Some Identities Involving Chebyshev Polynomial of Third Kind, Lucas Numbers and Fourth Kind, Fibonacci Numbers

Anu Verma, Pankaj Pandey, *Member, IAENG*, Vipin Verma

Abstract - In this article, we demonstrate certain identities involving both the third-fourth kinds of the Chebyshev polynomials taking into account the Fibonacci and Lucas numbers. This study brings to light some significant results and defines the relationship between these polynomials. We have used mathematical induction to establish the relation between third-fourth kinds of Chebyshev's polynomials, Fibonacci, and Lucas numbers. We also used the Binet formula and the second-order differential equation to establish their relationship. We also present some results for Fibonacci and Lucas numbers, particularly by using the second-order derivative of third-kind Chebyshev polynomials. We prove some results that connect the fourth-kind of Chebyshev polynomials with Fibonacci and Lucas numbers. These findings will pave the way for further exploration of the third-fourth kinds of Chebyshev polynomials. In addition, we look at the practical application of Chebyshev's polynomials in approximation theory.

***Index Terms*- Chebyshev polynomials of third-fourth kinds; Fibonacci numbers; Lucas numbers.**

I. INTRODUCTION

A. *Chebyshev polynomials*

As usual the first kind of Chebyshev polynomials are defined as:

$$T_0(y) = 1\,, T_1(y) = y\,,$$
$$\text{For } \ h \geq 2,3,\ldots$$
$$T_h(y) = 2yT_{h-1}(y) - T_{h-2}(y). \qquad (1)$$

The second kind of Chebyshev polynomials are defined as:

$$U_0(y) = 1\,, U_1(y) = 2y\,,$$
$$\text{For } \ h \geq 2,3,\ldots$$
$$U_h(y) = 2yU_{h-1}(y) - U_{h-2}(y). \qquad (2)$$

The third kind of Chebyshev polynomials are defined as:

$$V_0(y) = 1\,, V_1(y) = 2y - 1\,,$$
$$\text{For } h \geq 2,3,\ldots$$
$$V_h(y) = 2yV_{h-1}(y) - V_{h-2}(y). \qquad (3)$$

The fourth kind of Chebyshev polynomials are defined as:

$$W_0(y) = 1\,, W_1(y) = 2y + 1,$$
$$\text{For } \ h \geq 2,3,\ldots$$
$$W_h(y) = 2yW_{h-1}(y) - W_{h-2}(y). \qquad (4)$$

For $y = cos\theta$ the Chebyshev polynomials (all four kinds) can also be structured as (trigonometric way):

$$T_h(y) = cosh\theta,$$
$$U_h(y) = \frac{sin(h+1)\theta}{sin\theta},$$
$$V_h(y) = \frac{cos(h+\frac{1}{2})\theta}{cos\frac{1}{2}\theta},$$
$$W_h(y) = \frac{sin(h+\frac{1}{2})\theta}{sin\frac{1}{2}\theta}.$$

B. Fibonacci Numbers

Leonardo of Pisa, an Italian mathematician, is renowned for his work on Fibonacci numbers, which are a sequence of numbers defined by the following equation:

$$F_h = F_{h-1} + F_{h-2} \ , h \geq 2,$$
$$with\ F_0 = 0\ and\ F_1 = 1. \qquad (5)$$

C. Lucas Numbers

Lucas numbers named the mathematician Francois Edourad Anatole Lucas. Lucas's numbers are closely related to Fibonacci numbers. The recurrence relation of both Lucas and Fibonacci sequences is the same, although the initial values differ. Lucas's sequence of numbers is defined by:

$$L_h = L_{h-1} + L_{h-2} \ , h \geq 2,$$
$$with\ L_0 = 2\ and\ L_1 = 1. \qquad (6)$$

This paper comprises three primary sections. The initial section provides an introduction and definition of Chebyshev polynomials, which comprehensively cover all four kinds, including the Fibonacci and Lucas numbers. The second section encompasses the research conducted and work done on deriving six theorems and six corollaries related to Chebyshev polynomials, Fibonacci, and Lucas numbers. In the last sections, we have discussed the significance of this work and delved into the vast range of applications that Chebyshev polynomials have to offer.

In the 19th century, Pafnuty Lvovich Chebyshev introduced the idea of Chebyshev polynomials. The analysis of the initial and subsequent Chebyshev polynomials was conducted by Mason and Wenpeng Zhang, who also presented several interesting identities [6, 19, 21]. The basic definitions of Chebyshev polynomials have been taken from a very notable scholarly work, the classical orthogonal polynomials [1]. Jonny Griffiths [5] presented many results which connected all four kinds of Chebyshev polynomials. Kamal Aghigh et al. [8], M.R. Eslahchi et al. [12], and Taekyun Kim et al. [15, 16, 17] gave many identities attributed to both the third-fourth kinds of

Manuscript received January 26, 2023; revised September 27, 2023.

Anu Verma is Ph.D. student in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab 144411, India. (e-mail: anuverma0107@gmail.com)

Pankaj Pandey is Associate Professor in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab 144411, India. (corresponding author phone: +91-9691099413; e-mail: pankaj.anvarat@gmail.com).

Vipin Verma is Associate Professor in NMIMS University, Mumbai, Maharashtra 400056, India, (e-mail: vipin_verma2406@rediffmail.com).

Chebyshev polynomials. Wenpeng Zhang [18] gave the foundational idea to solve the summation of recurrence relations and also studied some identities related to Fibonacci sequences. Chebyshev polynomials are widely studied by researchers and defined in various forms like recurrence relations and trigonometric formulae, etc. Fibonacci and Lucas numbers share a close relationship with Chebyshev polynomials. These recursive relationships are employed in counting. Zhang [19] found a number of properties related to the Chebyshev polynomials' derivatives and showed the relationship between them as follows:

"Let h, q be integers with $h \geq 0, q \geq 1$, we have:

$$\sum_{b_1+b_2+\cdots+b_{q+1}=i} \prod_{k=1}^{q+1} U_{b_k}(y) = \frac{1}{2^q q!} U_{h+q}{}^{(q)}(y)."$$

Sanjay Harne et al. [13] found identities related to the Chebyshev polynomials, Lucas and Fibonacci numbers at certain variables with their derivatives as follows:

$$"\sum_{b_1+b_2+\cdots+b_{q+1}=i} F_{4(2b_1+1)} F_{4(2b_2+1)} \dots F_{4(2b_{q+1}+1)} = \frac{3^{q+1}}{2^q q!} U_{h+q}{}^{q}\left(\frac{7}{2}\right),"$$

$$\sum_{b_1+b_2+\cdots+b_{q+1}=i} F_{6(2b_1+1)} F_{6(2b_2+1)} \dots F_{6(2b_{q+1}+1)} = \frac{2^{2q+3}}{q!} U_{h+q}{}^{q}(9)."$$

Kamal et al. [8] examined the Chebyshev polynomials of both third-fourth kinds with their applications and obtained highly advantageous outcomes. Li et al. [9] used the Chebyshev polynomials matrices to obtain numerical solutions with examples for *BVP* of the Caputo fractional derivative integro *D.E.* of fourth-order. T. Korkiatsakul et al. [14] looked at a new way of using the Chebyshev operating matrix to figure out how to solve a problem with integrals of non-linear Caputo fractions in a static beam. Frontczak, and Goy [3] obtained Chebyshev-Fibonacci polynomial relations using generating functions. Shoukralla [2] obtained a numerical solution to the first kind of Fredholm integral equation using the matrix form of the second-kind Chebyshev polynomials. Zhang [20] gave results linked to the Lucas and Fibonacci numbers. Yang Li [22] determined the connection between derivatives of the Chebyshev, and Fibonacci polynomials. In addition, Yang Li [23] analysed the both first and second kinds of Chebyshev polynomials and applied the elementary method to find the relationship between both kinds of Chebyshev polynomials connected to Fibonacci polynomials and finally found some results related to the Fibonacci and the Lucas numbers. L. Zhang and W. Zhan [10, 19] used mathematical induction, to solve the problem of Chebyshev polynomials sums of powers and obtained many other properties related to Chebyshev polynomials. T. Kim et al. [16] studied sums of finite products of first kind Chebyshev, and Lucas's polynomials represented each one in terms of all kinds of Chebyshev polynomials. Y. Zhang and Z. Chen [24] found identity related to the Chebyshev polynomials of the second-kind by using combinatorial.

M. Arya and V. Verma [11] found a special type of representation of the Chebyshev polynomials. Han and Lv [4] studied the Chebyshev polynomials to explore some new identities. Zhang and Han [21] provided identities for reciprocal sums of the Chebyshev polynomials through mathematical induction and the properties of symmetrical polynomial sequences. Kim et al. [17] obtained the sums of Chebyshev polynomials' finite products of two different types and each one of them represented a linear combination of all types. Robert et al. [3] studied the first and second kinds of Chebyshev polynomials as well as the polynomials of the Fibonacci sequence and gave new connections and combinatorial identities by using generating functions. Jugal Kishore and Vipin Verma [7] used a computational method to give identities concerning product's finite sums of the Lucas, Fibonacci, and complex Fibonacci numbers. T. Kim et al. [15] studied the sum related to finite products of both third-fourth kinds of Chebyshev polynomials and have obtained the following relation among the fourth kind of the Chebyshev polynomials and their derivatives as follows:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} (-1)^{h-i} \binom{q-1+h-i}{q-1} W_{b_1}(y)\, W_{b_2}(y) \dots W_{b_{q+1}}(y) = \frac{1}{2^q q!} W_{h+q}{}^{(q)}(y), \tag{7}$$

here h, q be integers with $h \geq 0, q \geq 1$.

For obtaining the relationship between the third-kind of the Chebyshev polynomials and Lucas numbers, we also used the result obtained by Kim et. al. [15] and W. Zhang [19]:

$$"\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \binom{q-1+h-i}{q-1} V_{b_1}(y)\, V_{b_2}(y) \dots V_{b_{q+1}}(y) = \frac{1}{2^q q!} V_{h+q}{}^{(q)}(y)," \tag{8}$$

inside sum covers all non-negative integers $b_{1,}, b_2 \dots b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = i$, h, q be integers with $h \geq 0, q \geq 1$."
$V_h{}^{q}(y)$, $W_h{}^{q}(y)$ denotes the qth derivative of $V_h(y)$, $W_n(y)$.

II. SOME SIMPLE LEMMAS

To prove our primary outcome, we require numerous lemmas. For $h \geq 0$, we have these identities:

$$W_h(161) = \tfrac{1}{8} F_{6(2h+1)},$$
$$W_h\left(\tfrac{47}{2}\right) = \tfrac{1}{3} F_{4(2h+1)},$$
$$W_h\left(\frac{123}{2}\right) = \frac{1}{11} L_{5(2h+1),}$$
$$V_h\left(\tfrac{47}{2}\right) = u^{-1} \tfrac{1}{2} L_{4(2h+1)},$$
$$V_h(161) = u^{-1} \tfrac{1}{2} L_{6(2h+1)},$$
$$V_h(-9) = u^{-1} \frac{(-i)^{2h+1}}{2} L_{3(2h+1)}.$$

Lemma 1: The following identity holds true for all $h \geq 0$:

$$W_h(161) = \tfrac{1}{8} F_{6(2h+1)}.$$

Proof: To prove lemma 1, take $y = 161, u = \sqrt{\frac{1+y}{2}}$.

Utilizing the identity,

$$"U_h(u) = \frac{1}{8} F_{6(h+1),}"$$

$$\Rightarrow "U_{2h}(u) = \frac{1}{8} F_{6(2h+1)}."$$

We also utilizing,

$$"U_{2h}(u) = W_h(y)", \text{ to get}$$

$$W_h(161) = \frac{1}{8} F_{6(2h+1).}$$

This demonstrates lemma 1.

Lemma 2: The following identity holds true for all $h \geq 0$:

$$W_h\left(\frac{47}{2}\right) = \frac{1}{3} F_{4(2h+1).}$$

Proof: To prove lemma 2, take $y = \frac{47}{2}, u = \sqrt{\frac{1+y}{2}}$.

We utilize the identity,

$$"U_h(u) = \frac{1}{3} F_{4(h+1),}"$$

$$\Rightarrow U_{2h}(u) = \frac{1}{3} F_{4(2h+1)}.$$

Also utilizing,

$"U_{2h}(u) = W_h(y)"$, to get

$$W_h\left(\frac{47}{2}\right) = \frac{1}{3} F_{4(2h+1)}.$$

This demonstrates lemma 2.

Lemma 3: The following identity holds true for all $h \geq 0$:

$$W_h\left(\frac{123}{2}\right) = \frac{1}{11} L_{5(2h+1).}$$

Proof: To prove lemma 3, take $y = \frac{123}{2}, u = \sqrt{\frac{1+y}{2}}$.

Utilizing the identity,

$$"U_h(u) = \frac{1}{11} L_{5(h+1),}"$$

$$\Rightarrow "U_{2h}(u) = \frac{1}{11} L_{5(2h+1).}"$$

We also utilizing,

$"U_{2h}(u) = W_h(y)"$, to get

$$W_h\left(\frac{123}{2}\right) = \frac{1}{11} L_{5(2h+1).}$$

This demonstrates lemma 3.

Lemma 4: The following identity holds true for all $h \geq 0$:

$$V_h\left(\frac{47}{2}\right) = u^{-1} \frac{1}{2} L_{4(2h+1)}.$$

Proof: To prove lemma 4, take $y = \frac{47}{2}, u = \sqrt{\frac{1+y}{2}}$.

Using the identity,

$$"T_h(u) = \frac{1}{2} L_{4h},"$$

$$\Rightarrow "T_{2h+1}(u) = \frac{1}{2} L_{4(2h+1)}."$$

We also use,

$"V_h(y) = u^{-1} T_{2h+1}(u)"$, to get

$$V_h\left(\frac{47}{2}\right) = u^{-1} \frac{1}{2} L_{4(2h+1).}$$

This demonstrates lemma 4.

Lemma 5: The following identity holds true for all $h \geq 0$:

$$V_h(161) = u^{-1} \frac{1}{2} L_{6(2h+1).}$$

Proof: To prove lemma 5, take $y = 161, u = \sqrt{\frac{1+y}{2}}$.

We utilizing the identity,

$$"T_h(u) = \frac{1}{2} L_{6h},"$$

$$\Rightarrow "T_{2h+1}(u) = \frac{1}{2} L_{6(2h+1)}."$$

We also using,

$"V_h(y) = u^{-1} T_{2h+1}(u)"$, to get

$$V_h(161) = u^{-1} \frac{1}{2} L_{6(2h+1).}$$

This demonstrates lemma 5.

Lemma 6: The following identity holds true for all $h \geq 0$:

$$V_h(-9) = u^{-1} \frac{(-i)^{2h+1}}{2} L_{3(2h+1).}$$

Proof: To prove lemma 6, take $y = -9, u = \sqrt{\frac{1+y}{2}}$,

We utilizing the identity,

$$"T_h(-2i) = \frac{(-i)^h}{2} L_{3h}",$$

$$\Rightarrow T_{2h+1}(-2i) = \frac{(-i)^{2h+1}}{2} L_{3(2h+1),}$$

We also utilizing,

$"V_h(y) = u^{-1} T_{2h+1}(u)"$, to get

$$V_h(-9) = u^{-1} \frac{(-i)^{2h+1}}{2} L_{3(2h+1)}.$$

This demonstrates lemma 6.

III. MAIN RESULTS

A. Relation between fourth-kind of the Chebyshev polynomials, Fibonacci numbers, and Lucas numbers.

Theorem 1 and theorem 2 present the relation between Fibonacci numbers and fourth-kind Chebyshev polynomials. While theorem 3 explains the relationship between Lucas numbers and fourth-kind Chebyshev polynomials

Theorem 1. Let h, q be integers with $h \geq 0, q \geq 1$, F_h be the h^{th} Fibonacci number, we have:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} (-1)^{h-i} \binom{q-1+h-i}{q-1} F_{6(2b_1+1)}$$

$$F_{6(2b_2+1)} \ldots F_{6(2b_{q+1}+1)} = \frac{2^{2q+3}}{q!} W_{h+q}{}^{(q)}(161),$$

inside sum covers all non-negative integers $b_{1,}, b_2 \ldots b_{q+1}$ with

$$b_1 + b_2 + \cdots + b_{q+1} = i.$$

Theorem 2. Let h, q be integers with $h \geq 0, q \geq 1$, and F_h be the h^{th} Fibonacci number, we have:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} (-1)^{h-i} \binom{q-1+h-i}{q-1} F_{4(2b_1+1)}$$

$$F_{4(2b_2+1)} \ldots F_{4(2b_{q+1}+1)} = \frac{3^{q+1}}{2^q q!} W_{h+q}{}^{(q)}\left(\frac{47}{2}\right),$$

inside sum covers all non-negative integers $b_{1,}, b_2 \ldots b_{q+1}$ with

$$b_1 + b_2 + \cdots + b_{q+1} = i.$$

Theorem 3. Let h, q be integers with $h \geq 0, q \geq 1, L_h$ be the h^{th} Lucas number, then the following identity we can state:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} (-1)^{h-i} \binom{q-1+h-i}{q-1} L_{5(2b_1+1)}$$

$$L_{5(2b_2+1)} \ldots L_{5(2b_{q+1}+1)} = \frac{11^{q+1}}{2^q q!} W_{n+q}{}^{(q)}\left(\frac{123}{2}\right),$$

inside sum covers all non-negative integers $b_{1,}, b_2 \dots b_{q+1}$ with

$$b_1 + b_2 + \cdots + b_{q+1} = i.$$

B. Relation between the third- kind Chebyshev polynomials, Lucas numbers.

Here, we have obtained three theorems. We present the relationship between Lucas numbers and third kind Chebyshev polynomials at certain points.

Theorem 4. Let h, q be integer with $h \geq 0, q \geq 1$, L_h be the h^{th} Lucas number, we have the following:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \binom{q-1+h-i}{q-1} L_{4(2b_1+1)} L_{4(2b_2+1)} \cdots L_{4(2b_{q+1}+1)} = \frac{2}{q!} u^{q+1} V_{h+q}{}^{(q)}\left(\frac{47}{2}\right),$$

inside sum covers all non-negative integers $b_{1,}, b_2 \dots b_{q+1}$ with

$$b_1 + b_2 + \cdots + b_{q+1} = i, u = \sqrt{\tfrac{1+y}{2}}.$$

Theorem 5. Let h, q be integers with $h \geq 0, q \geq 1$, L_h be the h^{th} Lucas number, we have:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \binom{q-1+h-i}{q-1} L_{6(2b_1+1)} L_{6(2b_2+1)} \cdots L_{6(2b_{q+1}+1)} = \frac{2}{q!} u^{q+1} V_{h+q}{}^{(q)}(161),$$

inside sum covers all non-negative integers $b_{1,}, b_2 \dots b_{q+1}$ with

$$b_1 + b_2 + \cdots + b_{q+1} = i, u = \sqrt{\tfrac{1+y}{2}}.$$

Theorem 6. Let h, q be integers with $h \geq 0, q \geq 1$, L_h be the h^{th} Lucas number, we have:

$$\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \binom{q-1+h-i}{q-1} (-i)^{2b_1+1} L_{3(2b_1+1)} (-i)^{2b_2+1} L_{3(2b_2+1)} \dots (-i)^{2b_{q+1}+1} L_{3(2b_{q+1}+1)} = \frac{2}{q!} u^{q+1} V_{h+q}{}^{(q)}(-9),$$

inside sum covers all non-negative integers $b_{1,}, b_2 \dots b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = i,\ u = \sqrt{\tfrac{1+y}{2}}.$

We can draw the following six Corollary from the above six theorems:

Corollary 1. Let h, q be integers with $h \geq 0, q \geq 1$, the resulting output is as follows:

for $q = 2$,

$$\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} (-1)^{h-i} \binom{1+h-i}{1} F_{6(2b_1+1)} F_{6(2b_2+1)} F_{6(2b_3+1)} = \frac{\begin{bmatrix} 646(2h+5)\{F_{6(2h+3)} - F_{6(2h+7)}\} + 209304 F_{6(2h+5)} + \\ 207360(h+2)(h+3) F_{6(2h+5)} \end{bmatrix}}{(25920)^2}.$$

Corollary 2. Let h, q be integers with $h \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} (-1)^{h-i} \binom{1+h-i}{1} F_{4(2b_1+1)} F_{4(2b_2+1)} F_{4(2b_3+1)} = \frac{\begin{bmatrix} 216(2h+5)\{F_{4(2h+3)} - F_{4(2h+7)}\} + 10584 F_{4(2h+5)} + \\ \frac{19845}{2}(h+2)(h+3) F_{4(2h+5)} \end{bmatrix}}{(2205)^2}.$$

Corollary 3. Let h, q be integers with $h \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} (-1)^{h-i} \binom{1+h-i}{1} L_{5(2b_1+1)} L_{5(2b_2+1)} L_{5(2b_3+1)} = \frac{\begin{bmatrix} 7502(2h+5)\{L_{5(2h+3)} - L_{5(2h+7)}\} + 937750 L_{5(2h+5)} + \\ \frac{20131375}{22}(h+2)(h+3) L_{5(2h+5)} \end{bmatrix}}{(15125)^2}.$$

Corollary 4. Let h, q be integers with $h \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \binom{1+h-i}{1} L_{4(2b_1+1)} L_{4(2b_2+1)} L_{4(2b_3+1)} = \frac{\begin{bmatrix} u^2[92(2h+5)\{L_{4(2h+3)} - L_{4(2h+7)}\} + 4140 L_{4(2h+5)} + \\ 4410(h+2)(h+3) L_{4(2h+5)} \end{bmatrix}}{(2205)^2}.$$

Corollary 5. Let h, q be integers with $h \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{h} \sum_{b_1+b_2+b_3=i} \binom{1+h-i}{1} L_{6(2b_1+1)} L_{6(2b_2+1)} L_{6(2b_3+1)} = \frac{u^2[\frac{321}{8}(2h+5)\{L_{6(2h+3)} - L_{6(2h+7)}\} + 12840 L_{6(2h+5)} + 12960(h+2)(h+3) L_{6(2h+5)}]}{(25920)^2}.$$

Corollary 6. Let h, q be integers with $h \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \binom{1+h-i}{1} (-i)^{2b_1+1} L_{3(2b_1+1)} (-i)^{2b_2+1} L_{3(2b_2+1)} (-i)^{2b_3+1} L_{3(2b_3+1)} = \frac{u^{-1} \begin{bmatrix} \frac{-19}{8}(2h+5)\{(-i)^{2h+3} L_{3(2h+3)} - (-i)^{2h+7} L_{3(2h+7)}\} \\ + \frac{95}{2} (-i)^{2h+5} L_{3(2h+5)} \\ +800(h+2)(h+3)(-i)^{2h+5} L_{3(2h+5)} \end{bmatrix}}{6400}.$$

<u>Proof of the Theorems</u>

Theorems and corollaries will be proved in this section.

Theorem 1: Proof. Let $W_h(y)$ be defined by (4), then for any h, q be integers with $h \geq 0, q \geq 1$, by (7)

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}(-1)^{h-i}\binom{q-1+h-i}{q-1}\prod_{k=1}^{q+1}W_{b_k}(y)$$
$$=\frac{1}{2^q q!}W_{h+q}{}^{(q)}(y), \qquad (9)$$

From lemma 1, we have the identity between Chebyshev polynomials of fourth kind and Fibonacci numbers,

$$W_h(161)=\tfrac{1}{8}F_{6(2h+1).} \qquad (10)$$

From (10), we have

$$W_{b_1}(161)=\tfrac{1}{8}F_{6(2b_1+1)},$$
$$W_{b_2}(161)=\frac{1}{8}F_{6(2b_2+1)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$W_{b_{k+1}}(161)=\frac{1}{8}F_{6(2b_{k+1}+1)}.$$

Using above equations in (9), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}(-1)^{h-i}\binom{q-1+h-i}{q-1}F_{6(2b_1+1)}$$
$$F_{6(2b_2+1)}\ldots F_{6(2b_{q+1}+1)}=\frac{2^{2q+3}}{q!}W_{h+q}{}^{(q)}(161). \qquad (11)$$

Hence theorem 1 is formulated in this manner.

Theorem 2: Proof. Let $W_h(y)$ be defined by (4), then for any h,q be integers with $h\geq 0, q\geq 1$, by (7) yields

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots b_{q+1}=i}(-1)^{h-i}\binom{q-1+h-i}{q-1}\prod_{k=1}^{q+1}W_{b_k}(y)$$
$$=\frac{1}{2^q q!}W_{h+q}{}^{(q)}(y). \qquad (12)$$

From lemma 2, we have the identity between Chebyshev polynomials of fourth kind and Fibonacci numbers i.e.

$$W_h\left(\tfrac{47}{2}\right)=\tfrac{1}{3}F_{4(2h+1)}. \qquad (13)$$

From (13), we have

$$W_{b_1}\left(\tfrac{47}{2}\right)=\tfrac{1}{3}F_{4(2b_1+1)},$$
$$W_{b_2}\left(\frac{47}{2}\right)=\frac{1}{3}F_{4(2b_2+1),}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$W_{b_{k+1}}\left(\frac{47}{2}\right)=\frac{1}{3}F_{4(2b_{k+1}+1).}$$

Using above equations in (12), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}(-1)^{h-i}\binom{q-1+h-i}{q-1}F_{4(2b_1+1)}$$
$$F_{4(2b_2+1)}\ldots F_{4(2b_{q+1}+1)}=\frac{3^{q+1}}{2^q q!}W_{h+q}{}^{(q)}\left(\frac{47}{2}\right). \quad (14)$$

Thus theorem 2 established through proof.

Theorem 3: Proof. Let $W_h(y)$ be defined by (4), then for any h,q be integers with $h\geq 0, q\geq 1$, by using (7),

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+l}=i}(-1)^{h-i}\binom{q-1+h-i}{q-1}\prod_{k=1}^{q+1}W_{b_k}(y)$$
$$=\frac{1}{2^q q!}W_{h+q}{}^{(q)}(y). \qquad (15)$$

From lemma 3, we have the identity between Chebyshev polynomials of fourth kind and Lucas numbers, i.e.

$$W_h\left(\tfrac{123}{2}\right)=\tfrac{1}{11}L_{5(2h+1).} \qquad (16)$$

From (16), we have

$$W_{b_1}\left(\tfrac{123}{2}\right)=\tfrac{1}{11}L_{5(2b_1+1)},$$
$$W_{b_2}\left(\frac{123}{2}\right)=\frac{1}{11}L_{5(2b_2+1),}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$W_{b_{k+1}}\left(\frac{123}{2}\right)=\frac{1}{11}L_{5(2b_{k+1}+1).}$$

Using above equations in (15), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}(-1)^{h-i}\binom{q-1+h-i}{q-1}L_{5(2b_1+1)}$$
$$L_{5(2b_2+1)}\ldots L_{5(2b_{q+1}+1)}=\tfrac{11^{q+1}}{2^q q!}W_{n+q}{}^{(q)}\left(\tfrac{123}{2}\right). \qquad (17)$$

Hence, proof is established for theorem 3.

Theorem 4: Proof. Let $V_h(y)$ be defined by (3), then for any h,q be integers with $h\geq 0, q\geq 1$, by using (8)

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}\binom{q-1+h-i}{q-1}\prod_{k=1}^{q+1}V_{b_k}(y)$$
$$=\frac{1}{2^q q!}V_{h+q}{}^{(q)}(y). \qquad (18)$$

From lemma 4, we have the identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

$$V_h\left(\tfrac{47}{2}\right)=u^{-1}\tfrac{1}{2}L_{4(2h+1).} \qquad (19)$$

From (19), we have

$$V_{b_1}\left(\tfrac{47}{2}\right)=u^{-1}\tfrac{1}{2}L_{4(2b_1+1)},$$
$$V_{b_2}\left(\frac{47}{2}\right)=u^{-1}\frac{1}{2}L_{4(2b_2+1)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$V_{b_{k+1}}\left(\frac{47}{2}\right)=u^{-1}\frac{1}{2}L_{4(2b_{k+1}+1)}.$$

Using above equations in (18), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}\binom{q-1+h-i}{q-1}L_{4(2b_1+1)}L_{4(2b_2+1)}\cdots$$
$$L_{4(2b_{q+1}+1)}=\frac{2}{q!}u^{q+1}V_{h+q}{}^{(q)}\left(\frac{47}{2}\right). \qquad (20)$$

Hence, theorem $4's$ proof is now complete.

Theorem 5: Proof. Let $V_h(y)$ be defined by (3), then for any h,q be integers with $h\geq 0, q\geq 1$, by (8)

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}\binom{q-1+h-i}{q-1}\prod_{k=1}^{q+1}V_{b_k}(y)$$
$$=\frac{1}{2^q q!}V_{h+q}{}^{(q)}(y). \qquad (21)$$

From lemma 5, we have the identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

$$V_h(161)=u^{-1}\tfrac{1}{2}L_{6(2h+1)}. \qquad (22)$$

From (22), we have

$$V_{b_1}(161)=u^{-1}\tfrac{1}{2}L_{6(2b_1+1)},$$

$$V_{b_2}(161) = u^{-1}\frac{1}{2}L_{6(2b_2+1)},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$V_{b_{k+1}}(161) = u^{-1}\frac{1}{2}L_{6(2b_{k+1}+1)}.$$

Using above equation in (21), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}\binom{q-1+h-i}{q-1}L_{6(2b_1+1)}L_{6(2b_2+1)}\cdots L_{6(2b_{q+1}+1)} = \frac{2}{q!}u^{q+1}V_{h+q}^{(q)}(161). \qquad (23)$$

The 5 theorem proof is now complete.

Theorem 6: Proof. Let $V_h(y)$ be defined by (3), then for any h, q be integers with $h \geq 0, q \geq 1$, by (8) yields

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}\binom{q-1+h-i}{q-1}\prod_{k=1}^{q+1}V_{b_k}(y) = \frac{1}{2^q q!}V_{h+q}^{(q)}(y). \qquad (24)$$

From lemma 6, we have the identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

$$V_h(-9) = u^{-1}\frac{(-i)^{2h+1}}{2}L_{3(2h+1)}. \qquad (25)$$

From (25), we have

$$V_{b_1}(-9) = u^{-1}\frac{(-i)^{2b_1+1}}{2}L_{3(2b_1+1)},$$
$$V_{b_2}(-9) = u^{-1}\frac{(-i)^{2b_2+1}}{2}L_{3(2b_2+1)},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$V_{b_{k+1}}(-9) = u^{-1}\frac{(-i)^{2b_{k+1}+1}}{2}L_{3(2b_{k+1}+1)},$$

Using above equation in (24), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+\cdots+b_{q+1}=i}\binom{q-1+h-i}{q-1}(-i)^{2b_1+1}L_{3(2b_1+1)}(-i)^{2b_2+1}L_{3(2b_2+1)}\dots(-i)^{2b_{q+1}+1}L_{3(2b_{q+1}+1)} = \frac{2}{q!}u^{q+1}V_{h+q}^{(q)}(-9), \qquad (26)$$

All of our theorem now has through proof.

Proof of the Corollaries

Proof of Corollary 1: We know that $W_h{}^y$ satisfies the differential equations:

$$(1-y^2)W_h'(y) = \frac{1}{2}\left(h+\frac{1}{2}\right)\left(W_{h-1}(y)-W_{h+1}(y)\right) + \frac{1}{2}(1+y)W_h(y). \qquad (27)$$

and

$$(1-y^2)W_h''(y) = (1+2y)W_h'(y) - h(h+1)W_h(y). \qquad (28)$$

Putting $q=2$ in (11), by (5), (27), (28),
We get,

$$\sum_{i=0}^{h}\sum_{b_1+b_2+b_3=i}(-1)^{h-i}\binom{1+h-i}{1}F_{6(2b_1+1)}F_{6(2b_2+1)}F_{6(2b_3+1)} =$$
$$64\left[\frac{323}{(25920)^2}\left\{\frac{(2h+5)}{4}\left(\frac{1}{8}F_{6(2h+3)}-\frac{1}{8}F_{6(2h+7)}\right) + 81\left(\frac{1}{8}F_{6(2h+5)}\right\} + \frac{(h+2)(h+3)\frac{1}{8}F_{6(2h+5)}}{25920}\right]$$
$$= \frac{\begin{bmatrix}646(2h+5)\{F_{6(2h+3)}-F_{6(2h+7)}\}+209304F_{6(2h+5)}\\ +207360(h+2)(h+3)F_{6(2h+5)}\end{bmatrix}}{(25920)^2}.$$

Proof of Corollary 2:
Put $q=2$ in (14), and by (5), (27), (28),
We get,

$$\sum_{i=0}^{h}\sum_{b_1+b_2+b_3=i}(-1)^{h-i}\binom{1+h-i}{1}F_{4(2b_1+1)}F_{4(2b_2+1)}F_{4(2b_3+1)}$$
$$= \frac{27}{8}\left[\frac{192}{(2205)^2}\left\{(2h+5)\left(\frac{1}{3}F_{4(2h+3)} - \frac{1}{3}F_{4(2h+7)}\right) + 49\left(\frac{1}{3}F_{4(2h+5)}\right)\right\} + \frac{4(h+2)(h+3)\frac{1}{3}F_{4(2h+5)}}{2205}\right]$$
$$= \frac{\begin{bmatrix}216(2h+5)\{F_{4(2h+3)}-F_{4(2h+7)}\}+10584F_{4(2h+5)}\\ +\frac{19845}{2}(h+2)(h+3)F_{4(2h+5)}\end{bmatrix}}{(2205)^2}.$$

Proof of Corollary 3:
Put $q=2$ in (17) and by (6), (27), (28), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+b_3=i}(-1)^{h-i}\binom{1+h-i}{1}L_{5(2b_1+1)}L_{5(2b_2+1)}L_{5(2b_3+1)} =$$
$$\frac{1331}{8}\left[\frac{496}{(15125)^2}\left\{(2h+5)\left(\frac{1}{11}L_{5(2h+3)}-\frac{1}{11}L_{5(2h+7)}\right) + 125\left(\frac{1}{11}L_{5(2h+5)}\right\} + \frac{4}{15125}(h+2)(h+3)\frac{1}{11}L_{5(2h+5)}\right]$$
$$= \frac{\begin{bmatrix}7502(2h+5)\{L_{5(2h+3)}-L_{5(2h+7)}\}+\\ 937750L_{5(2h+5)}+\frac{20131375}{22}(h+2)(h+3)L_{5(2h+5)}\end{bmatrix}}{(15125)^2}.$$

Proof of Corollary 4: $(1-y^2)V_h'(y) = \frac{1}{2}\left(h+\frac{1}{2}\right)\left(V_{h-1}(y)-V_{h+1}(y)\right) - \frac{1}{2}(1-y)V_h(y).$ (29)

$$(1-y^2)V_h''(y) = -(1-2y)V_h'(y) - h(h+1)V_h(y). \qquad (30)$$

Put $q=2$ in (20), and by (6), (29), (30),
We get,

$$\sum_{i=0}^{h}\sum_{b_1+b_2+b_3=i}\binom{1+h-i}{1}L_{4(2b_1+1)}L_{4(2b_2+1)}L_{4(2b_3+1)}$$

$$= \frac{\begin{bmatrix} u^2[92(2h+5)\{L_{4(2h+3)}-L_{4(2h+7)}\}+4140L_{4(2h+5)}+ \\ 4410(h+2)(h+3)L_{4(2h+5)} \end{bmatrix}}{(2205)^2}.$$

Proof of Corollary 5:
Put $q=2$ in (23), and by (6), (29), (30), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+b_3=i}\binom{1+h-i}{1}L_{6(2b_1+1)}L_{6(2b_2+1)}L_{6(2b_3+1)} = \frac{\begin{bmatrix} u^2[\frac{321}{8}(2h+5)\{L_{6(2h+3)}-L_{6(2h+7)}\}+12840L_{6(2h+5)} \\ +12960(h+2)(h+3)L_{6(2h+5)}] \end{bmatrix}}{(25920)^2}.$$

Proof of Corollary 6:
Put $q=2$ in (26), and by (6), (29), (30), we get

$$\sum_{i=0}^{h}\sum_{b_1+b_2+b_3=i}\binom{1+h-i}{1}(-i)^{2b_1+1}L_{3(2b_1+1)} (-i)^{2b_2+1}L_{3(2b_2+1)}(-i)^{2b_3+1}L_{3(2b_3+1)} = \frac{u^{-1}\begin{bmatrix} \frac{-19}{8}(2h+5)\{(-i)^{2h+3}L_{3(2h+3)}-(-i)^{2h+7}L_{3(2h+7)}\} \\ +\frac{95}{2}(-i)^{2h+5}L_{3(2h+5)} \\ +800(h+2)(h+3)(-i)^{2h+5}L_{3(2h+5)}. \end{bmatrix}}{6400}$$

IV. APPLICATIONS

Chebyshev polynomials are widely used in various fields like mathematics, computer science, engineering and physics etcetera. Chebyshev polynomials fall under the category of recurrence relations and are extensively used to improve the advanced techniques for counting. These polynomials are used to study the integer function that allows us to establish new relationships among other polynomials. Chebyshev polynomials are of significant importance when it comes to solving other polynomials for obtaining novel trigonometric identities, finding the solutions to second-order differential equations, interpolating large data and in approximation theory. With the help of Chebyshev polynomials, approximate numerical solutions can be obtained for differential and integral equations. Chebyshev polynomials play a key role in signal processing, primarily in the design of filters known as Chebyshev filters. They are in high demand in the field of computer graphics to generate a variety of shapes, surfaces, and curves.

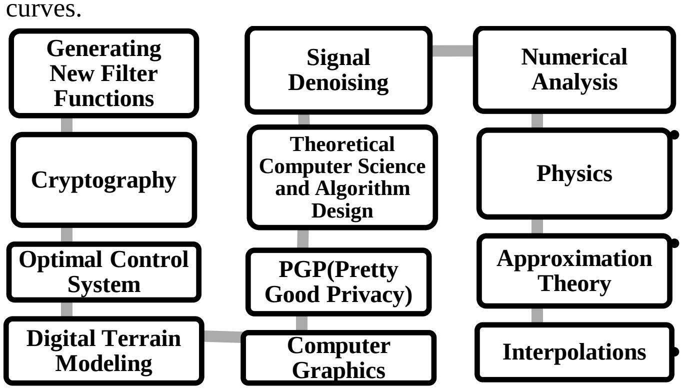

Figure 1: Flow Chart of Chebyshev Polynomial Application

V. PRATICAL APPLICATIONS

Express $y^4-4y^3-2y^2+3y-1$ in terms of third-kind of the Chebyshev polynomials.

Sol: The first four third-kind Chebyshev polynomials are:

$$V_0(y)=1,$$
$$V_1(y)=2y-1,$$
$$V_2(y)=4y^2-2y-1,$$
$$V_3(y)=8y^3-4y^2-4y+1,$$
$$V_4(y)=16y^4-8y^3-12y^2+4y+1,$$

From above equations, we get

$$y=\tfrac{1}{2}[V_0(y)+V_1(y)],$$
$$y^2=\tfrac{1}{4}[2V_0(y)+V_1(y)+V_2(y)],$$
$$y^3=\tfrac{1}{8}[3V_0(y)+3V_1(y)+V_2(y)+V_3(y)],$$
$$y^4=\tfrac{1}{16}[6V_0(y)-4V_1(y)+4V_2(y)+V_3(y)+V_4(y)].$$

Putting above values in $y^4-4y^3-2y^2+3y-1$,
We get,

$$\tfrac{1}{16}[6V_0(y)-4V_1(y)+4V_2(y)+V_3(y)+V_4(y)]-\tfrac{1}{2}[3V_0(y)+3V_1(y)+V_2(y)+V_3]-\tfrac{1}{2}[2V_0(y)+V_1(y)+V_2(y)]+\tfrac{3}{2}[V_0(y)+V_1(y)]-V_0(y).$$

Hence,

$$-\frac{13}{8}V_0(y)-\frac{3}{4}V_1(y)-\frac{3}{4}V_2(y)-\frac{7}{16}V_3(y)+\frac{1}{16}V_4(y).$$

VI. CONCLUSION

In relation to both the third-fourth kinds of Chebyshev polynomials with Fibonacci numbers and Lucas numbers, we have discovered six theorems and six corollaries. The first two theorems present the relationship between fourth-kind Chebyshev polynomials and Fibonacci numbers. Theorem 3 clearly demonstrates the strong correlation between the Lucas numbers and fourth-kind Chebyshev polynomials, emphasizing the significance of the relationship. Theorems 4,5 and 6 provided the link between Lucas numbers and third-kind Chebyshev polynomials at certain points. The six corollaries are merely particular cases associated with our six theorems. These findings are useful for understanding the properties and identities of Chebyshev polynomials with Fibonacci and Lucas numbers.

VII. SIGNIFICANCE OF THE WORK

Here, we illustrate the significance of the present work in the following elements:

Obtaining connections between fourth-kind of the Chebyshev polynomials and Fibonacci numbers with some variables: $x=161,\frac{47}{2}$.

Obtaining connections between fourth-kind of the Chebyshev polynomials and Lucas numbers at a particular variable: $x=\frac{123}{2}$.

Obtaining connections between third-kind of the Chebyshev polynomials and Lucas numbers with some variables:

$$x = 161, \frac{47}{2}, -9.$$

It is worth mentioning here that the above-achieved results and analysis are fruitful. Some of their presumed uses are given below:

- These results strengthen the correlation of Chebyshev polynomials to Fibonacci and Lucas.
- They are also beneficial in studying problems connected to calculating general summations.
- They help study integer sequences.
- These polynomials are fruitful in solving convolution sum problems.
- These polynomials can be used to solve differential equations, whether they are linear or non-linear.
- To acquire numerical answers to differential equations, whether linear or nonlinear.
- The connections between the Chebyshev polynomials, Fibonacci, and Lucas numbers are highly helpful in obtaining the identities related to them.
- The Chebyshev polynomials are fruitful in approximation theory.

REFERENCES

[1] B.G.S. Doman, "The classical orthogonal polynomials", *World Scientific Publishing Co. Pte. Ltd*, 2016, pp, 87-109.

[2] E.S. Shoukralla," Application of Chebyshev polynomials of the second kind to the numerical solution of weakly singular Fredholm integral equations of the first kind", *IAENG International Journal of Applied Mathematics*, vol. 51, no. 1, pp 66-81, 2021.

[3] R. Frontczak, T. Goy, "Chebyshev-Fibonacci polynomial relations using generating functions", *Integers,* vol. 21, pp. 1-15, 2021. https://doi.org/10.48550/arXiv.2103.08015.

[4] D. Han, X. Lv, "On the Chebyshev polynomials and some of their new identities", *Advances in Difference Equations*, vol. 2020, no.1, pp 1-8, 2020. https://doi.org/10.1186/s13662-020-02550-y.

[5] J. Griffiths, "Identities connecting the Chebyshev polynomials", *The Mathematical Gazette*, vol. 100, no. 549, pp 450-459, 2016. https://doi:10.1017/mag.2016.110

[6] J.C. Mason, "Chebyshev polynomials of the second kind, third and fourth kinds in approximation, indefinite integration, and integral transforms", *Journal of Computational and Applied Mathematics*, vol. 49, pp 169-178, 1993, https://doi:.org/10.1016/0377-0427(93)90148-5

[7] J. Kishore, V. Verma, "Some identities on finite sums of product of Fibonacci and Lucas numbers in Chebyshev polynomials of second kind", *Journal of Algebraic Statistics*, vol. 13, no.3, pp 3566-3570, 2022.

[8] K. Aghigh, M. M. Jamei, and M. Dehghan, "A survey on third and fourth kind of Chebyshev polynomials and their applications", *Applied Mathematics and Computation*, vol. 199, no.1, pp 1-12, 2008.

[9] L. Li, Y. Li, and Q. Huang, "Fractional order Chebyshev cardinal functions for solving two classes of fractional differential equations", *Engineering Letters*, vol.30, no.1, pp 208-213, 2022.

[10] L. Zhang, W. Zhang, "On the sums of powers of Chebyshev polynomials and their divisible properties", *Mathematical Problems in Engineering*, vol. 2017, no. 2, pp (1-6), Article ID 9363680, 2017, https://doi.org/10.1155/2017/9363680

[11] M. Arya, V. Verma, "A special representation of Chebyshev polynomials", *Journal of Advanced Research in Dynamical & Control Systems*, vol.11, no. 07, pp 795-798, 2019.

[12] M.R. Eslahchi, M. Dehghan, and S. Amani, "The third and fourth kinds of Chebyshev polynomials and best uniform approximation", *Mathematical and Computer Modelling*, vol.55, pp 1746-1762, 2012, https://doi.org/10.1016/j.mcm.2011.11.023

[13] S. Harne, V.H. Badshah, and V. Verma, "Some identities involving the Chebyshev polynomials, Fibonacci numbers and Lucas numbers", *International Journal of Contemporary Mathematical Sciences*, vol. 9, no.12, pp 553-561, 2014.

[14] T. Korkiatsakul, S. Koonprasert, and K. Neamprem, "A new operational matrix method for solving problems via Chebyshev polynomials", *Proceedings of the International Multi Conference of Engineers and Computer Scientists*, vol. 1, pp 450-455, 2018.

[15] T. Kim, D.S. Kim, D.V. Dolgy, and J. Kwon, "Sums of finite products of Chebyshev polynomials of third and fourth kinds", *Advances in Difference Equations*, vol. 2018, no.1, pp 1-17, 2018. https://doi.org/10.1186/s13662-018-1747-z.

[16] T. Kim, D.S. Kim, D. V. Dolgy, and J. Kwon, "Representing sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials by Chebyshev polynomials", *Mathematics*, vol. 7, no. 1, pp 1-15, 2018. http://doi:10.3390/math7010026.

[17] T. Kim, D.S. Kim, D.V. Dolgy, and J. Kwon, "Sums of finite products of Chebyshev polynomials of two different types", *AIMS Mathematics*, vol. 6, no. 11, pp. 12528-12542, 2021.

[18] W. Zhang, "Some Identities involving the Fibonacci numbers", *The Fibonacci Quarterly*, vol. 35, no. 3, pp 225-229, 1997.

[19] W. Zhang, "On Chebyshev polynomials and Fibonacci numbers", *The Fibonacci Quarterly*, vol. 40, no. 5, pp 424-428, 2002.

[20] W. Zhang, "Some identities involving the Fibonacci numbers and Lucas numbers," *The Fibonacci Quarterly*, vol. 42, no. 2, pp 149-154, 2004.

[21] W. Zhang, D. Han, "On the Chebyshev polynomial and some of their reciprocal sums", *Symmetry*, vol. 12, no. 5, pp (1-9), 2020. https://doi:10.3390/sym12050704.

[22] Y. Li, "On Chebyshev polynomials, Fibonacci polynomials and their derivatives", *Journal of Applied Mathematics*, vol. 2014, pp 1-8, Article ID 451953, 2014. http://dx.doi.org/10.1155/2014/451953

[23] Y. Li, "Some properties of Fibonacci and Chebyshev polynomials", *Advances in Difference Equations*, vol. 2015, no. 1, pp 1-12, 2015. http://dx.doi.org/10.1186/s13662-015-0420-z

[24] Y. Zhang, Z. Chen, "A new identity involving the Chebyshev polynomials", *Mathematics*, vol. 6, no. 1, pp 1-8, 2018. https://doi.org/10.3390/math6110244