(3), we have

$$-\bigcap_{\beta\in(-t_2)B_V}(\beta+C)-t_1B_V-C\subseteq-(t_1+t_2)B_V-C\,.$$

Together with (13) means that $M_1+M_2\subseteq N_1+N_2-(t_1+t_2)B_V-C$. From Proposition 1(1), $h_C(M_1,N_1)+h_C(M_2,N_2)\le t_1+t_2$ holds.

If $h_C(M_1,N_1)+h_C(M_2,N_2)<0$, in a general way, suppose that $t_1+t_2<0$, then $-t_2>t_1\ge 0$. Due to Lemma 8(4), we obain

$$-\bigcap_{\beta\in(-t_2)B_V}(\beta+C)-t_1B_V-C\subseteq-\bigcap_{\beta\in(-t_1-t_2)B_V}(\beta+C).$$

It follows from (12) that

$$M_1+M_2\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1-t_2)B_V}(\beta+C)$$
$$\subseteq N_1+N_2-\bigcap_{\beta\in(-t_1-t_2)B_V^0}(\beta+C)\,.$$

By Proposition 3(1), we get $h_C(M_1,N_1)+h_C(M_2,N_2)\le t_1+t_2$. Then,

$$h_C(M_1+M_2,N_1+N_2)\le t_1+t_2$$
$$\le h_C(M_1,N_1)+h_C(M_2,N_2)+2\varepsilon\,.$$

By the arbitrariness of $\varepsilon>0$, we have

$$h_C(M_1+M_2,N_1+N_2)\le h_C(M_1,N_1)+h_C(M_2,N_2)\,. \quad \square$$

Theorem 3 Suppose M, N and D are C-bounded. Then

(1) $h_C(coM,coN)\le h_C(M,N)$;

(2) $h_C(M+D,N+D)\le h_C(M,N)$.

Proof (1) Noting that $\eta=h_C(M,N)$. Two cases are considered here.

Case 1. $\eta\ge 0$. By Theorem 1(1), for any $\varepsilon>0$, there is $\eta\le\bar{t}<\eta+\varepsilon$ such that $M\subseteq N-C-\bar{t}B_V\subseteq coN-C-\bar{t}B_V$. Let $coN-C-\bar{t}B_V$ is convex, $coM\subseteq coN-C-\bar{t}B_V$ holds. From Proposition 1(1), we get $h_C(coM,coN)\le\bar{t}<\eta+\varepsilon=h_C(M,N)+\varepsilon$.

Case 2. $\eta<0$. For any $\varepsilon>0$, from Theorem 2(2), there is $t_0<0$ so that $\eta\le t_0<\eta+\varepsilon$ and $M\subseteq N-\bigcap_{\beta\in(-t_0)B_V}(\beta+C)\subseteq coN-\bigcap_{\beta\in(-t_0)B_V}(\beta+C)$. Due to $coN-\bigcap_{\beta\in(-t_0)B_V}(\beta+C)$ is convex, we get $coM\subseteq coN-\bigcap_{\beta\in(-t_0)B_V}(\beta+C)$. Together with Proposition 3(1) means that

$$h_C(coM,coN)\le t_0<\eta+\varepsilon=h_C(M,N)+\varepsilon\,.$$

Therefore, it follows from the arbitrariness of $\varepsilon>0$ that $h_C(coM,coN)\le h_C(M,N)$.

(2) Noting $\eta=h_C(M,N)$. Two cases are considered as follows.

Case 1. $\eta\ge 0$. By Theorem 1(1), for any $\varepsilon>0$, there is $\eta\le\bar{t}<\eta+\varepsilon$ so that $M\subseteq N-C-\bar{t}B_V$. Let M, N and D

are C -bounded, then $M+D\subseteq N+D-C-\bar{t}B_V$. Due to Proposition 1(1), we get $h_C(M+D,N+D)\le\bar{t}<\eta+\varepsilon=h_C(M,N)+\varepsilon$.

Case 2. $\eta<0$. For any $\varepsilon>0$, from Theorem 2(2), there is $t_0<0$ so that $\eta\le t_0<\eta+\varepsilon$ and $M\subseteq N-\bigcap_{\beta\in(-t_0)B_V}(\beta+C)$. Because of M , N and D are C -bounded, the following is obtained

$$M+D\subseteq N+D-\bigcap_{\beta\in(-t_0)B_V}(\beta+C).$$

Together with Proposition 3(1) we have $h_C(M+D,N+D)\le t_0<\eta+\varepsilon=h_C(M,N)+\varepsilon$.

Therefore, it follows from the arbitrariness of $\varepsilon>0$ that $h_C(M+D,N+D)\le h_C(M,N)$. □

IV. Application to Set Optimization Problems by Dini Directional Derivatives

In what follows, the optimality conditions of set optimization problems are derived by the Dini directional derivatives. Therefore, the definitions of the Dini directional derivatives are given below.

Definition 5 Noting that $G:U\to 2^V$ is a set-valued mapping. At x in direction l where $x,l\in U$, the upper Dini directional derivative of G is defined as

$$G^{\uparrow}(x,l)=\limsup_{t\downarrow 0}\frac{1}{t}h_C(G(x+tl),G(x)) =\inf_{s>0}\sup_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x)),$$

and the lower Dini directional derivative of G is defined as

$$G^{\downarrow}(x,l)=\liminf_{t\downarrow 0}\frac{1}{t}h_C(G(x+tl),G(x)) =\sup_{s>0}\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x)),$$

and the Dini directional derivative of G is denoted as $G'(x,l)$, and $G'(x,l)=G^{\uparrow}(x,l)=G^{\downarrow}(x,l)$ holds.

Obviously, $G^{\uparrow}(x,l)\ge G^{\downarrow}(x,l)$. Then, $G'(x,l)$ exists iff $G^{\uparrow}(x,l)\le G^{\downarrow}(x,l)$.

Theorem 4 Suppose G is C -convex and C -bounded values on U , and nonempty. Thus

(1) the Dini derivative of G at $x\in U$ exists for all $l\in U$ and

$$G'(x,l)=G^{\uparrow}(x,l)=G^{\downarrow}(x,l)=\inf_{0<s}\frac{1}{s}h_C(G(x+sl),G(x));$$

(2) $\forall x,l\in U$, $G'(x,\xi l)=\xi G'(x,l)$ for all $\xi>0$;

(3) $\forall x\in U$, $G'(x,\cdot)$ is a convex function, namely for any $l_1,l_2\in U$ and $\lambda\in[0,1]$,

$$G'(x,\lambda l_1+(1-\lambda)l_2)\le\lambda G'(x,l_1)+(1-\lambda)G'(x,l_2).$$

Proof (1) Firstly, $\forall t,r\in R$, and $0<t\le r$, prove (14) holds, which is

$$\frac{1}{t}h_C(G(x+tl),G(x))\le\frac{1}{r}h_C(G(x+rl),G(x)). \quad (14)$$

As G is C -convex on U , there is

$$G(x+tl)\subseteq\frac{r-t}{r}G(x)+\frac{t}{r}G(x+rl)-C. \quad (15)$$

Noting that $\eta=h_C(G(x+rl),G(x))$. Considering two cases as follows.

Case 1. $\eta\ge 0$. For any $\varepsilon>0$, there is $\delta\in R$ such that $\eta<\delta\le\eta+\varepsilon$. By Proposition 2(1) and $h_C(G(x+rl),G(x))=\eta<\delta$, there is

$$G(x+rl)\subseteq G(x)-C-\delta B_V^0\subseteq G(x)-C-\delta B_V. \quad (16)$$

From (16), there is $\frac{t}{r}G(x+rl)\subseteq\frac{t}{r}G(x)-\frac{t}{r}C-\frac{t}{r}\delta B_V$, therefore

$$\frac{t}{r}G(x+rl)+\frac{r-t}{r}G(x)\subseteq\frac{t}{r}G(x)-\frac{t}{r}C-\frac{t}{r}\delta B_V+\frac{r-t}{r}G(x) \subseteq G(x)-\frac{t}{r}C-\frac{t}{r}\delta B_V.$$

Applying (15)

$$G(x+tl)\subseteq\frac{t}{r}G(x+rl)+\frac{r-t}{r}G(x)-C \subseteq G(x)-C-\frac{t}{r}C-\frac{t}{r}\delta B_V\subseteq G(x)-C-\frac{t}{r}\delta B_V.$$

Due to Proposition 1(1), there is

$$h_C(G(x+tl),G(x))\le\frac{t}{r}\delta\le\frac{t}{r}(\eta+\varepsilon).$$

By the arbitrariness of $\varepsilon>0$, we get $h_C(G(x+tl),G(x))\le\frac{t}{r}\eta$, this together with $\eta=h_C(G(x+rl),G(x))$ and $0<t\le r$, therefore (14) holds.

Case 2. $\eta<0$. For any $\varepsilon>0$ with $\eta+\varepsilon<0$, there is $\delta>0$ such that $\eta<-\delta\le\eta+\varepsilon$. From Proposition 4(1) there is

$$G(x+rl)\subseteq G(x)-\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C). \quad (17)$$

From (17), there is $\frac{t}{r}G(x+rl)\subseteq\frac{t}{r}G(x)-\frac{t}{r}\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C)$, therefore

$$\frac{t}{r}G(x+rl)+\frac{r-t}{r}G(x)\subseteq\frac{t}{r}G(x)+\frac{r-t}{r}G(x)-\frac{t}{r}\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C).$$

Applying (15)

$$G(x+tl)\subseteq\frac{t}{r}G(x+rl)+\frac{r-t}{r}G(x)-C-\frac{t}{r}\bigcap_{\beta\in\delta B_V}(\beta+\operatorname{int}C) \subseteq G(x)-\bigcap_{\beta\in\frac{t}{r}\delta B_V}(\beta+\operatorname{int}C).$$

By Proposition 3(1), there is

$$h_C(G(x+tl),G(x))\le\frac{t}{r}(-\delta)\le\frac{t}{r}(\eta+\varepsilon).$$

By the arbitrariness of $\varepsilon>0$, we get $h_C(G(x+tl),G(x))\le\frac{t}{r}\eta$, this together with $\eta=h_C(G(x+rl),G(x))$ and $0<t\le r$, so (14) holds.

From (14), for any $s>0$, we have

$$\sup_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))=\frac{1}{s}h_C(G(x+sl),G(x)),$$

thus

$$G^{\uparrow}(x,l)=\inf_{0<s}\frac{1}{s}h_C(G(x+sl),G(x)).$$

Therefore, for any $s>0$,

$$\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))=\inf_{0<r}\frac{1}{r}h_C(G(x+rl),G(x)) \quad (18)$$

Clearly,

$$\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))\ge\inf_{0<r}\frac{1}{r}h_C(G(x+rl),G(x)).$$

Assume that there is $\xi\in R$ such that

$$\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))>\xi>\inf_{0<r}\frac{1}{r}h_C(G(x+rl),G(x)).$$

Therefore there is $r_0>0$ so that

$$\frac{1}{r_0}h_C(G(x+r_0l),G(x))<\xi. \quad (19)$$

If $0<r_0\le s$, then

$$\frac{1}{r_0}h_C(G(x+r_0l),G(x))\ge\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))>\xi,$$

which contradicts (19). Then $r_0>s$. From (13), there is

$$\begin{aligned}\frac{1}{r_0}h_C(G(x+r_0l),G(x))&\ge\frac{1}{s}h_C(G(x+sl),G(x))\\&\ge\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))>\xi\end{aligned},$$

which contradicts (19). Thus, the following results hold:

$$\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))=\inf_{0<r}\frac{1}{r}h_C(G(x+rl),G(x)),$$

and

$$G^{\downarrow}(x,l)=\sup_{s>0}\inf_{0<t\le s}\frac{1}{t}h_C(G(x+tl),G(x))=\inf_{0<r}\frac{1}{r}h_C(G(x+rl),G(x)).$$

Therefore

$$G'(x,l)=G^{\uparrow}(x,l)=G^{\downarrow}(x,l)=\inf_{0<s}\frac{1}{s}h_C(G(x+sl),G(x)).$$

(2) For any $\xi>0$, there is

$$G'(x,\xi l)=\inf_{0<s}\frac{1}{s}h_C(G(x+s\xi l),G(x)). \quad (20)$$

Noting that $r=s\xi$. Thus $\frac{1}{s}=\frac{\xi}{r}$ where $s>0\Leftrightarrow r>0$. Due to (19), we get

$$\begin{aligned}G'(x,\xi l)&=\inf_{0<r}\frac{\xi}{r}h_C(G(x+rl),G(x))\\&=\xi\inf_{0<r}\frac{1}{r}h_C(G(x+rl),G(x))=\xi G'(x,l)\end{aligned}.$$

(3) Let $l_1,l_2\in U$ and $\lambda\in(0,1)$. For any $\varepsilon>0$, noting that

$$G'(x,l_1)=\inf_{0<s}\frac{1}{s}h_C(G(x+sl_1),G(x)),$$

and

$$G'(x,l_2)=\inf_{0<s}\frac{1}{s}h_C(G(x+sl_2),G(x)).$$

There are $s_1>0$ and $s_2>0$, such that

$$\frac{1}{s_1}h_C(G(x+s_1l_1),G(x))<G'(x,l_1)+\varepsilon \quad (21)$$

and

$$\frac{1}{s_2}h_C(G(x+s_2l_2),G(x))<G'(x,l_2)+\varepsilon. \quad (22)$$

Let $s_0=\min\{s_1,s_2\}>0$. Based on (14), (21) and (22), we can get $\delta_1,\delta_2\in R$ satisfying

$$\begin{aligned}\frac{1}{s_0}h_C(G(x+s_0l_1),G(x))&<\frac{1}{s_1}h_C(G(x+s_1l_1),G(x))\\&<\delta_1<G'(x,l_1)+\varepsilon\end{aligned} \quad (23)$$

and

$$\begin{aligned}\frac{1}{s_0}h_C(G(x+s_0l_2),G(x))&<\frac{1}{s_2}h_C(G(x+s_2l_2),G(x))\\&<\delta_2<G'(x,l_2)+\varepsilon\end{aligned}. \quad (24)$$

Bcause G is C-convex on U and

$$\lambda(x+s_0l_1)+(1-\lambda)(x+s_0l_2)=x+s_0(\lambda l_1+(1-\lambda)l_2),$$

we get

$$\begin{aligned}&G(x+s_0(\lambda l_1+(1-\lambda)l_2))\\&\subseteq\lambda G(x+s_0l_1)+(1-\lambda)G(x+s_0l_2)-C\end{aligned}. \quad (25)$$

Four cases will be discussed as follows.

Case 1: $G'(x,l_1)\ge 0$ and $G'(x,l_2)\ge 0$. From (23), (24) and Proposition 2(1), there are

$$G(x+s_0l_1)\subseteq G(x)-C-s_0\delta_1B_V^0,$$

and

$$G(x+s_0l_2)\subseteq G(x)-C-s_0\delta_2B_V^0.$$

Together with (25), we have

$$\begin{aligned}&G(x+s_0(\lambda l_1+(1-\lambda)l_2))\subseteq\lambda G(x+s_0l_1)+(1-\lambda)G(x+s_0l_2)-C\\&\subseteq\lambda G(x)-\lambda C-\lambda s_0\delta_1B_V^0+(1-\lambda)G(x)-(1-\lambda)C-(1-\lambda)s_0\delta_2B_V^0-C\\&\subseteq G(x)-C-s_0(\lambda\delta_1+(1-\lambda)\delta_2)B_V^0\end{aligned}$$

From Proposition 1(1), there is

$$h_C(G(x+s_0(\lambda l_1+(1-\lambda)l_2)),G(x))\le s_0(\lambda\delta_1+(1-\lambda)\delta_2). \quad (26)$$

Based on (23), (24) and (26), we have

$$\begin{aligned}G'(x,\lambda l_1+(1-\lambda)l_2)&\le\frac{1}{s_0}h_C(G(x+s_0(\lambda l_1+(1-\lambda)l_2)),G(x))\\&\le\lambda\delta_1+(1-\lambda)\delta_2\\&<\lambda G'(x,l_1)+(1-\lambda)G'(x,l_2)+\varepsilon.\end{aligned} \quad (27)$$

By the arbitrariness of $\varepsilon>0$, from (27) there is

$$G'(x,\lambda l_1+(1-\lambda)l_2)\le\lambda G'(x,l_1)+(1-\lambda)G'(x,l_2) \quad (28)$$

Case 2: $G'(x,l_1)<0$ and $G'(x,l_2)<0$. Without loss of generality, suppose $\delta_1<G'(x,l_1)+\varepsilon<0$ and $\delta_2<G'(x,l_2)+\varepsilon<0$. By (23), (24) and Proposition 4(1), we have

$$G(x+s_0l_1)\subseteq G(x)-\bigcap_{\beta\in s_0(-\delta_1)B_V}(\beta+\operatorname{int}C)$$

and

$$G(x+s_0l_2)\subseteq G(x)-\bigcap_{\beta\in s_0(-\delta_2)B_V}(\beta+\operatorname{int}C).$$

Together with (25) and Lemma 8(1), there is

$$\begin{aligned}&G(x+s_0(\lambda l_1+(1-\lambda)l_2))\subseteq\lambda G(x+s_0l_1)+(1-\lambda)G(x+s_0l_2)-C\\&\subseteq\lambda G(x)-\lambda\bigcap_{\beta\in s_0(-\delta_1)B_V}(\beta+\operatorname{int}C)+(1-\lambda)G(x)-(1-\lambda)\bigcap_{\beta\in s_0(-\delta_2)B_V}(\beta+\operatorname{int}C)\\&\subseteq G(x)-\bigcap_{\beta\in s_0(\lambda(-\delta_1)+(1-\lambda)(-\delta_2))B_V}(\beta+\operatorname{int}C).\end{aligned}$$

Due to Proposition 3(1), there is

$$h_C(G(x+s_0(\lambda l_1+(1-\lambda)l_2)),G(x))\le s_0(\lambda\delta_1+(1-\lambda)\delta_2).$$

Therefore, we have

$G'(x,\lambda l_1+(1-\lambda)l_2)\le\lambda G'(x,l_1)+(1-\lambda)G'(x,l_2)$ easily.

Case 3. $G'(x,l_1)\ge 0$ and $G'(x,l_2)<0$. In a general way, suppose $\delta_2<G'(x,l_2)+\varepsilon<0$. Based on (23) and Proposition

2(1), there is

$$G(x+s_0 l_1) \subseteq G(x) - C - s_0\delta_1 B_V^0 \subseteq G(x) - C - s_0\delta_1 B_V \,. \quad (29)$$

Together with (24) and Proposition 4(1), there is

$$\begin{aligned} G(x+s_0 l_2) &\subseteq G(x) - \bigcap_{\beta\in s_0(-\delta_2)B_V} (\beta + \operatorname{int} C) \\ &\subseteq G(x) - \bigcap_{\beta\in s_0(-\delta_2)B_V} (\beta + C) \end{aligned} \,. \quad (30)$$

Based on (25), (29), (30) and Lemma 8(1), there is

$$\begin{aligned} &G(x+s_0(\lambda l_1 + (1-\lambda)l_2)) \subseteq \lambda G(x+s_0 l_1) + (1-\lambda)G(x+s_0 l_2) - C \\ &\subseteq \lambda G(x) - \lambda C - \lambda s_0\delta_1 B_V + (1-\lambda)G(x) - (1-\lambda)\bigcap_{\beta\in s_0(-\delta_2)B_V}(\beta + C) - C \\ &\subseteq G(x) - C - \lambda s_0\delta_1 B_V - \bigcap_{\beta\in s_0(1-\lambda)(-\delta_2)B_V}(\beta + C) \,. \end{aligned} \quad (31)$$

If $\lambda s_0\delta_1 \geq (1-\lambda)s_0(-\delta_2) > 0$, then from (31) and Lemma 8(3) there exists

$$G(x+s_0(\lambda l_1 + (1-\lambda)l_2)) \subseteq G(x) - C - (\lambda s_0\delta_1 + (1-\lambda)s_0\delta_2)B_V \,.$$

By Proposition 1(1), there is

$$h_C(G(x+s_0(\lambda l_1 + (1-\lambda)l_2)), G(x)) \leq s_0(\lambda\delta_1 + (1-\lambda)\delta_2) \,.$$

If $0 < \lambda s_0\delta_1 < (1-\lambda)s_0(-\delta_2)$, from (31) and Lemma 8(4) we obtain

$$G(x+s_0(\lambda l_1 + (1-\lambda)l_2)) \subseteq G(x) - C - \bigcap_{\beta\in(\lambda s_0(-\delta_1)+(1-\lambda)s_0(-\delta_2))B_V}(\beta + C) \,.$$

By Proposition 3(1), we get

$$h_C(G(x+s_0(\lambda l_1 + (1-\lambda)l_2)), G(x)) \leq s_0(\lambda\delta_1 + (1-\lambda)\delta_2) \,.$$

Thus, (28) holds. By the arbitrariness of $\varepsilon > 0$, from (28) there is

$$G'(x, \lambda l_1 + (1-\lambda)l_2) \leq \lambda G'(x, l_1) + (1-\lambda)G'(x, l_2) \,.$$

Case 4. $G'(x,l_1) < 0$ and $G'(x,l_2) \geq 0$. In a general way, suppose $\delta_1 < G'(x,l_1) + \varepsilon < 0$. Based on (23) and Proposition 2(1), there is

$$G(x+s_0 l_2) \subseteq G(x) - C - s_0\delta_2 B_V^0 \subseteq G(x) - C - s_0\delta_2 B_V \,. \quad (32)$$

Together with (24) and Proposition 4(1), there is

$$G(x+s_0 l_1) \subseteq G(x) - \bigcap_{\beta\in s_0(-\delta_1)B_V}(\beta + \operatorname{int} C) \subseteq G(x) - \bigcap_{\beta\in s_0(-\delta_1)B_V}(\beta + C) \,. \quad (33)$$

Based on (26), (32), (33) and Lemma 8(1), there is

$$\begin{aligned} &G(x+s_0(\lambda l_1 + (1-\lambda)l_2)) \subseteq \lambda G(x+s_0 l_1) + (1-\lambda)G(x+s_0 l_2) - C \\ &\subseteq \lambda G(x) - \lambda \bigcap_{\beta\in s_0(-\delta_1)B_V}(\beta + C) + (1-\lambda)G(x) - (1-\lambda)C - (1-\lambda)s_0\delta_2 B_V - C \\ &\subseteq G(x) - C - \bigcap_{\beta\in s_0\lambda(-\delta_1)B_V}(\beta + C) - (1-\lambda)s_0\delta_2 B_V \,. \end{aligned} \quad (34)$$

If $\lambda s_0(-\delta_1) \geq (1-\lambda)s_0\delta_2 > 0$, then from (34) and Lemma 8(3) there exists

$$G(x+s_0(\lambda l_1 + (1-\lambda)l_2)) \subseteq G(x) - C - (\lambda s_0\delta_1 + (1-\lambda)s_0\delta_2)B_V \,.$$

By Proposition 1(1), there is

$$h_C(G(x+s_0(\lambda l_1 + (1-\lambda)l_2)), G(x)) \leq s_0(\lambda\delta_1 + (1-\lambda)\delta_2) \,.$$

If $0 < \lambda s_0(-\delta_1) < (1-\lambda)s_0\delta_2$, from (34) and Lemma 8(4) we obtain

$$G(x+s_0(\lambda l_1 + (1-\lambda)l_2)) \subseteq G(x) - C - \bigcap_{\beta\in(\lambda s_0(-\delta_1)+(1-\lambda)s_0(-\delta_2))B_V}(\beta + C) \,.$$

By Proposition 3(1), we get

$$h_C(G(x+s_0(\lambda l_1 + (1-\lambda)l_2)), G(x)) \leq s_0(\lambda\delta_1 + (1-\lambda)\delta_2) \,.$$

Thus, (28) holds. By the arbitrariness of $\varepsilon > 0$, from (28) there is

$$G'(x, \lambda l_1 + (1-\lambda)l_2) \leq \lambda G'(x, l_1) + (1-\lambda)G'(x, l_2) \,. \quad \square$$

Theorem 5 Suppose D is a C -bounded values nonempty convex set, G is a C -convex function on D. $\forall x_0 \in D$, if $G'(x_0, l) > 0$ for all $l \in U$ with $x_0 + l \in D$ and $l \neq 0$, therefore $x_0 \in E_u(G,D)$.

Proof $\forall \bar{x} \in D$ and $\bar{x} \neq x_0$, based on the assumption, there is $G'(x_0, \bar{x} - x_0) > 0$. From Theorem 4(1) there is

$$0 < G'(x_0, \bar{x} - x_0) = \inf_{s>0} \frac{1}{s} h_C(G(x_0 + s(\bar{x} - x_0)), G(x_0)) \leq h_C(G(\bar{x}), G(x_0)) \,.$$

Due to Corollary 3(1), there has $G(\bar{x}) \underset{\neq}{<}^u G(x_0)$ for all $\bar{x} \in D$ where $\bar{x} \neq x_0$. This implies $x_0 \in E_u(G,D)$. $\square$

Theorem 6 Suppose D is a convex set, and G is a C -convex function on D with nonempty and C -bounded values. Denote point $x_0 \in D$ so that $G(x_0)$ is C -compact. Thus $x_0 \in E_u(G,D)$ iff $G'(x_0, l) \geq 0$ for all $l \in U$ where $x_0 + l \in D$.

Proof (1) Sufficiency. For any $x' \in D$, there is $G'(x_0, x' - x_0) \geq 0$. Based on Theorem 4(1), there we have

$$0 \leq G'(x_0, x' - x_0) = \inf_{s>0} \frac{1}{s} h_C(G(x_0 + s(x' - x_0)), G(x_0)) \leq h_C(G(x'), G(x_0)) \,.$$

Using Corollary 3, for any $x' \in D$, $G(x') \ll^u G(x_0)$ is not true, so $x_0 \in E_u(G,D)$.

(2) Necessity. Assume $x_0 \in E_u(G,D)$, thus by Lemma 5 there does not exist $\bar{x} \in D$ satisfying $G(\bar{x}) \ll^u G(x_0)$. Together with Corollary means that

$$h_C(G(x'), G(x_0)) \geq 0 \,, \ \forall x' \in D \,. \quad (35)$$

$\forall l \in U$ where $x_0 + l \in D$, and $\forall t \in (0,1]$, the convexity of D means $x_0 + tl \in D$.

By (35), there is $\frac{1}{t} h_C(G(x_0 + tl), G(x_0)) \geq 0$, and thus

$$\inf_{0<t\leq 1} \frac{1}{t} h_C(G(x_0 + tl), G(x_0)) \geq 0 \,. \quad (36)$$

Due to (20), (36) and Theorem 4(1), there is

$$G'(x_0, l) = \inf_{r>0} \frac{1}{r} h_C(G(x_0 + rl), G(x_0)) = \inf_{0<t\leq 1} \frac{1}{t} h_C(G(x_0 + tl), G(x_0)) \geq 0 \quad .\square$$

V. Conclusions

The present paper introduces the sup-inf set scalarization function, which is used to define the Dini directional derivatives of set mapping. The optimality conditions of set optimization problems are then derived by utilizing the Dini directional derivatives. Further research will focus on defining more general non-linear scalarization functions and developing optimality conditions under different order relations.

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