On the Reverse Minkowski's, Reverse Hölder's and Other Fractional Integral Inclusions Arising from Interval-Valued Mappings

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***Abstract*—To investigate integral inclusions through interval-valued mappings, we utilize the notion of the interval-valued fractional integrals having exponential kernels. We develop the reverse Minkowsky's, including with the reverse Hölder's fractional integral inclusions in association with interval-valued mappings. Other fractional integral inclusions in relation to the Hermite–Hadamard-type inequalities are reported as well. Moreover, we provide some examples concerning interval-valued mappings, to enucleate the correctness of the inclusion relations presented here.**

***Index Terms*—Interval-valued mappings; fractional integral; fractional integral inclusions; Hermite–Hadamard's inequalities**

I. Introduction and preliminaries

THE convexity of mappings is an impressive tool, in particular, which is applicable in several distinct areas of engineering mathematical and applied analysis. Recently, a large number of researchers, including with mathematicians, engineers and scientists, have devoted all their energies to researching the integral inequalities in accordance with functional convexities within a few varieties of directions, see the published articles [1], [2], [3], [4] and the references cited in them. Among them, one of the outstanding integral inequalities, as respects convex mappings, is named as the Hermite–Hadamard's integral inequalities, which is employed diffusely in plenty of other branch of learning of applied analysis, especially in the field of optimality analysis. Let us retrospect it as follows.

Given that the mapping $h : \Lambda \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the real-valued interval Λ, for all $c, z \in \Lambda$ together with $c \neq z$. Then, we have the successive Hermite–Hadamard's integral inequalities:

$$h\left(\frac{c+z}{2}\right) \leq \frac{1}{z-c}\int_c^z h(\zeta)\mathrm{d}\zeta \leq \frac{h(c)+h(z)}{2}. \quad (1)$$

The extraordinary Hermite–Hadamard's integral inequalities, which have set off considerable attention from quite a few mathematicians, provide upper and lower bounds for the mean value involving with a continuous convex mapping $h : [c, z] \to \mathbb{R}$. There have been a large amount of the published articles regarding the Hermite–Hadamard-type integral inequalities. Hereby, we enumerate some existing ones on the basis of different groups of convex mappings. For example, one can refer to Dragomir and Nikodem [5] for strongly convex mappings, to Kórus [6] for s-convex mappings, to Kadakal et al. [7] for exponential trigonometric convex mappings, to Delavar and De La Sen [8] for h-convex mappings, to Andrić and Pečarić [9] for $(h, g; m)$-convex mappings and so on. For recent outcomes with regard to this topical subject, we recommend the minded readers to consult the published articles [10], [11], [12], [13], [14] and the bibliographies quoted in them.

On the other hand, not a few researchers have contributed to the study of the Hermite–Hadamard-type inequalities in the sense of different types of fractional integral operators. Here we mention some outcomes of [15], [16], [17] and the corresponding references cited therein. In 2019, Ahmad et al. put forward a family of fractional integral operators, that is, the fractional integral operators having exponential kernels in the coming way.

Definition 1.1: [18] It is assumed that the mapping $h \in L^1([c, z])$. The fractional integral operators together with exponential-type kernels $\mathcal{I}^{\alpha}_{c^+}h$ and $\mathcal{I}^{\alpha}_{z^-}h$ of order $\alpha \in (0, 1)$ are, correspondingly, defined by the successive expressions

$$\mathcal{I}^{\alpha}_{c^+}h(x) = \frac{1}{\alpha}\int_c^x \exp\left(-\frac{1-\alpha}{\alpha}(x-\zeta)\right)h(\zeta)\mathrm{d}\zeta, \ x > c,$$

and

$$\mathcal{I}^{\alpha}_{z^-}h(x) = \frac{1}{\alpha}\int_x^z \exp\left(-\frac{1-\alpha}{\alpha}(\zeta-x)\right)h(\zeta)\mathrm{d}\zeta, \ x < z.$$

From Definition 1.1, we can also readily observe that

$$\lim_{\alpha\to 1^-}\mathcal{I}^{\alpha}_{c^+}h(x) = \int_c^x h(\zeta)\mathrm{d}\zeta, \ \lim_{\alpha\to 1^-}\mathcal{I}^{\alpha}_{z^-}h(x) = \int_x^z h(\zeta)\mathrm{d}\zeta.$$

If we consider taking $c = 0$ only, for the sake of brevity, then we will write

$$\mathcal{I}^{\alpha}h(x) = \frac{1}{\alpha}\int_0^x \exp\left(-\frac{1-\alpha}{\alpha}(x-\zeta)\right)h(\zeta)\mathrm{d}\zeta, \ x > 0.$$

Ahmad et al. also established the coming outcome.

Theorem 1.1: [18] It is assumed that the convex mapping $h : [c, z] \to \mathbb{R}$ is positive along with $0 \leq c < z$. If the mapping $h \in L^1([c, z])$, then the coming inequalities via fractional integrals having exponential-type kernels hold true:

$$\begin{aligned} h\left(\frac{c+z}{2}\right) &\leq \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{c^+}h(z) + \mathcal{I}^{\alpha}_{z^-}h(c)\right] \\ &\leq \frac{h(c)+h(z)}{2}, \end{aligned} \quad (2)$$

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where

$$\rho = \frac{1-\alpha}{\alpha}(z-c), (0 < \alpha < 1).$$

Fractional calculus, as a rather advantageous tool, has proven to be an vital cornerstone in engineering mathematics and applied analyses. For example, Li et al. in Ref. [19] studied the fractional order Chebyshev cardinal function and presented solutions for two different forms of fractional order delay differential equations. In Ref. [20], the author considered the oscillation of a class fractional differential equations and established some oscillatory criteria for the equations. Moreover, Alomari and Massoun [21] investigated time-fractional coupled Korteweg-de Vries differential equations and proposed an efficient solution method based on Caputo definition. For more application results with regard to this issue, see the published articles [22], [23] and the references cited in them.

On the other hand, the research of the fractional calculus theory has attracted extensive interest of many scholars, especially in the field of fractional integral inequality. In consequence, certain brilliant integral inequalities, in accordance with a fruitful interaction of various approaches of fractional calculus, were brought into force by mass learned men, containing with Mohammed [24] in the study of the Hermite–Hadamard-type integral inequalities, and Set, et al. [25] in the Simpson-type integral inequalities in terms of Riemann–Liouville fractional integrals, Chen and Katugampola [26] in the Fejér–Hermite–Hadamard-type integral inequalities by virtue of Katugampola fractional integrals, Meftah, et al. [27] in the Maclaurin-type integral inequalities via local fractional integrals, Du, et al. [28] in the extensions of the trapezoid-type integral inequalities considering with k-fractional integrals, as well as Mohammed and Hamasalh [29] in the Hermite–Hadamard's integral inequalities through conformable fractional integrals. For more findings in relation to the fractional integral operators, we recommend the minded readers to glance over the published articles [30], [31], [32], [33], [34] and the bibliographies quoted in them.

Later, we will begin with evoking the axioms of interval analysis, which are employed throughout the present study.

It is assumed that the set Δ is closed, bounded real-valued interval subset of $\mathbb{R}$, which is represented as

$$\Delta = [\underline{\Delta}, \overline{\Delta}] = \Big\{\phi \in \mathbb{R} : \underline{\Delta} \le \phi \le \overline{\Delta}\Big\},$$

in which, $\underline{\Delta}, \overline{\Delta} \in \mathbb{R}$ together with $\underline{\Delta} \le \overline{\Delta}$. The left as well as right endpoints of the real-valued interval Δ are the real numbers $\underline{\Delta}$ and $\overline{\Delta}$, correspondingly. If we consider that the condition $\underline{\Delta} = \Upsilon = \overline{\Delta}$ holds true, then the real-valued interval Δ is known as the degradation, in this situation, we consider adopting the form $\Delta = \Upsilon = [\Upsilon, \Upsilon]$. In addition, if the real number $\underline{\Delta} > 0$, then the real-valued interval Δ is positive or if the real number $\overline{\Delta} < 0$, then the real-valued interval Δ is negative. We appoint $\mathbb{R}_{\mathcal{I}}$, $\mathbb{R}_{\mathcal{I}}^{-}$ together with $\mathbb{R}_{\mathcal{I}}^{+}$ to denote the sets of all closed real-valued intervals of $\mathbb{R}$, the sets of all closed negative real-valued intervals of $\mathbb{R}$, as well as the sets of all closed positive real-valued intervals of $\mathbb{R}$, correspondingly. Based upon these symbols, the Hausdorff–Pompeiu distance, considering with the real-valued intervals Δ and N, is represented as

$$d(\Delta, N) = d\Big([\underline{\Delta}, \overline{\Delta}], [\underline{N}, \overline{N}]\Big) = \max\Big\{\big|\underline{\Delta} - \underline{N}\big|, \big|\overline{\Delta} - \overline{N}\big|\Big\}.$$

Clearly, $(\mathbb{R}_{\mathcal{I}}, d)$ belongs to a complete metric space.

Concerning to the real-valued intervals Δ and N, their interval arithmetic operations are the coming ones:

$$\Delta + N = [\underline{\Delta} + \underline{N}, \overline{\Delta} + \overline{N}],$$
$$\Delta - N = [\underline{\Delta} - \overline{N}, \overline{\Delta} - \underline{N}],$$
$$\Delta \cdot N = [\min S, \max S],$$

where $S = \{\underline{\Delta}\ \underline{N}, \underline{\Delta}\overline{N}, \overline{\Delta}\underline{N}, \overline{\Delta}\ \overline{N}\}$,

$$\Delta / N = [\min T, \max T] (0 \notin N),$$

where $T = \{\underline{\Delta}/\underline{N}, \underline{\Delta}/\overline{N}, \overline{\Delta}/\underline{N}, \overline{\Delta}/\overline{N}\}$.

The power operation regarding the real-valued interval Δ is characterized as

$$\Delta^n = \Big[\min\{\underline{\Delta}^n, \overline{\Delta}^n\}, \max\{\underline{\Delta}^n, \overline{\Delta}^n\}\Big]$$
$$= [\underline{\Delta}^n, \overline{\Delta}^n], (\underline{\Delta} > 0, n \in \mathbb{R}).$$

The scalar multiplicative operation pertaining to the real-valued interval Δ is described by the coming expressions:

$$\eta\Delta = \eta[\underline{\Delta}, \overline{\Delta}] = \begin{cases} [\eta\underline{\Delta}, \eta\overline{\Delta}], & \eta > 0, \\ 0, & \eta = 0, \\ [\eta\overline{\Delta}, \eta\underline{\Delta}], & \eta < 0, \end{cases}$$

in which the number $\eta \in \mathbb{R}$. In concerning to the opposite element for the real-valued interval Δ, one knows that

$$-\Delta := (-1)\Delta = [-\overline{\Delta}, -\underline{\Delta}].$$

The subtracting operation is adopted by the subsequent ways:

$$\Delta - N = \Delta + (-N) = [\underline{\Delta} - \overline{N}, \overline{\Delta} - \underline{N}].$$

Here, what we want to point out is that $-\Delta$ is not additive inverse for Δ, that is, $\Delta - \Delta \neq 0$. If the condition $\underline{\Delta} = \overline{\Delta}$ holds true, in this case, then the equality is satisfied.

We next begin to state a nonempty closed interval set $\mathbb{R}_{\mathcal{I}}$, together with addition operations $(+)$, as well as multiplicative operations $(\cdot)$, which is often referred as to a quasilinear space, if the below-mentioned laws hold true.

(a) $(\Delta + N) + H = \Delta + (N + H)$ for each $\Delta, N, H \in \mathbb{R}_{\mathcal{I}}$,
(b) $\Delta + 0 = 0 + \Delta = \Delta$ for each $\Delta \in \mathbb{R}_{\mathcal{I}}$,
(c) $\Delta + N = N + \Delta$ for each $\Delta, N \in \mathbb{R}_{\mathcal{I}}$,
(d) $\Delta + N = H + N \Rightarrow \Delta = H$ for any $\Delta, N, H \in \mathbb{R}_{\mathcal{I}}$,
(e) $(\Delta \cdot N) \cdot H = \Delta \cdot (N \cdot H)$ for each $\Delta, N, H \in \mathbb{R}_{\mathcal{I}}$,
(f) $\Delta \cdot N = N \cdot \Delta$ for each $\Delta, N \in \mathbb{R}_{\mathcal{I}}$,
(g) $Z \cdot 1 = 1 \cdot Z = Z$ for any $Z \in \mathbb{R}_{\mathcal{I}}$,
(h) $\xi(\eta\Delta) = (\xi\eta)\Delta$ for each $\Delta \in \mathbb{R}_{\mathcal{I}}$ and each $\xi, \eta \in \mathbb{R}$,
(i) $\eta(\Delta + N) = \eta\Delta + \eta N$ for each $\Delta, N \in \mathbb{R}_{\mathcal{I}}$ and every $\eta \in \mathbb{R}$,
(j) $(\xi + \eta)\Delta = \xi\Delta + \eta N$ for each $\Delta, N \in \mathbb{R}_{\mathcal{I}}$ and each $\xi, \eta \in \mathbb{R}$ with $\xi\eta \ge 0$.

It is worthy of notice that the distributive law, in the interval analysis context, is not always valid. To illustrate this fact, we provide an example here.

$$\Delta = [2, 6], N = [-3, -1], H = [1, 2],$$
$$\Delta \cdot (N + H) = [-12, 6],$$

whereas

$$\Delta \cdot N + \Delta \cdot H = [-16, 10].$$

However, the distributive law can be caught hold of in certain cases. For instance, if we consider adding the condition $N \cdot H > 0$, then we find out that

$$\Delta \cdot (N + H) = \Delta \cdot N + \Delta \cdot H.$$

The inclusion relation, with respect to the real-valued intervals Δ and N, is represented as below.

$$\Delta \subseteq N \Leftrightarrow \underline{N} \leq \underline{\Delta} \text{ and } \overline{\Delta} \leq \overline{N}.$$

Concerning to the arithmetic operations, as well as the inclusion relations, one has the successive characteristic, which is referred to as the inclusion isotone regarding interval operations.

The addition, subtraction, multiplication or division is represented by the symbol $\odot$. It is assumed that Δ, N, H along with K are all the real-valued intervals such that

$$\Delta \subseteq N \text{ and } H \subseteq K.$$

In view of this, the subsequent inclusion relation holds true:

$$\Delta \odot H \subseteq N \odot K.$$

Let us retrospect the following important conception, which was considered by Moore et al. in [35].

Definition 1.2: [35] It is assumed that $H(\zeta) = [\underline{H}(\zeta), \overline{H}(\zeta)]$, $\zeta \in I^\circ$, in which I° is the interior of $I \subseteq \mathbb{R}$. We call that the interval-valued mapping $H(\zeta)$ is Lebesgue integrable, if the mappings $\underline{H}(\zeta)$ and $\overline{H}(\zeta)$ are both measurable and Lebesgue integrable defined on the real-valued interval I°. Moreover, the interval-valued integral $\int_c^z H(\zeta)\mathrm{d}\zeta$ is given by the coming expression:

$$\int_c^z H(\zeta)\mathrm{d}\zeta = \left[\int_c^z \underline{H}(\zeta)\mathrm{d}\zeta, \int_c^z \overline{H}(\zeta)\mathrm{d}\zeta\right].$$

Breckner took into account the conception of interval-valued convexity in the article [36].

Definition 1.3: [36] It is assumed that the mapping $H : I \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued mapping, in which $I \subseteq \mathbb{R}$ is a convex set. We say that H is a convex interval-valued mapping when, and only when

$$H(\xi\chi + (1-\xi)\eta) \supseteq \xi H(\chi) + (1-\xi)H(\eta)$$

holds true for all $\chi, \eta \in I$ and $\xi \in [0, 1]$.

In the article [37], Sadowska generalized the Hermite–Hadamard's integral inequalities to the form of interval integrals as below:

$$H\left(\frac{c+z}{2}\right) \supseteq \frac{1}{z-c}\int_c^z H(\zeta)\mathrm{d}\zeta \supseteq \frac{H(c)+H(z)}{2}, \quad (3)$$

in which $H : I \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{I}}$ is a convex interval-valued mapping on the real-valued interval I, for any $c, z \in I$ with $c < z$.

Next, let us call to mind the coming left- and right-sided of the interval-valued fractional integrals including with exponential kernels, which were taken into account by Zhou et al. in the article [38].

Definition 1.4: [38] Given that $H : [c, z] \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued mapping satisfying that $H(\zeta) = \left[\underline{H}(\zeta), \overline{H}(\zeta)\right]$, in which $\underline{H}(\zeta), \overline{H}(\zeta)$ are both Riemannian integrable defined on the real-valued interval $[c, z]$. The interval-valued left- and right-sided fractional integrals, including with exponential-type kernels with regard to the mapping H, correspondingly, are given in the coming expressions:

$$\mathcal{I}_{c^+}^{\alpha} H(\zeta) = \frac{1}{\alpha}\int_c^{\zeta} \exp\left(-\frac{1-\alpha}{\alpha}(\zeta-\theta)\right)H(\theta)\mathrm{d}\theta, \ \zeta > c,$$

and

$$\mathcal{I}_{z^-}^{\alpha} H(\zeta) = \frac{1}{\alpha}\int_{\zeta}^{z} \exp\left(-\frac{1-\alpha}{\alpha}(\theta-\zeta)\right)H(\theta)\mathrm{d}\theta, \ \zeta < z,$$

along with $\alpha \in (0, 1)$. Apparently, we can readily observe that

$$\mathcal{I}_{c^+}^{\alpha} H(\zeta) = \left[\mathcal{I}_{c^+}^{\alpha} \underline{H}(\zeta), \mathcal{I}_{c^+}^{\alpha} \overline{H}(\zeta)\right],$$

and

$$\mathcal{I}_{z^-}^{\alpha} H(\zeta) = \left[\mathcal{I}_{z^-}^{\alpha} \underline{H}(\zeta), \mathcal{I}_{z^-}^{\alpha} \overline{H}(\zeta)\right].$$

Interval analysis theories, as a special case of set-valued analysis, take advantage of the methods of mathematical analysis and the ideas of general topology to study the characteristics of set inclusions. It is undeniable that the interval-valued functions have a great influence on both engineering mathematical and applied analysis. A primordial application of the interval analysis is to consider the error bounds of the numerical solution of finite state machines. As one of the approaches to solve interval uncertainty, interval analysis has been used in mathematical and computer models in the past fifty years. It is to be mentioned here that a good few key application problems in connection with interval analysis, especially in computer graphics [39], automatic error analysis [40] as well as neural network output [41], have been reported in the literature. On the other hand, the study of quite a few integral inequalities, as a natural integration with the interval-valued functions, has been implemented by some authors, containing with Kara et al. [42], Li et al. [43], Nikodem [44] and Nwaeze [45] in the study of Hermite–Hadamard-type set inclusions considering with the interval-valued two-dimensional co-ordinated convexity, the interval-valued generalized p-convexity, the set-valued (k, h)-convexity as well as the interval-valued m-polynomial harmonically convexity, correspondingly. And Khan et al. [46] further considered the fractional Hermite–Hadamard-type integral inequalities in accordance with the interval-valued harmonically convexity. by virtue of interval-valued h-convexity and harmonically h-convexity, Kalsoom et al. [47] gave the fractional Hermite–Hadamard–Fejér-type set inclusions. In the meantime, Kara et al. [48] presented the fractional Hermite–Hadamard–Mercer-type set inclusions for interval-valued functions. Under the setting of interval-valued multiplicative calculus, Ali et al. [49] established the Hermite–Hadamard-type multiplicative integral inclusion relations. In addition, the authors of the paper [50], by virtue of gH-differentiable interval-valued mappings, investigated some Ostrowski-type integral inequalities. And the authors of the paper [51] also discussed the Wirtinger-type integral inequalities involving with interval-valued functions and so on. For more findings in accordance with interval-valued mappings, we invite the minded readers to consult [52], [53], [54], [55], [56] and the references therein.

Enlightened by the above-mentioned outcomes, in particular, these developed in [57] and [58], the present paper, by virtue of the interval-valued fractional integrals, is devoted to establishing set inclusion relations, which are related to the reverse Minkowsky's, as well as the reverse Hölder's integral inequalities. To achieve this objective, by taking advantage of the left interval-valued fractional integral operators along with exponential kernels only, we develop the reverse Minkowsky's, as well as reverse Hölder's fractional integral inclusions via interval-valued mappings. Certain other fractional integral inclusions, in association with the Hermite–Hadamard-type inequalities, are discussed as well.

II. Main results

In view of the interval-valued fractional integral operators having exponential kernels, our first outcome is the successive reverse Minkowski's integral inclusion relation.

Theorem 2.1: If $G, H : [c, z] \to \mathbb{R}^+_{\mathcal{I}}$ both belong to two interval-valued mappings along with $c < z$ satisfying that $G(\tau) = \left[\underline{G}(\tau), \overline{G}(\tau)\right]$, $H(\tau) = \left[\underline{H}(\tau), \overline{H}(\tau)\right]$, $\mathcal{I}^{\alpha}_{c^+} G^p(\tau) < \infty$ and $\mathcal{I}^{\alpha}_{c^+} H^p(\tau) < \infty$, then we have the coming inclusion relation

$$\begin{aligned}&\left[\mathcal{I}^{\alpha}_{c^+} G^p(z)\right]^{\frac{1}{p}} + \left[\mathcal{I}^{\alpha}_{c^+} H^p(z)\right]^{\frac{1}{p}}\\ &\subseteq \left[\frac{1+r(R+2)}{(1+r)(1+R)}, \frac{1+R(r+2)}{(1+r)(1+R)}\right]\\ &\quad\times \left[\mathcal{I}^{\alpha}_{c^+} (G+H)^p(z)\right]^{\frac{1}{p}},\end{aligned} \tag{4}$$

where $0 < r \le \frac{\underline{G}(\tau)}{\underline{H}(\tau)} \le R$, $0 < r \le \frac{\overline{G}(\tau)}{\overline{H}(\tau)} \le R$, $\tau \in [c, z]$ and $0 < \alpha < 1$ with $p \ge 1$.

Proof. Since $\frac{\overline{G}(\tau)}{\overline{H}(\tau)} \le R$, $\tau \in [c, z]$ with $c < z$, we find out that

$$(R+1)^p \overline{G}^p(\tau) \le R^p (\overline{G} + \overline{H})^p(\tau). \tag{5}$$

Multiplying both sides of (5) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation with regard to τ from c to z, it yields that

$$\begin{aligned}&\frac{(R+1)^p}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\overline{G}^p(\tau)\mathrm{d}\tau\\ &\le \frac{R^p}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)(\overline{G}+\overline{H})^p(\tau)\mathrm{d}\tau,\end{aligned}$$

which is equivalent to

$$\left[\mathcal{I}^{\alpha}_{c^+} \overline{G}^p(z)\right]^{\frac{1}{p}} \le \frac{R}{R+1}\left[\mathcal{I}^{\alpha}_{c^+} (\overline{G}+\overline{H})^p(z)\right]^{\frac{1}{p}}. \tag{6}$$

Taking advantage of the condition $\frac{\underline{G}(\tau)}{\underline{H}(\tau)} \ge r$, we have that

$$(r+1)^p \underline{G}^p(\tau) \ge r^p (\underline{G} + \underline{H})^p(\tau). \tag{7}$$

Multiplying both sides of (7) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation with regard to τ on the interval $[c, z]$, we deduce that

$$\begin{aligned}&\frac{(r+1)^p}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\underline{G}^p(\tau)\mathrm{d}\tau\\ &\ge \frac{r^p}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)(\underline{G}+\underline{H})^p(\tau)\mathrm{d}\tau,\end{aligned}$$

which is equivalent to

$$\left[\mathcal{I}^{\alpha}_{c^+} \underline{G}^p(z)\right]^{\frac{1}{p}} \ge \frac{r}{r+1}\left[\mathcal{I}^{\alpha}_{c^+} (\underline{G}+\underline{H})^p(z)\right]^{\frac{1}{p}}. \tag{8}$$

Owing to $[\mathcal{I}^{\alpha}_{c^+} G^p(z)]^{\frac{1}{p}} = \left[[\mathcal{I}^{\alpha}_{c^+} \underline{G}^p(z)]^{\frac{1}{p}}, [\mathcal{I}^{\alpha}_{c^+} \overline{G}^p(z)]^{\frac{1}{p}}\right]$, by means of (6) and (8), we know that

$$\begin{aligned}[\mathcal{I}^{\alpha}_{c^+} G^p(z)]^{\frac{1}{p}} &= \left[[\mathcal{I}^{\alpha}_{c^+} \underline{G}^p(z)]^{\frac{1}{p}}, [\mathcal{I}^{\alpha}_{c^+} \overline{G}^p(z)]^{\frac{1}{p}}\right]\\ &\subseteq \Bigg[\frac{r}{r+1}\left[\mathcal{I}^{\alpha}_{c^+} (\underline{G}+\underline{H})^p(z)\right]^{\frac{1}{p}},\\ &\qquad \frac{R}{R+1}\left[\mathcal{I}^{\alpha}_{c^+} (\overline{G}+\overline{H})^p(z)\right]^{\frac{1}{p}}\Bigg]\\ &= \left[\frac{r}{r+1}, \frac{R}{R+1}\right]\left[\mathcal{I}^{\alpha}_{c^+} (G+H)^p(z)\right]^{\frac{1}{p}}.\end{aligned} \tag{9}$$

Besides, for the another function $[\mathcal{I}^{\alpha}_{c^+} H^p(z)]^{\frac{1}{p}}$, in view of the condition $\frac{\overline{G}(\tau)}{\overline{H}(\tau)} \ge r$, we acquire that

$$\left(1+\frac{1}{r}\right)^p \overline{H}^p(\tau) \le \frac{1}{r^p}(\overline{G}+\overline{H})^p(\tau). \tag{10}$$

Multiplying both sides of (10) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation regarding τ on the interval $[c, z]$, we gain that

$$\begin{aligned}&\left(1+\frac{1}{r}\right)^p\frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\overline{H}^p(\tau)\mathrm{d}\tau\\ &\le \frac{1}{r^p}\frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)(\overline{G}+\overline{H})^p(\tau)\mathrm{d}\tau.\end{aligned}$$

It yields that

$$\left[\mathcal{I}^{\alpha}_{c^+} \overline{H}^p(z)\right]^{\frac{1}{p}} \le \frac{1}{r+1}\left[\mathcal{I}^{\alpha}_{c^+} (\overline{G}+\overline{H})^p(z)\right]^{\frac{1}{p}}. \tag{11}$$

Analogously, utilizing the condition $\frac{\underline{G}(\tau)}{\underline{H}(\tau)} \le R$, it follows that

$$\left(1+\frac{1}{R}\right)^p \underline{H}^p(\tau) \ge \frac{1}{R^p}(\underline{G}+\underline{H})^p(\tau). \tag{12}$$

Multiplying both sides of (12) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation pertaining to τ on the interval $[c, z]$, we get that

$$\begin{aligned}&\left(1+\frac{1}{R}\right)^p\frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\underline{H}^p(\tau)\mathrm{d}\tau\\ &\ge \frac{1}{R^p}\frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)(\underline{G}+\underline{H})^p(\tau)\mathrm{d}\tau.\end{aligned}$$

This implies that we have

$$\left[\mathcal{I}^{\alpha}_{c^+} \underline{H}^p(z)\right]^{\frac{1}{p}} \ge \frac{1}{R+1}\left[\mathcal{I}^{\alpha}_{c^+} (\underline{G}+\underline{H})^p(z)\right]^{\frac{1}{p}}. \tag{13}$$

In terms of (11) and (13), we obtain that

$$\begin{aligned}[\mathcal{I}^{\alpha}_{c^+} H^p(z)]^{\frac{1}{p}} &= \left[[\mathcal{I}^{\alpha}_{c^+} \underline{H}^p(z)]^{\frac{1}{p}}, [\mathcal{I}^{\alpha}_{c^+} \overline{H}^p(z)]^{\frac{1}{p}}\right]\\ &\subseteq \Bigg[\frac{1}{R+1}\left[\mathcal{I}^{\alpha}_{c^+} (\underline{G}+\underline{H})^p(z)\right]^{\frac{1}{p}},\\ &\qquad \frac{1}{r+1}\left[\mathcal{I}^{\alpha}_{c^+} (\overline{G}+\overline{H})^p(z)\right]^{\frac{1}{p}}\Bigg]\\ &= \left[\frac{1}{R+1}, \frac{1}{r+1}\right]\left[\mathcal{I}^{\alpha}_{c^+} (G+H)^p(z)\right]^{\frac{1}{p}}.\end{aligned} \tag{14}$$

Summing (9) and (14), we receive the desired inclusion relation in (4). This concludes the proof.

Remark 2.1: According to Theorem 2.1, we have the succedent findings.

(1) If we consider taking $\alpha \to 1$, then we have that

$$\left(\int_c^z G^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}} + \left(\int_c^z H^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}} \subseteq \left[\frac{1+r(R+2)}{(1+r)(1+R)}, \frac{1+R(r+2)}{(1+r)(1+R)}\right] \times \left(\int_c^z (G+H)^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}}.$$

(2) If the mappings fulfill the conditions $\underline{G} = \overline{G}$ and $\underline{H} = \overline{H}$, then one deduces that

$$\begin{aligned}\frac{1+r(R+2)}{(1+r)(1+R)}&\left[\mathcal{I}_{c^+}^{\alpha}(G+H)^p(z)\right]^{\frac{1}{p}} \\ &\leq \left[\mathcal{I}_{c^+}^{\alpha} G^p(z)\right]^{\frac{1}{p}} + \left[\mathcal{I}_{c^+}^{\alpha} H^p(z)\right]^{\frac{1}{p}} \\ &\leq \frac{1+R(r+2)}{(1+r)(1+R)}\left[\mathcal{I}_{c^+}^{\alpha}(G+H)^p(z)\right]^{\frac{1}{p}}.\end{aligned}$$

(3) If we consider letting $\alpha \to 1$, $\underline{G} = \overline{G}$ and $\underline{H} = \overline{H}$, then we derive that

$$\begin{aligned}\frac{1+r(R+2)}{(1+r)(1+R)}&\left(\int_c^z (G+H)^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}} \\ &\leq \left(\int_c^z G^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}} + \left(\int_c^z H^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}} \\ &\leq \frac{1+R(r+2)}{(1+r)(1+R)}\left(\int_c^z (G+H)^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}},\end{aligned}$$

which is a generalized form of the reverse Minkowski's integral inequality in [59, Theorem 1.2].

We now establish the subsequent two inclusion relations, which are related to the reverse Hölder's fractional integral inequalities.

Theorem 2.2: If $G, H : [c,z] \to \mathbb{R}_{\mathcal{I}}^+$ are both two interval-valued mappings with $c < z$ satisfying that $G(\tau) = \left[\underline{G}(\tau), \overline{G}(\tau)\right]$, $H(\tau) = \left[\underline{H}(\tau), \overline{H}(\tau)\right]$, $\mathcal{I}_{c^+}^{\alpha} G(\tau) < \infty$ together with $\mathcal{I}_{c^+}^{\alpha} H(\tau) < \infty$, then we have the successive set inclusion relation

$$\begin{aligned}&\left[\mathcal{I}_{c^+}^{\alpha} G(z)\right]^{\frac{1}{p}}\left[\mathcal{I}_{c^+}^{\alpha} H(z)\right]^{\frac{1}{q}} \\ &\subseteq \left[\left(\frac{r}{R}\right)^{\frac{1}{pq}}, \left(\frac{R}{r}\right)^{\frac{1}{pq}}\right]\mathcal{I}_{c^+}^{\alpha}\left[(G(z))^{\frac{1}{p}}(H(z))^{\frac{1}{q}}\right],\end{aligned} \tag{15}$$

where $0 < r \leq \frac{\underline{G}(\tau)}{\underline{H}(\tau)} \leq R$, $0 < r \leq \frac{\overline{G}(\tau)}{\overline{H}(\tau)} \leq R$, $\tau \in [c,z]$ and $\alpha \in (0,1)$ with $\frac{1}{p} + \frac{1}{q} = 1,\ p > 1$.

Proof. On account of $\frac{\overline{G}(\tau)}{\overline{H}(\tau)} \leq R$, $\tau \in [c,z]$ with $c < z$, we find out that

$$R^{\frac{1}{q}}\left[\overline{H}(\tau)\right]^{\frac{1}{q}} \geq \left[\overline{G}(\tau)\right]^{\frac{1}{q}},$$

and

$$R^{\frac{1}{q}}\left[\overline{H}(\tau)\right]^{\frac{1}{q}}\left[\overline{G}(\tau)\right]^{\frac{1}{p}} \geq \left[\overline{G}(\tau)\right]^{\frac{1}{q}}\left[\overline{G}(\tau)\right]^{\frac{1}{p}} = \overline{G}(\tau). \tag{16}$$

Multiplying both sides of (16) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation with regard to τ on the interval $[c,z]$, we get that

$$\begin{aligned}R^{\frac{1}{q}}\frac{1}{\alpha}\int_c^z &\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\left[\overline{H}(\tau)\right]^{\frac{1}{q}}\left[\overline{G}(\tau)\right]^{\frac{1}{p}}\mathrm{d}\tau \\ &\geq \frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\overline{G}(\tau)\mathrm{d}\tau,\end{aligned}$$

which is equivalent to

$$\left[\mathcal{I}_{c^+}^{\alpha}\overline{G}(z)\right]^{\frac{1}{p}} \leq R^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\overline{H}(z)\right]^{\frac{1}{q}}\left[\overline{G}(z)\right]^{\frac{1}{p}}\right)^{\frac{1}{p}}. \tag{17}$$

In the light of the condition $\frac{\underline{G}(\tau)}{\underline{H}(\tau)} \geq r$, we gain that

$$\left[\underline{G}(\tau)\right]^{\frac{1}{q}} \geq r^{\frac{1}{q}}\left[\underline{H}(\tau)\right]^{\frac{1}{q}},$$

and

$$r^{\frac{1}{q}}\left[\underline{H}(\tau)\right]^{\frac{1}{q}}\left[\underline{G}(\tau)\right]^{\frac{1}{p}} \leq \left[\underline{G}(\tau)\right]^{\frac{1}{q}}\left[\underline{G}(\tau)\right]^{\frac{1}{p}} = \underline{G}(\tau). \tag{18}$$

Multiplying both sides of (18) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation with regard to τ on the interval $[c,z]$, we derive that

$$\begin{aligned}r^{\frac{1}{q}}\frac{1}{\alpha}\int_c^z &\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\left[\underline{H}(\tau)\right]^{\frac{1}{q}}\left[\underline{G}(\tau)\right]^{\frac{1}{p}}\mathrm{d}\tau \\ &\leq \frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\underline{G}(\tau)\mathrm{d}\tau.\end{aligned}$$

It means that we have

$$\left[\mathcal{I}_{c^+}^{\alpha}\underline{G}(z)\right]^{\frac{1}{p}} \geq r^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\underline{H}(z)\right]^{\frac{1}{q}}\left[\underline{G}(z)\right]^{\frac{1}{p}}\right)^{\frac{1}{p}}. \tag{19}$$

Due to $\left[\mathcal{I}_{c^+}^{\alpha} G(z)\right]^{\frac{1}{p}} = \left[\left[\mathcal{I}_{c^+}^{\alpha}\underline{G}(z)\right]^{\frac{1}{p}}, \left[\mathcal{I}_{c^+}^{\alpha}\overline{G}(z)\right]^{\frac{1}{p}}\right]$, by means of (17) and (19), we deduce that

$$\begin{aligned}\left[\mathcal{I}_{c^+}^{\alpha} G(z)\right]^{\frac{1}{p}} &= \left[\left[\mathcal{I}_{c^+}^{\alpha}\underline{G}(z)\right]^{\frac{1}{p}}, \left[\mathcal{I}_{c^+}^{\alpha}\overline{G}(z)\right]^{\frac{1}{p}}\right] \\ &\subseteq \left[r^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\underline{G}(z)\right]^{\frac{1}{p}}\left[\underline{H}(z)\right]^{\frac{1}{q}}\right)^{\frac{1}{p}},\right. \\ &\qquad \left. R^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\overline{G}(z)\right]^{\frac{1}{p}}\left[\overline{H}(z)\right]^{\frac{1}{q}}\right)^{\frac{1}{p}}\right] \\ &= \left[r^{\frac{1}{pq}}, R^{\frac{1}{pq}}\right]\mathcal{I}_{c^+}^{\alpha}\left[(G(z))^{\frac{1}{p}}(H(z))^{\frac{1}{q}}\right]^{\frac{1}{p}}.\end{aligned} \tag{20}$$

For the another mapping $\left[\mathcal{I}_{c^+}^{\alpha} H(z)\right]^{\frac{1}{q}}$, on the basis of the condition $\frac{\overline{G}(\tau)}{\overline{H}(\tau)} \geq r$, we have that

$$\frac{1}{r^p}\left[\overline{G}(\tau)\right]^{\frac{1}{p}} \geq \left[\overline{H}(\tau)\right]^{\frac{1}{p}},$$

and arrive at

$$\frac{1}{r^p}\left[\overline{H}(\tau)\right]^{\frac{1}{q}}\left[\overline{G}(\tau)\right]^{\frac{1}{p}} \geq \left[\overline{H}(\tau)\right]^{\frac{1}{q}}\left[\overline{H}(\tau)\right]^{\frac{1}{p}} = \overline{H}(\tau). \tag{21}$$

Multiplying both sides of (21) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)$, and integrating the resulting relation pertaining to τ on the interval $[c,z]$, we achieve that

$$\begin{aligned}\frac{1}{r^p}\frac{1}{\alpha}\int_c^z &\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\left[\overline{H}(\tau)\right]^{\frac{1}{q}}\left[\overline{G}(\tau)\right]^{\frac{1}{p}}\mathrm{d}\tau \\ &\geq \frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\overline{H}(\tau)\mathrm{d}\tau.\end{aligned}$$

It yields that

$$\left[\mathcal{I}_{c^+}^{\alpha}\overline{H}(z)\right]^{\frac{1}{q}} \leq \frac{1}{r^{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\overline{H}(z)\right]^{\frac{1}{q}}\left[\overline{G}(z)\right]^{\frac{1}{p}}\right)^{\frac{1}{q}}. \tag{22}$$

Making use of the condition $\frac{\underline{G}(\tau)}{\underline{H}(\tau)} \leq R$, we acquire that

$$\frac{1}{R^p}\left[\underline{G}(\tau)\right]^{\frac{1}{p}} \leq \left[\underline{H}(\tau)\right]^{\frac{1}{p}},$$

and

$$\frac{1}{R^p}\left[\underline{H}(\tau)\right]^{\frac{1}{q}}\left[\underline{G}(\tau)\right]^{\frac{1}{p}} \leq \left[\underline{H}(\tau)\right]^{\frac{1}{q}}\left[\underline{H}(\tau)\right]^{\frac{1}{p}} = \underline{H}(\tau). \tag{23}$$

Multiplying both sides of (23) by $\frac{1}{\alpha}e^{-\frac{1-\alpha}{\alpha}(z-\tau)}$, and integrating the resulting relation with regard to τ on the interval $[c,z]$, we obtain that

$$\begin{aligned}\frac{1}{R^p}\frac{1}{\alpha}\int_c^z & e^{-\frac{1-\alpha}{\alpha}(z-\tau)}\left[\underline{H}(\tau)\right]^{\frac{1}{q}}\left[\underline{G}(\tau)\right]^{\frac{1}{p}}\mathrm{d}\tau \\ &\leq \frac{1}{\alpha}\int_c^z e^{-\frac{1-\alpha}{\alpha}(z-\tau)}\underline{H}(\tau)\mathrm{d}\tau.\end{aligned}$$

It follows that

$$\left[\mathcal{I}_{c^+}^{\alpha}\underline{H}(z)\right]^{\frac{1}{q}} \geq \frac{1}{R^{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\underline{H}(z)\right]^{\frac{1}{q}}\left[\underline{G}(z)\right]^{\frac{1}{p}}\right)^{\frac{1}{q}}. \tag{24}$$

According to (22) and (24), we deduce that

$$\begin{aligned}\left[\mathcal{I}_{c^+}^{\alpha}H(z)\right]^{\frac{1}{q}} &= \left[\left[\mathcal{I}_{c^+}^{\alpha}\underline{H}(z)\right]^{\frac{1}{q}},\left[\mathcal{I}_{c^+}^{\alpha}\overline{H}(z)\right]^{\frac{1}{q}}\right] \\ &\subseteq \Big[\frac{1}{R^{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\underline{H}(z)\right]^{\frac{1}{q}}\left[\underline{G}(z)\right]^{\frac{1}{p}}\right)^{\frac{1}{q}}, \\ &\qquad \frac{1}{r^{pq}}\mathcal{I}_{c^+}^{\alpha}\left(\left[\overline{H}(z)\right]^{\frac{1}{q}}\left[\overline{G}(z)\right]^{\frac{1}{p}}\right)^{\frac{1}{q}}\Big] \\ &= \left[\frac{1}{R^{pq}},\frac{1}{r^{pq}}\right]\mathcal{I}_{c^+}^{\alpha}\left[(G(z))^{\frac{1}{p}}(H(z))^{\frac{1}{p}}\right]^{\frac{1}{q}}.\end{aligned} \tag{25}$$

Multiplying the relations (20) and (25), we receive the required set inclusion relation in (15). The proof is now completed.

Remark 2.2: In the light of Theorem 2.2, we have the succeeding findings.

(1) If we consider taking $\alpha \to 1$, then we have that

$$\begin{aligned}&\left(\int_c^z G(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_c^z H(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}} \\ &\subseteq \left[\left(\frac{r}{R}\right)^{\frac{1}{pq}},\left(\frac{R}{r}\right)^{\frac{1}{pq}}\right]\int_c^z (G(\tau))^{\frac{1}{p}}(H(\tau))^{\frac{1}{q}}\mathrm{d}\tau.\end{aligned}$$

(2) If the mappings meet the conditions $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then one gains that

$$\begin{aligned}&\left(\frac{r}{R}\right)^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left[(G(z))^{\frac{1}{p}}(H(z))^{\frac{1}{q}}\right] \\ &\leq \left[\mathcal{I}_{c^+}^{\alpha}G(z)\right]^{\frac{1}{p}}\left[\mathcal{I}_{c^+}^{\alpha}H(z)\right]^{\frac{1}{q}} \\ &\leq \left(\frac{R}{r}\right)^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left[(G(z))^{\frac{1}{p}}(H(z))^{\frac{1}{q}}\right].\end{aligned}$$

(3) If we consider letting $\alpha \to 1$, $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then we get that

$$\begin{aligned}&\left(\frac{r}{R}\right)^{\frac{1}{pq}}\int_c^z (G(\tau))^{\frac{1}{p}}(H(\tau))^{\frac{1}{q}}\mathrm{d}\tau \\ &\leq \left(\int_c^z G(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_c^z H(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}} \\ &\leq \left(\frac{R}{r}\right)^{\frac{1}{pq}}\int_c^z (G(\tau))^{\frac{1}{p}}(H(\tau))^{\frac{1}{q}}\mathrm{d}\tau.\end{aligned}$$

Theorem 2.3: If $G,H:[c,z]\to\mathbb{R}_{\mathcal{I}}^{+}$ are both two interval-valued mappings along with $c<z$ such that $G(\tau)=\left[\underline{G}(\tau),\overline{G}(\tau)\right]$, $H(\tau)=\left[\underline{H}(\tau),\overline{H}(\tau)\right]$, $\mathcal{I}_{c^+}^{\alpha}G^p(\tau)<\infty$ and $\mathcal{I}_{c^+}^{\alpha}H^q(\tau)<\infty$, then we have the succeeding set inclusion relation

$$\begin{aligned}&\left[\mathcal{I}_{c^+}^{\alpha}G^p(z)\right]^{\frac{1}{p}}\left[\mathcal{I}_{c^+}^{\alpha}H^q(z)\right]^{\frac{1}{q}} \\ &\subseteq \left[\left(\frac{r}{R}\right)^{\frac{1}{pq}},\left(\frac{R}{r}\right)^{\frac{1}{pq}}\right]\mathcal{I}_{c^+}^{\alpha}\left[G(z)H(z)\right],\end{aligned}$$

where $0<r\leq\frac{\underline{G}^p(\tau)}{\underline{H}^q(\tau)}\leq R$, $0<r\leq\frac{\overline{G}^p(\tau)}{\overline{H}^q(\tau)}\leq R$, $\tau\in[c,z]$ and $\alpha\in(0,1)$ with $\frac{1}{p}+\frac{1}{q}=1,\ p>1$.

Proof. With respect to the proof of Theorem 2.3, we omit the details involved, because it is essentially in common with that of Theorem 2.2.

Remark 2.3: In accordance with Theorem 2.3, we derive the subsequent findings.

(1) If we consider letting $\alpha\to 1$, then we have that

$$\begin{aligned}&\left(\int_c^z G^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_c^z H^q(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}} \\ &\subseteq \left[\left(\frac{r}{R}\right)^{\frac{1}{pq}},\left(\frac{R}{r}\right)^{\frac{1}{pq}}\right]\int_c^z G(\tau)H(\tau)\mathrm{d}\tau.\end{aligned}$$

(2) If the mappings fulfill the conditions $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then we deduce that

$$\begin{aligned}\left(\frac{r}{R}\right)^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left[G(z)H(z)\right] &\leq \left[\mathcal{I}_{c^+}^{\alpha}G^p(z)\right]^{\frac{1}{p}}\left[\mathcal{I}_{c^+}^{\alpha}H^q(z)\right]^{\frac{1}{q}} \\ &\leq \left(\frac{R}{r}\right)^{\frac{1}{pq}}\mathcal{I}_{c^+}^{\alpha}\left[G(z)H(z)\right].\end{aligned}$$

(3) If we consider taking $\alpha\to 1$, $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then we derive that

$$\begin{aligned}&\left(\frac{r}{R}\right)^{\frac{1}{pq}}\int_c^z G(\tau)H(\tau)\mathrm{d}\tau \\ &\leq \left(\int_c^z G^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_c^z H^q(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}} \\ &\leq \left(\frac{R}{r}\right)^{\frac{1}{pq}}\int_c^z G(\tau)H(\tau)\mathrm{d}\tau,\end{aligned}$$

which is a generalized form pertaining to the reverse Hölder's integral inequality.

We continue to discuss other fractional integral inclusions in relation to the Hermite–Hadamard-type inequalities.

Theorem 2.4: If $G,H:[c,z]\to\mathbb{R}_{\mathcal{I}}^{+}$ are both two convex interval-valued mapping together with $c<z$ such that $G(\tau)=\left[\underline{G}(\tau),\overline{G}(\tau)\right]$ and $H(\tau)=\left[\underline{H}(\tau),\overline{H}(\tau)\right]$, then we have the coming set inclusion relation

$$\begin{aligned}&\frac{\alpha}{z-c}\left[\mathcal{I}_{c^+}^{\alpha}G(z)H(z)\right] \\ &\supseteq \frac{2-(\rho^2+2\rho+2)\exp(-\rho)}{\rho^3}G(c)H(c) \\ &\quad + \frac{\rho^2-2\rho+2-2\exp(-\rho)}{\rho^3}G(z)H(z) \\ &\quad + \frac{\rho-2+(\rho+2)\exp(-\rho)}{\rho^3}\Big[G(c)H(z)+G(z)H(c)\Big],\end{aligned} \tag{26}$$

where

$$\rho=\frac{1-\alpha}{\alpha}(z-c),\ (0<\alpha<1).$$

Proof. As the mappings G and H are both interval-valued convex on $[c, z]$, it yields that

$$G\big(\tau c+(1-\tau)z\big) \supseteq \tau G(c)+(1-\tau)G(z), \tag{27}$$

and

$$H\big(\tau c+(1-\tau)z\big) \supseteq \tau H(c)+(1-\tau)H(z). \tag{28}$$

If we multiply both sides of (27) with corresponding parts of (28), and notice that all these terms are nonnegative, then we obtain that

$$\begin{aligned} &G\big(\tau c+(1-\tau)z\big)H\big(\tau c+(1-\tau)z\big) \\ &\quad\supseteq \tau^2 G(c)H(c)+(1-\tau)^2 G(z)H(z) \\ &\qquad+(1-\tau)\tau\left[G(c)H(z)+G(z)H(c)\right]. \end{aligned} \tag{29}$$

Multiplying both sides (29) by $\exp(-\rho\tau)$, and integrating the resulting relation with regard to τ on $[0, 1]$, we gain that

$$\begin{aligned} &\int_0^1 \exp(-\rho\tau)G\big(\tau c+(1-\tau)z\big)H\big(\tau c+(1-\tau)z\big)\mathrm{d}\tau \\ &\quad\supseteq \int_0^1 \exp(-\rho\tau)\tau^2 G(c)H(c)\mathrm{d}\tau \\ &\qquad+\int_0^1 \exp(-\rho\tau)(1-\tau)^2 G(z)H(z)\mathrm{d}\tau \\ &\qquad+\int_0^1 \exp(-\rho\tau)(1-\tau)\tau\left[G(c)H(z)+G(z)H(c)\right]\mathrm{d}\tau. \end{aligned} \tag{30}$$

By virtue of Definition 1.4, we achieve that

$$\begin{aligned} &\int_0^1 \exp(-\rho\tau)G\big(\tau c+(1-\tau)z\big)H\big(\tau c+(1-\tau)z\big)\mathrm{d}\tau \\ &\quad=\frac{\alpha}{z-c}\mathcal{I}^{\alpha}_{c^+}G(z)H(z). \end{aligned} \tag{31}$$

Direct computation yields that

$$\int_0^1 \exp(-\rho\tau)\tau^2\mathrm{d}\tau=\frac{2-(\rho^2+2\rho+2)\exp(-\rho)}{\rho^3}, \tag{32}$$

$$\int_0^1 \exp(-\rho\tau)(1-\tau)^2\mathrm{d}\tau=\frac{\rho^2-2\rho+2-2\exp(-\rho)}{\rho^3}, \tag{33}$$

and

$$\int_0^1 \exp(-\rho\tau)(\tau(1-\tau))\mathrm{d}\tau=\frac{\rho-2+(\rho+2)\exp(-\rho)}{\rho^3}. \tag{34}$$

Employing (31)–(34) in (30), we receive the desired set inclusion relation. This concludes the proof.

Remark 2.4: In the light of Theorem 2.4, we deduce the successive findings.

(1) If we consider letting $\alpha \to 1$, i.e. $\rho \to 0$, then we have

$$\lim_{\alpha\to 1}\frac{2-(\rho^2+2\rho+2)\exp(-\rho)}{\rho^3}=\frac{1}{3},$$

$$\lim_{\alpha\to 1}\frac{\rho^2-2\rho+2-2\exp(-\rho)}{\rho^3}=\frac{1}{3},$$

and

$$\lim_{\alpha\to 1}\frac{\rho-2+(\rho+2)\exp(-\rho)}{\rho^3}=\frac{1}{6}.$$

As a result, Theorem 2.4 is transformed to

$$\begin{aligned} \frac{1}{z-c}\int_c^z G(\tau)H(\tau)\mathrm{d}\tau \supseteq &\frac{1}{3}\left[G(c)H(c)+G(z)H(z)\right] \\ &+\frac{1}{6}[G(c)H(z)+G(z)H(c)], \end{aligned}$$

which is Theorem 4.5 provided by Zhao et al. in [60] with $h(t) = t$.

(2) If the mappings fulfill the conditions $\underline{G} = \overline{G}$ and $\underline{H} = \overline{H}$, then we have the subsequent inequality

$$\begin{aligned} &\frac{\alpha}{z-c}\left[\mathcal{I}^{\alpha}_{c^+}G(z)H(z)\right] \\ &\quad\leq \frac{2-(\rho^2+2\rho+2)\exp(-\rho)}{\rho^3}G(c)H(c) \\ &\quad+\frac{\rho^2-2\rho+2-2\exp(-\rho)}{\rho^3}G(z)H(z) \\ &\quad+\frac{\rho-2+(\rho+2)\exp(-\rho)}{\rho^3}[G(c)H(z)+G(z)H(c)]. \end{aligned}$$

(3) If we consider taking $\alpha \to 1$, $\underline{G} = \overline{G}$ and $\underline{H} = \overline{H}$, then we have the successive inequality

$$\begin{aligned} \frac{1}{z-c}\int_c^z G(\tau)H(\tau)\mathrm{d}\tau \leq &\frac{1}{3}\left[G(c)H(c)+G(z)H(z)\right] \\ &+\frac{1}{6}[G(c)H(z)+G(z)H(c)], \end{aligned}$$

which is the first outcome within Theorem 1 proposed by Pachpatte in [61].

We further have the succeeding set inclusion relation.

Theorem 2.5: It is assumed that $G, H : [c, z] \to \mathbb{R}^+_{\mathcal{I}}$ are both two interval-valued mappings together with $c < z$ satisfying that $G(\tau) = \big[\underline{G}(\tau), \overline{G}(\tau)\big]$ and $H(\tau) = \big[\underline{H}(\tau), \overline{H}(\tau)\big]$. If G^p, H^q are both two convex interval-valued mappings defined on the real-valued interval $[c, z]$, and $G^p(c+z-\tau) = G^p(\tau)$ with $H^q(c+z-\tau) = H^q(\tau)$, then we have the coming set inclusion relation

$$\begin{aligned} &\Big[2^{2-p-q}((\underline{G}(c)+\underline{G}(z))^p(\underline{H}(c)+\underline{H}(z))^q, \\ &\quad(\overline{G}(c)+\overline{G}(z))^p(\overline{H}(c)+\overline{H}(z))^q\Big] \\ &\quad\times\left[\mathcal{I}^{\alpha}_{c^+}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right)\right]^2 \\ &\subseteq 4\mathcal{I}^{\alpha}_{c^+}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right] \\ &\quad\times\mathcal{I}^{\alpha}_{c^+}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right], \end{aligned} \tag{35}$$

in which $0 < p < 1$, $0 < q < 1$ and $0 < \alpha < 1$.

To prove Theorem 2.5 above, we propose the succeeding lemma.

Lemma 2.1: It is assumed that $w : [c, z] \to \mathbb{R}^+_{\mathcal{I}}$ is a convex interval-valued mapping. Then we have the coming set inclusion relation

$$2w\left(\frac{c+z}{2}\right) \supseteq w(c+z-x)+w(x) \supseteq w(c)+w(z). \tag{36}$$

Proof. As the mapping w is interval-valued convex on $[c, z]$, we find out that

$$\begin{aligned} w\left(\frac{c+z}{2}\right)&=w\left(\frac{c+z-x+x}{2}\right) \\ &\supseteq \frac{1}{2}w(c+z-x)+\frac{1}{2}w(x). \end{aligned} \tag{37}$$

If we change the substitution $x = \lambda c + (1-\lambda)z, \lambda \in (0,1)$, and take advantage of the interval-valued convexity of the mapping w, then we gain that

$$\begin{aligned}\frac{1}{2}w(c+z-\lambda c-(1-\lambda)z)+\frac{1}{2}w(\lambda c+(1-\lambda)z)\\ =\frac{1}{2}w((1-\lambda)c+\lambda z)+\frac{1}{2}w(\lambda c+(1-\lambda)z)\\ \supseteq \frac{1}{2}\left[w(c)+w(z)\right].\end{aligned} \tag{38}$$

Making use of the relations (37) and (38), we receive the desired set inclusion relation (36). This concludes the proof.

Proof of Theorem 2.5. As the mappings G^p and H^q are both interval-valued convex on $[c,z]$, and by Lemma 2.1, we find out that

$$2G^p\left(\frac{c+z}{2}\right)\supseteq G^p(c+z-\tau)+G^p(\tau)\supseteq G^p(c)+G^p(z), \tag{39}$$

and

$$2H^q\left(\frac{c+z}{2}\right)\supseteq H^q(c+z-\tau)+H^q(\tau)\supseteq H^q(c)+H^q(z). \tag{40}$$

Multiplying both sides of (39) and (40) by $\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}z\right)$, and integrating the resulting inclusion relation regarding τ over $[c,z]$, together with certain integral calculations, we infer that

$$\begin{aligned}&\frac{2G^p\left(\frac{c+z}{2}\right)}{\alpha}\int_c^z \varepsilon(\tau)\mathrm{d}\tau\\ &\supseteq \frac{1}{\alpha}\int_c^z \varepsilon(\tau)G^p(c+z-\tau)\mathrm{d}\tau+\frac{1}{\alpha}\int_c^z \varepsilon(\tau)G^p(\tau)\mathrm{d}\tau\\ &\supseteq \frac{G^p(c)+G^p(z)}{\alpha}\int_c^z \varepsilon(\tau)\mathrm{d}\tau,\end{aligned} \tag{41}$$

and

$$\begin{aligned}&\frac{2H^q\left(\frac{c+z}{2}\right)}{\alpha}\int_c^z \varepsilon(\tau)\mathrm{d}\tau\\ &\supseteq \frac{1}{\alpha}\int_c^z \varepsilon(\tau)H^q(c+z-\tau)\mathrm{d}\tau+\frac{1}{\alpha}\int_c^z \varepsilon(\tau)H^q(\tau)\mathrm{d}\tau\\ &\supseteq \frac{H^q(c)+H^q(z)}{\alpha}\int_c^z \varepsilon(\tau)\mathrm{d}\tau,\end{aligned} \tag{42}$$

where,

$$\begin{aligned}\varepsilon(\tau):&=\exp\left(-\frac{1-\alpha}{\alpha}z\right)\\ &=\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\exp\left(-\frac{1-\alpha}{\alpha}\tau\right).\end{aligned}$$

Taking into consideration that $G^p(c+z-\tau)=G^p(\tau)$ and the inclusion relations (41), we acquire that

$$\begin{aligned}&2G^p\left(\frac{c+z}{2}\right)\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right]\\ &\supseteq 2\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right]\\ &\supseteq (G^p(c)+G^p(z))\left[\mathcal{I}_{c^+}^{\alpha}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right)\right].\end{aligned} \tag{43}$$

Analogously,

$$\begin{aligned}&2H^q\left(\frac{c+z}{2}\right)\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right]\\ &\supseteq 2\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right]\\ &\supseteq (H^q(c)+H^q(z))\left[\mathcal{I}_{c^+}^{\alpha}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right)\right].\end{aligned} \tag{44}$$

The inclusion relations (43) and (44) imply that we have

$$\begin{aligned}&4\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right]\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right]\\ &\supseteq (G^p(c)+G^p(z))(H^q(c)+H^q(z))\\ &\times\left[\mathcal{I}_{c^+}^{\alpha}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right)\right]^2.\end{aligned} \tag{45}$$

It can be easily seen that

$$\begin{aligned}&(G^p(c)+G^p(z))(H^q(c)+H^q(z))\\ &=\Big[(\underline{G}^p(c)+\underline{G}^p(z))(\underline{H}^q(c)+\underline{H}^q(z)),\\ &\quad(\overline{G}^p(c)+\overline{G}^p(z))(\overline{H}^q(c)+\overline{H}^q(z))\Big].\end{aligned}$$

In view of the inequalities $(x+y)^r \le (x^r+y^r) \le 2^{1-r}(x+y)^r$, $0<r<1$ and $x,y>0$, we derive that

$$\begin{aligned}&(\underline{G}^p(c)+\underline{G}^p(z))(\underline{H}^q(c)+\underline{H}^q(z))\\ &\le 2^{2-p-q}(\underline{G}(c)+\underline{G}(z))^p(\underline{H}(c)+\underline{H}(z))^q,\end{aligned}$$

and

$$\begin{aligned}&(\overline{G}^p(c)+\overline{G}^p(z))(\overline{H}^q(c)+\overline{H}^q(z))\\ &\ge (\overline{G}(c)+\overline{G}(z))^p(\overline{H}(c)+\overline{H}(z))^q.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}&\Big[2^{2-p-q}(\underline{G}(c)+\underline{G}(z))^p(\underline{H}(c)+\underline{H}(z))^q,\\ &(\overline{G}(c)+\overline{G}(z))^p(\overline{H}(c)+\overline{H}(z))^q\Big]\\ &\subseteq (G^p(c)+G^p(z))(H^q(c)+H^q(z)).\end{aligned} \tag{46}$$

Employing the set inclusion relations (45) and (46), it yields the desired set inclusion relation. This completes the proof.

Remark 2.5: In the light of Theorem 2.5, we have the coming findings.

(1) If we consider taking $\alpha \to 1$, then we deduce the subsequent inclusion relation

$$\begin{aligned}&(z-c)^2\Big[2^{2-p-q}((\underline{G}(c)+\underline{G}(z))^p(\underline{H}(c)+\underline{H}(z))^q,\\ &(\overline{G}(c)+\overline{G}(z))^p(\overline{H}(c)+\overline{H}(z))^q\Big]\\ &\subseteq 4\int_c^z G^p(\tau)\mathrm{d}\tau\int_c^z H^q(\tau)\mathrm{d}\tau.\end{aligned}$$

(2) If the mappings fulfill the conditions $\underline{G}=\overline{G}$ and $\underline{H}=$

$\overline{H}$, then we have the successive inequalities

$$\begin{aligned}&(G(c)+G(z))^p(H(c)+H(z))^q\left[\mathcal{I}_{c^+}^{\alpha}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right)\right]^2\\&\quad\le 4\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right]\\&\qquad\times\mathcal{I}_{c^+}^{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right]\\&\quad\le 2^{2-p-q}(G(c)+G(z))^p(H(c)+H(z))^q\\&\qquad\times\left[\mathcal{I}_{c^+}^{\alpha}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)\right)\right]^2.\end{aligned}$$

(3) If we consider taking $\alpha\to 1$, $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then we achieve the coming inequalities

$$\begin{aligned}&(z-c)^2(G(c)+G(z))^p(H(c)+H(z))^q\\&\quad\le 4\int_c^z G^p(\tau)\mathrm{d}\tau\int_c^z H^q(\tau)\mathrm{d}\tau\\&\quad\le 2^{2-p-q}(z-c)^2(G(c)+G(z))^p(H(c)+H(z))^q.\end{aligned}$$

Finally, on the interval $[0,z]$, we identify the relationship between two interval-valued mappings $G(\tau)$, $H(\tau)$ with two other convex interval-valued mappings $G^p(\tau)$, $H^q(\tau)$, correspondingly. We have the succeeding outcome.

Theorem 2.6: It is assumed that $G,H:[0,z]\to\mathbb{R}_{\mathcal{I}}^{+}$ are both two interval-valued mappings together with $z>0$ satisfying that $G(\tau)=\left[\underline{G}(\tau),\overline{G}(\tau)\right]$ and $H(\tau)=\left[\underline{H}(\tau),\overline{H}(\tau)\right]$. If $G^p(\tau),H^q(\tau)$ are both two interval-valued convex mappings defined on the real-valued interval $[0,z]$, then we have the subsequent set inclusion relation

$$\begin{aligned}&\Big[2^{2-p-q}((\underline{G}(0)+\underline{G}(z))^p(\underline{H}(0)+\underline{H}(z))^q,\\&(\overline{G}(0)+\overline{G}(z))^p(\overline{H}(0)+\overline{H}(z))^q\Big]\left[\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)\right)\right]^2\\&\quad\subseteq\left[\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right)\right.\\&\qquad\left.+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)G^p(z)\right)\right]\\&\qquad\times\left[\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right)\right.\\&\qquad\left.+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)H^q(z)\right)\right],\end{aligned}\tag{47}$$

where $p\in(0,1)$, $q\in(0,1)$ and $\alpha,\beta\in(0,1)$.

Proof. Owing to the interval-valued positive convexity of the mappings G^p and H^q defined on the real-valued interval $[0,z]$, and by Lemma 2.1, we know that

$$2G^p\left(\frac{z}{2}\right)\supseteq G^p(z-\tau)+G^p(\tau)\supseteq G^p(0)+G^p(z),\tag{48}$$

and

$$2H^q\left(\frac{z}{2}\right)\supseteq H^q(z-\tau)+H^q(\tau)\supseteq H^q(0)+H^q(z).\tag{49}$$

Multiplying both sides of (48) and (49) by $\kappa(\tau):=\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}(z-\tau)\right)\exp\left(-\frac{1-\beta}{\beta}\tau\right)$, and integrating the resulting inclusion relation pertaining to τ on $[0,z]$, one gains that

$$\begin{aligned}&2G^p\left(\frac{z}{2}\right)\int_0^z\kappa(\tau)\mathrm{d}\tau\\&\quad\supseteq\int_0^z\kappa(\tau)G^p(z-\tau)\mathrm{d}\tau+\int_0^z\kappa(\tau)G^p(\tau)\mathrm{d}\tau\\&\quad\supseteq[G^p(0)+G^p(z)]\int_0^z\kappa(\tau)\mathrm{d}\tau,\end{aligned}\tag{50}$$

and

$$\begin{aligned}&2H^q\left(\frac{z}{2}\right)\int_0^z\kappa(\tau)\mathrm{d}\tau\\&\quad\supseteq\int_0^z\kappa(\tau)H^q(z-\tau)\mathrm{d}\tau+\int_0^z\kappa(\tau)H^q(\tau)\mathrm{d}\tau\\&\quad\supseteq[H^q(0)+H^q(z)]\int_0^z\kappa(\tau)\mathrm{d}\tau.\end{aligned}\tag{51}$$

Taking advantage of the variable substitution, we acquire that

$$\begin{aligned}&\frac{\beta}{\alpha\beta}\int_0^z\kappa(\tau)G^p(z-\tau)\mathrm{d}\tau\\&\quad=\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right),\end{aligned}\tag{52}$$

and

$$\begin{aligned}&\frac{\beta}{\alpha\beta}\int_0^z\kappa(\tau)H^q(z-\tau)\mathrm{d}\tau\\&\quad=\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right).\end{aligned}\tag{53}$$

By means of (50) and (52), we derive that

$$\begin{aligned}&2G^p\left(\frac{z}{2}\right)\mathcal{I}^{\alpha}\left[\exp\left(-\frac{1-\beta}{\beta}z\right)\right]\\&\quad\supseteq\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^p(z)\right)\\&\qquad+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)G^p(z)\right)\\&\quad\supseteq(G^p(0)+G^p(z))\left[\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)\right)\right],\end{aligned}\tag{54}$$

and by virtue of (51) and (53), we deduce that

$$\begin{aligned}&2H^q\left(\frac{z}{2}\right)\mathcal{I}^{\alpha}\left[\exp\left(-\frac{1-\beta}{\beta}z\right)\right]\\&\quad\supseteq\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^q(z)\right)\\&\qquad+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)H^q(z)\right)\\&\quad\supseteq(H^q(0)+H^q(z))\left[\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)\right)\right].\end{aligned}\tag{55}$$

The inclusion relations (54) and (55) imply that we obtain

$$\begin{aligned}&\left[\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^{p}(z)\right)\right.\\&\quad\left.+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)G^{p}(z)\right)\right]\\&\quad\times\left[\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^{q}(z)\right)\right.\\&\qquad\left.+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)H^{q}(z)\right)\right]\\&\quad\supseteq(G^{p}(0)+G^{p}(z))(H^{q}(0)+H^{q}(z))\\&\qquad\times\left[\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)\right)\right]^{2}.\end{aligned}\tag{56}$$

Combining the relations (46) with (56), it deduces the desired set inclusion relation. This completes the proof.

Remark 2.6: In the light of Theorem 2.6, we deduce the coming findings.

(1) If we consider taking $\alpha\to 1$ and $\beta\to 1$, then we acquire the subsequent inclusion relation

$$\begin{aligned}z^{2}&\Big[2^{2-p-q}((\underline{G}(0)+\underline{G}(z))^{p}(\underline{H}(0)+\underline{H}(z))^{q},\\&\quad(\overline{G}(0)+\overline{G}(z))^{p}(\overline{H}(0)+\overline{H}(z))^{q}\Big]\\&\subseteq 4\int_{0}^{z}G^{p}(\tau)\mathrm{d}\tau\int_{0}^{z}H^{q}(\tau)\mathrm{d}\tau.\end{aligned}$$

(2) If the mappings fulfill the conditions $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then we have the successive inequalities

$$\begin{aligned}&(G(0)+G(z))^{p}(H(0)+H(z))^{q}\\&\quad\times\left[\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)\right)\right]^{2}\\&\quad\leq\left[\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)G^{p}(z)\right)\right.\\&\qquad\left.+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)G^{p}(z)\right)\right]\\&\quad\times\left[\frac{\beta}{\alpha}\mathcal{I}^{\beta}\left(\exp\left(-\frac{1-\alpha}{\alpha}z\right)H^{q}(z)\right)\right.\\&\qquad\left.+\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)H^{q}(z)\right)\right]\\&\quad\leq 2^{2-p-q}(G(0)+G(z))^{p}(H(0)+H(z))^{q}\\&\qquad\times\left[\mathcal{I}^{\alpha}\left(\exp\left(-\frac{1-\beta}{\beta}z\right)\right)\right]^{2}.\end{aligned}$$

(3) If we consider taking $\alpha\to 1$, $\beta\to 1$, $\underline{G}=\overline{G}$ and $\underline{H}=\overline{H}$, then we achieve the coming inequalities

$$\begin{aligned}&z^{2}(G(0)+G(z))^{p}(H(0)+H(z))^{q}\\&\quad\leq 4\int_{0}^{z}G^{p}(\tau)\mathrm{d}\tau\int_{0}^{z}H^{q}(\tau)\mathrm{d}\tau.\\&\quad\leq 2^{2-p-q}z^{2}(G(0)+G(z))^{p}(H(0)+H(z))^{q}.\end{aligned}$$

Remark 2.7: If the mappings $G^{p}(\tau)$ and $H^{q}(\tau)$ don't satisfy the symmetry conditions, and we consider taking $c=0, z>0$ within Theorem 2.5, then the set inclusion relation given by Theorem 2.6 is a generalized form of the proposed outcome in Theorem 2.5 with $\alpha=\beta$.

Remark 2.8: The set inclusion relations pertaining to the interval-valued mappings, involving with the right-sided fractional integral operators having exponential kernels , can be also established on the basis of the analogous approaches adopted in Theorems 2.1– 2.6. Here, we omit its details.

III. Examples

Example 3.1: Consider the interval-valued mappings $G,H:[1,3]\to\mathbb{R}_{\mathcal{I}}^{+}$, given by $G(x)=[x^{2},xe^{x}]$, as well as $H(x)=[2x,e^{x}]$, respectively. If we attempt to take $c=1, z=3$, $p=2$, $r=\frac{1}{2}$, $R=3$ and $\alpha=\frac{1}{2}$, then all requirements raised in Theorem 2.1 are satisfied.

With respect to the left-hand side part of (4), we have that

$$\begin{aligned}&\left[\mathcal{I}_{c^{+}}^{\alpha}G^{p}(z)\right]^{\frac{1}{p}}+\left[\mathcal{I}_{c^{+}}^{\alpha}H^{p}(z)\right]^{\frac{1}{p}}\\&=\left(2\int_{1}^{3}e^{-(3-x)}[x^{2},xe^{x}]^{2}\mathrm{d}x\right)^{\frac{1}{2}}\\&\quad+\left(2\int_{1}^{3}e^{-(3-x)}[2x,e^{x}]^{2}\mathrm{d}x\right)^{\frac{1}{2}}\\&=\left[66-18e^{-2},\frac{130e^{6}-10}{27}\right]^{\frac{1}{2}}\\&\quad+\left[40-8e^{-2},\frac{2e^{6}-2}{3}\right]^{\frac{1}{2}}\\&\approx[14.2111,60.4483].\end{aligned}$$

For the right-hand side part of (4), we find out that

$$\left[\frac{1+r(R+2)}{(1+r)(1+R)},\frac{1+R(r+2)}{(1+r)(1+R)}\right]=\left[\frac{7}{12},\frac{17}{12}\right],$$

and

$$\begin{aligned}&\left[\frac{1+r(R+2)}{(1+r)(1+R)},\frac{1+R(r+2)}{(1+r)(1+R)}\right]\left[\mathcal{I}_{c^{+}}^{\alpha}(G+H)^{p}(z)\right]^{\frac{1}{p}}\\&=\left[\frac{7}{12},\frac{17}{12}\right]\left(2\int_{1}^{3}e^{-(3-x)}\left([x^{2},xe^{x}]+[2x,e^{x}]\right)^{2}\mathrm{d}x\right)^{\frac{1}{2}}\\&=\left[\frac{7}{12},\frac{17}{12}\right]\left[2\int_{1}^{3}e^{-(3-x)}\left(x^{2}+2x\right)^{2}\mathrm{d}x,\right.\\&\qquad\left.2\int_{1}^{3}e^{-(3-x)}\left(xe^{x}+e^{x}\right)^{2}\mathrm{d}x\right]^{\frac{1}{2}}\\&=\left[\frac{7}{12},\frac{17}{12}\right]\left[202-10e^{-2},\frac{244e^{6}-52}{27}\right]^{\frac{1}{2}}\\&\approx[8.2629,85.5164].\end{aligned}$$

Evidently,

$$[14.2111,60.4483]\subseteq[8.2629,85.5164],$$

which exhibits the correctness of the set inclusion relation asserted within Theorem 2.1.

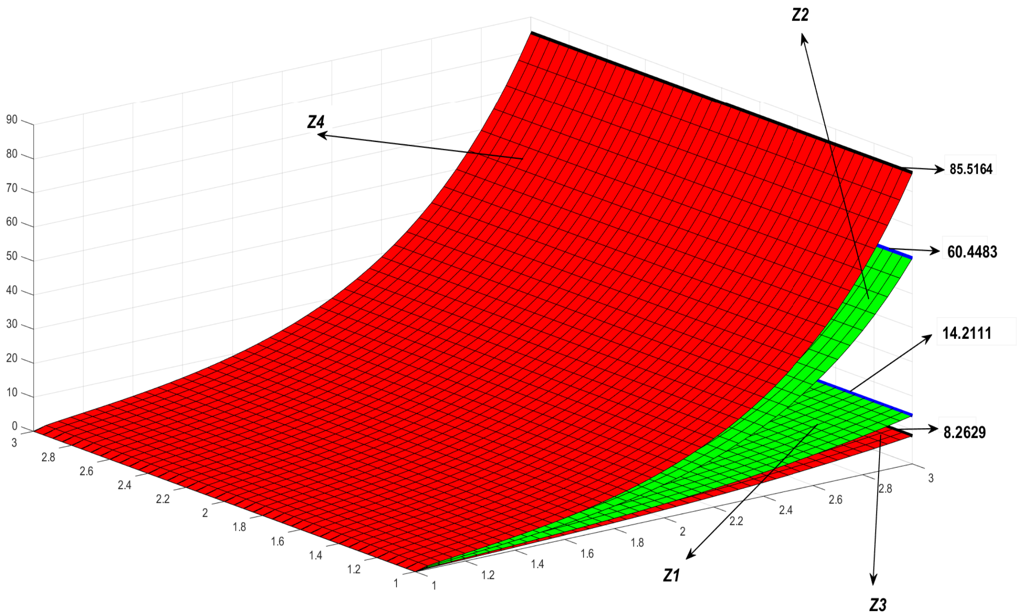

Fig. 1: Graphical representation for Example 3.1

For the sake of brevity, we give the subsequent notations with regard to $Z1$, $Z2$, $Z3$ and $Z4$.

$$
\begin{aligned}
Z1 = & \left(2\int_1^3 e^{-(3-x)}\left(x^2\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
& + \left(2\int_1^3 e^{-(3-y)}\left(2y\right)^2 \mathrm{d}y\right)^{\frac{1}{2}}, \\
Z2 = & \left(2\int_1^3 e^{-(3-x)}(xe^x)^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
& + \left(2\int_1^3 e^{-(3-y)}(e^y)^2 \mathrm{d}y\right)^{\frac{1}{2}}, \\
Z3 = & \frac{7}{12}\left(2\int_1^3 e^{-(3-x)}\left(x^2+2x\right)^2 \mathrm{d}x\right)^{\frac{1}{4}} \\
& \times \left(2\int_1^3 e^{-(3-y)}\left(y^2+2y\right)^2 \mathrm{d}x\right)^{\frac{1}{4}},
\end{aligned}
$$

and

$$
\begin{aligned}
Z4 = & \frac{17}{12}\left(2\int_1^3 e^{-(3-x)}\left(xe^x+e^x\right)^2 \mathrm{d}x\right)^{\frac{1}{4}} \\
& \times \left(2\int_1^3 e^{-(3-y)}\left(ye^y+e^y\right)^2 \mathrm{d}y\right)^{\frac{1}{4}}.
\end{aligned}
$$

Let us discretize the rectangular region $[1,3]\times[1,3]$, $x\in[1,3]$ and $y\in[1,3]$. For example, we consider taking the suitable step $\triangle x = 0.05$ and $\triangle y = 0.05$. Correspondingly, we achieve the discretized integral values with the corresponding parts of $Z1$, $Z2$, $Z3$ and $Z4$ above. Combining with the programming idea, we smooth the obtained discretized numerical integral values. In view of this, the three-dimensional figure is created. From the visual perspective of graphics, The created Fig. 1 vividly describes the set inclusion relation exhibited in Example 3.1. A graphical representation of the findings established in Theorem 2.1, in the meantime, is shown in Fig. 1.

From Fig. 1 above, we discretize the parameters in Theorem 2.1, and we can see that the surfaces Z_1 and Z_2 are always contained by the surfaces Z_3 and Z_4 in the range of the definition of each parameter. The results illustrated in Fig. 1 are consistent with the theoretical results established in Theorem 2.1.

Example 3.2: Consider the interval-valued mappings $G, H : [1,2] \to \mathbb{R}^+_{\mathcal{I}}$, given by $G(x) = [(x+1)^2, e^{2x}]$ and $H(x) = [(x+1)^2, 2e^x]$, respectively. If we consider taking $c=1, z=2$, $p=3$, $q=\frac{3}{2}$, $r=1$, $R=4$, as well as $\alpha=\frac{1}{2}$, then all requirements raised in Theorem 2.2 are satisfied.

With respect to the left-hand side part of (15), we have that

$$
\begin{aligned}
& \left[\mathcal{I}^{\alpha}_{c^+}G(z)\right]^{\frac{1}{p}}\left[\mathcal{I}^{\alpha}_{c^+}H(z)\right]^{\frac{1}{q}} \\
& = \left(2\int_1^2 e^{-(2-x)}[(x+1)^2, e^{2x}]\mathrm{d}x\right)^{\frac{1}{3}} \\
& \quad \times \left(2\int_1^2 e^{-(2-x)}[(x+1)^2, 2e^{x}]\mathrm{d}x\right)^{\frac{2}{3}} \\
& = \left[10-4e^{-1}, \frac{2e^4-2e}{3}\right]^{\frac{1}{3}}\left[10-4e^{-1}, 2e^2-2\right]^{\frac{2}{3}} \\
& \approx [8.5285, 17.8079].
\end{aligned}
$$

For the right-hand side part of (15), it follows that

$$
\left[\left(\frac{r}{R}\right)^{\frac{1}{pq}}, \left(\frac{R}{r}\right)^{\frac{1}{pq}}\right] = \left[\left(\frac{1}{4}\right)^{\frac{2}{9}}, 4^{\frac{2}{9}}\right],
$$

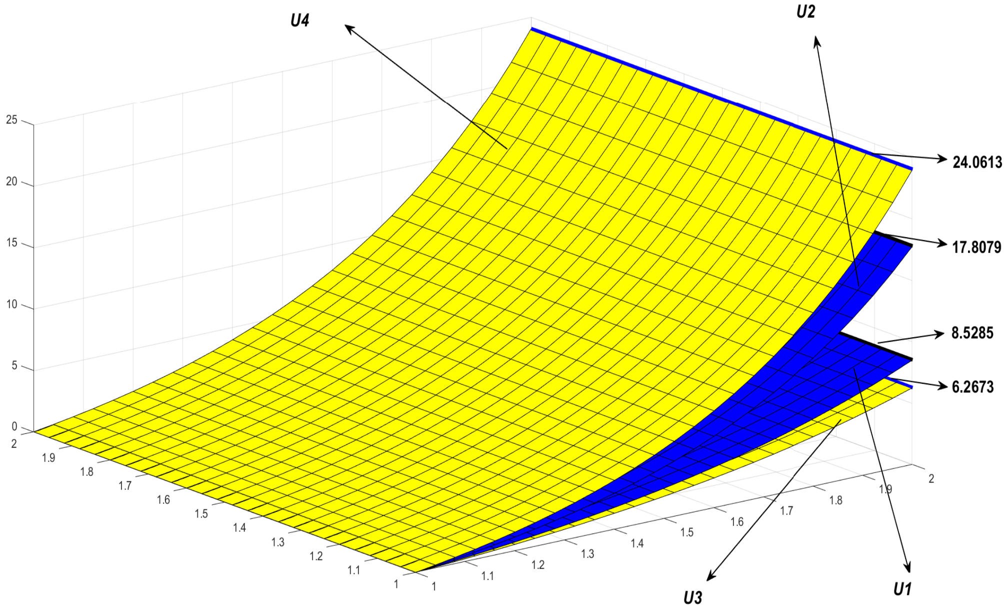

Fig. 2: Graphical representation for Example 3.2

and

$$\begin{aligned}
&\left[\left(\frac{r}{R}\right)^{\frac{1}{pq}}, \left(\frac{R}{r}\right)^{\frac{1}{pq}}\right] \mathcal{I}^{\alpha}_{a+}\left[(G(b))^{\frac{1}{p}}(H(b))^{\frac{1}{q}}\right] \\
&\quad = \left[\left(\frac{1}{4}\right)^{\frac{2}{9}}, 4^{\frac{2}{9}}\right] \\
&\quad \times 2\int_1^2 e^{-(2-x)}\left[(x+1)^2, e^{2x}\right]^{\frac{1}{3}}\left[(x+1)^2, 2e^{x}\right]^{\frac{2}{3}}\mathrm{d}x \\
&\quad = \left[\left(\frac{1}{4}\right)^{\frac{2}{9}}, 4^{\frac{2}{9}}\right] \\
&\quad \times \Big[\int_1^2 e^{-(2-x)}(x+1)^2\mathrm{d}x + \int_1^2 e^{-(2-x)}(x+1)^2\mathrm{d}x, \\
&\qquad \int_1^2 e^{-(2-x)}(2e^{2x})^{\frac{2}{3}}\mathrm{d}x + \int_1^2 e^{-(2-x)}(2e^{2x})^{\frac{2}{3}}\mathrm{d}x\Big] \\
&\quad \approx \left[\left(\frac{1}{4}\right)^{\frac{2}{9}}, 4^{\frac{2}{9}}\right]\left[10-4e^{-1}, 1.3606(e^{\frac{8}{3}}-e^{\frac{1}{3}})\right] \\
&\quad \approx [6.2673, 24.0631].
\end{aligned}$$

Evidently,

$$[8.5285, 17.8079] \subseteq [6.2673, 24.0631],$$

which illustrates the correctness of the inclusion relation described in Theorem 2.2.

For the sake of brevity, we give the successive notations with regard to $U1$, $U2$, $U3$ and $U4$.

$$\begin{aligned}
U1 &= \left(2\int_1^2 e^{-(2-x)}(x+1)^2\mathrm{d}x\right)^{\frac{1}{3}} \\
&\quad \times \left(2\int_1^2 e^{-(2-y)}(y+1)^2\mathrm{d}y\right)^{\frac{2}{3}}, \\
U2 &= \left(2\int_1^2 e^{-(2-x)}e^{2x}\mathrm{d}x\right)^{\frac{1}{3}} \\
&\quad \times \left(2\int_1^2 e^{-(2-y)}2e^{y}\mathrm{d}y\right)^{\frac{2}{3}}, \\
U3 &= \left(\frac{1}{4}\right)^{\frac{2}{9}}\Big(\int_1^2 e^{-(2-x)}(x+1)^2\mathrm{d}x \\
&\quad + \int_1^2 e^{-(2-y)}(y+1)^2\mathrm{d}y\Big),
\end{aligned}$$

and

$$\begin{aligned}
U4 &= 4^{\frac{2}{9}}\int_1^2 e^{-(2-x)}(2e^{2x})^{\frac{2}{3}}\mathrm{d}x \\
&\quad + \int_1^2 e^{-(2-y)}(2e^{2y})^{\frac{2}{3}}\mathrm{d}y.
\end{aligned}$$

Let us discretize the rectangular region $[1,2]\times[1,2]$, $x \in [1,2]$ and $y \in [1,2]$. For example, we consider choosing the appropriate step $\triangle x = 0.04$ and $\triangle y = 0.04$. Correspondingly, we acquire the discretized integral values with the corresponding parts of $U1$, $U2$, $U3$ and $U4$ above. Combining with the programming idea, we smooth the discretize numerical integral values. In view of this, the three-dimensional figure is generated. From the visual perspective of graphics, Fig. 2 vividly depicts the set inclusion relation exhibited in Example 3.2. A graphical representation of the findings presented in Theorem 2.2, in the meantime, is shown in Fig. 2.

From Fig. 2, we discretize the parameters in Theorem 2.2, and we can see that the surfaces U_2 and U_3 are always surrounded by the surfaces U_1 and U_4 in the range of the definition of each parameter. The results indicated in Fig. 2 are consistent with the theoretical results presented in Theorem 2.2.

Example 3.3: Consider the interval-valued mappings $G, H : [0,1] \to \mathbb{R}_{\mathcal{I}}^{+}$, given by $G(x) = [x^2, 2-x^2]$ and $H(x) = [x^2, x+1]$, respectively. If we consider taking $c=0, z=1$ and $\alpha \in (0,1)$, then all requirements raised in Theorem 2.4 are satisfied.

With respect to the left-hand side part of (26), we have that

$$\begin{aligned}
&\frac{\alpha}{z-c}\left[\mathcal{I}_{c^+}^{\alpha} G(z)H(z)\right] \\
&= \frac{\alpha}{z-c}\frac{1}{\alpha}\int_c^z \exp\left(-\frac{1-\alpha}{\alpha}(z-x)\right) G(x)H(x)\mathrm{d}x \\
&= \int_0^1 \exp\left(\frac{1-\alpha}{\alpha}(x-1)\right)[x^2, 2-x^2][x^2, x+1]\mathrm{d}x \\
&= \int_0^1 \exp\left(\frac{1-\alpha}{\alpha}(x-1)\right)[x^4, (2-x^2)(x+1)]\mathrm{d}x.
\end{aligned}$$

For the right-hand side part of (26), it yields that

$$\begin{aligned}
&\frac{2-(\rho^2+2\rho+2)e^{-\rho}}{\rho^3}G(c)H(c) \\
&+ \frac{\rho^2-2\rho+2-2e^{-\rho}}{\rho^3}G(z)H(z) \\
&+ \frac{\rho-2+(\rho+2)e^{-\rho}}{\rho^3}[G(c)H(z)+G(z)H(c)] \\
&= \frac{2-(\rho^2+2\rho+2)e^{-\rho}}{\rho^3}[0,2] + \frac{\rho^2-2\rho+2-2e^{-\rho}}{\rho^3}[1,2] \\
&+ \frac{\rho-2+(\rho+2)e^{-\rho}}{\rho^3}[0,5] \\
&= \Bigg[\frac{\rho^2-2\rho+2-2e^{-\rho}}{\rho^3}, 2\frac{2-(\rho^2+2\rho+2)e^{-\rho}}{\rho^3} \\
&+ 2\frac{\rho^2-2\rho+2-2e^{-\rho}}{\rho^3} + 5\frac{\rho-2+(\rho+2)e^{-\rho}}{\rho^3}\Bigg],
\end{aligned}$$

where

$$\rho = \frac{1-\alpha}{\alpha}(z-c) = \frac{1-\alpha}{\alpha}.$$

Four mappings with regard to the variable α, involving with the Upper-left-, Lower-left-, Upper-right-, as well as Lower-right-side parts given by the set inclusion relations (26), are correspondingly plotted in Fig. 3 against $\alpha \in (0,1)$. The mappings curves indicate that the outcome described in Theorem 2.4 is correct.

From Fig. 3 above, we discretize the parameters in Theorem 2.4, and we can see that the red curve keeps covering the blue curve in the range of the definition of each parameter. The results demonstrated in Fig. 3 are consistent with the theoretical results proposed in Theorem 2.4.

IV. Conclusions

Certain fractional integral inclusions, involving with the reverse Minkowsky's and Hölder's integral inequalities, are addressed in the study by taking advantage of the interval-valued fractional integrals along with exponential kernels.

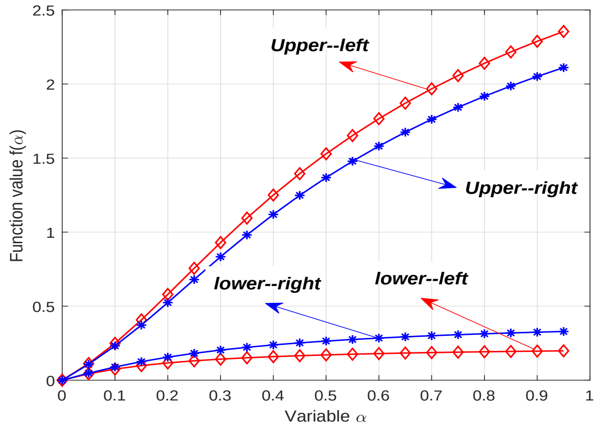

Fig. 3: Graphical representation for Example 3.3

Other set inclusion relations, in association with the fractional Hermite–Hadamard's integral inequality, are investigated as well. Moreover, some examples are proposed to bear out the correctness of the outcomes deduced in the paper. To exhibit the generalized characteristics of the presented set inclusion relations, we point out the connections between the findings obtained here with previously reported ones. We may reasonably conclude that the interval-valued analyses are widely employed in the realms of applied analysis, especially in the fields of optimality analysis, see the published articles [62], [63] and the bibliographies quoted in them. To some extent, the set inclusion relations, in the integration with the interval-valued fractional integral operators, is worth further exploration.

References

[1] S. R. Hwang, K. L. Tseng, K. C. Hsu, "New inequalities for fractional integrals and their applications," *Turkish J. Math.*, Vol. 40, pp. 471–486, 2016.

[2] T. S. Du, H. Wang, M. A. Khan, Y. Zhang, "Certain integral inequalities considering generalized m-convexity on fractal sets and their applications," *Fractals,* Vol. 27, pp. 1–17, 2019.

[3] J. G. Liao, S. H. Wu, T. S. Du, "The Sugeno integral with respect to α-preinvex functions," *Fuzzy Sets and Systems,* Vol. 379, pp. 102–114, 2020.

[4] T. S. Du, C. Y. Luo, Y. Bo, "Certain quantum estimates on the parameterized integral inequalities and their applications," *J. Math. Inequal.,* Vol. 15, pp. 201–228, 2021.

[5] S. S. Dragomir, K. Nikodem, "Jensen's and Hermite–Hadamard's type inequalities for lower and strongly convex functions on normed spaces," *Bull. Iranian Math. Soc.,* Vol. 44, pp. 1337–1349, 2018.

[6] P. Kórus, "An extension of the Hermite–Hadamard inequality for convex and s-convex functions," *Aequationes Math.*, Vol. 93, pp. 527–534, 2019.

[7] M. Kadakal, İ. İşcan, P. Agarwal, M. Jleli, "Exponential trigonometric convex function and Hermite–Hadamard type inequalities," *Math. Slovaca*, Vol. 71, pp. 43–56, 2021.

[8] M. R. Delavar, M. De La Sen, "A mapping associated to h-convex version of the Hermite–Hadamard inequality with applications," *J. Math. Inequal.*, Vol. 14, pp. 329–335, 2020.

[9] M. Andrić, J. Pečarić, "On $(h, g; m)$-convexity and the Hermite–Hadamard inequality," *J. Convex Anal.*, Vol. 29, pp. 257–268, 2022.

[10] İ. İşcan, "Weighted Hermite–Hadamard–Mercer type inequalities for convex functions," *Numer. Methods Partial Differential Eq.*, Vol. 37, pp. 118–130, 2021.

[11] M. A. Latif, "Weighted Hermite–Hadamard type inequalities for differentiable GA-convex and geometrically quasiconvex mappings," *Rocky Mountain J. Math.,* Vol. 51, pp. 1899–1908, 2021.

[12] D. Ş. Marinescu, M. Monea, "A very short proof of the Hermite–Hadamard inequalities," *Amer. Math. Monthly,* Vol. 127, pp. 850–851, 2020.

[13] M. Z. Sarikaya, M. A. Ali, "Hermite–Hadamard type inequalities and related inequalities for subadditive functions," *Miskolc Math. Notes,* Vol. 22, pp. 929–937, 2021.
[14] H. M. Srivastava, S. Mehrez, S. M. Sitnik, "Hermite–Hadamard-type integral inequalities for convex functions and their applications," *Mathematics*, Vol. 10, Art. ID 3127, 13 pages, 2022.
[15] A. Ekinci, M. E. Özdemir, "Some new integral inequalities via Riemann–Liouville integral operators," *Appl. Comput. Math.*, Vol. 18, pp. 288–295, 2019.
[16] M. A. Khan, T. Ali, S. S. Dragomir, M. Z. Sarikaya, "Hermite–Hadamard type inequalities for conformable fractional integrals," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, Vol. 112, pp. 1033–1048, 2018.
[17] C. Y. Luo, B. Yu, Y. Zhang, T. S. Du, "Certain bounds related to multi-parameterized k-fractional integral inequalities and their applications," *IEEE Access*, Vol. 7, pp.124662–124673, 2019.
[18] B. Ahmad, A. Alsaedi, M. Kirane, B. T. Torebek, "Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals," *J. Comput. Appl. Math.*, Vol. 353, pp. 120–129, 2019.
[19] L. N. Li, Y. Z. Li, Q. D. Huang, "Fractional order Chebyshev cardinal functions for solving two classes of fractional differential equations," *Engineering Letters,* Vol. 30, pp. 208–213, 2022.
[20] Q. H. Feng, "Several oscillatory results for a class of fractional differential equations with the fractional derivative in conformable sense," *IAENG International Journal of Computer Science*, Vol. 49, pp. 1041–1047, 2022.
[21] A. K. Alomari, Y. Massoun, "Numerical solution of time fractional coupled Korteweg-de Vries equation with a Caputo fractional derivative in two parameters," *IAENG International Journal of Computer Science,* Vol. 50, pp. 388–393, 2023.
[22] K. Kankhunthodl, V. Kongratana, A. Numsomran, and V. Tipsuwanporn, "Self-balancing robot control using fractional-order PID controller," *Proceedings of the International MultiConference of Engineers and Computer Scientists 2019, 13-15 March*, Hong Kong, pp. 77–82, 2019.
[23] J. Blackledge, "A new definition, a generalisation and an approximation for a fractional derivative with applications to stochastic time series modeling," *Engineering Letters,* Vol. 29, pp. 138–150, 2021.
[24] P. O. Mohammed, "Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals of a convex function with respect to a monotone function," *Math. Meth. Appl. Sci.*, Vol. 44, pp. 2314–2324, 2021.
[25] E. Set, A. O. Akdemir, M. E. Özdemir, "Simpson type integral inequalities for convex functions via Riemann–Liouville integrals," *Filomat,* Vol. 31, pp. 4415–4420, 2017.
[26] H. Chen, U. N. Katugampola, "Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for generalized fractional integrals," *J. Math. Anal. Appl.*, Vol. 446, pp. 1274–1291, 2017.
[27] B. Meftah, A. Souahi, M. Merad, "Some local fractional Maclaurin type inequalities for generalized convex functions and their applications," *Chaos Solitons and Fractals*, Vol. 162, Art. ID 112504, 2022.
[28] T. S. Du, M. U. Awan, A. Kashuri, S. S. Zhao, "Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m, h)-preinvexity," *Appl. Anal.*, Vol. 100, pp. 642–662, 2021.
[29] P. O. Mohammed, F. K. Hamasalh, "New conformable fractional integral inequalities of Hermite–Hadamard type for convex functions," *Symmetry,* Vol. 11, Art. ID 263, 13 pages, 2019.
[30] S. I. Butt, S. Yousaf, A. O. Akdemir, M. A. Dokuyucu, "New Hadamard-type integral inequalities via a general form of fractional integral operators," *Chaos, Solitons and Fractals,* Vol. 148, Art. ID 111025, 2021.
[31] M. Jleli, D. O'Regan, B. Samet, "On Hermite–Hadamard type inequalities via generalized fractional integrals," *Turkish J. Math.*, Vol. 40, pp. 1221–1230, 2016.
[32] M. Kunt, İ. İşcan, S.Turhan, D. Karapinar, "Improvement of fractional Hermite–Hadamard type inequality for convex functions," *Miskolc Math. Notes,* Vol. 19, pp. 1007–1017, 2018.
[33] B. Meftah, M. Benssaad, W. Kaidouchi, S. Ghomrani, "Conformable fractional Hermite–Hadamard type inequalities for product of two harmonic s-convex functions," *Proc. Amer. Math. Soc.*, Vol. 149, pp. 1495–1506, 2021.
[34] K. Mehrez, P. Agarwal, "New Hermite–Hadamard type integral inequalities for convex functions and their applications," *J. Comput. Appl. Math.*, Vol. 350, pp. 274–285, 2019.
[35] R. E. Moore, R. B. Kearfott, M. J. Cloud, "Introduction to interval analysis," *Society for Industrial and Applied Mathematics (SIAM), Philadelphia,* 2009.
[36] W. W. Breckner, "Continuity of generalized convex and generalized concave set-valued functions," *Rev. Anal. Numér. Théor. Approx.*, Vol. 22, pp. 39–51, 1993.
[37] E. Sadowska, "Hadamard inequality and a refinement of Jensen inequality for set-valued functions," *Result Math.*, Vol. 32, pp. 332–337, 1997.
[38] T. C. Zhou, Z. R. Yuan, T. S. Du, "On the fractional integral inclusions having exponential kernels for interval-valued convex functions," *Math. Sci.,* Vol. 17, pp. 107–120, 2023.
[39] J. M. Snyder, "Interval analysis for computer graphics," *SIGGRAPH Comput. Graph,* Vol. 26, pp. 121–130, 1992.
[40] E. J. Rothwell, M. J. Cloud, "Automatic error analysis using intervals," *IEEE Trans. Ed.,* Vol. 55, pp. 9–15, 2012.
[41] E. de Weerdt, Q. P. Chu, J. A. Mulder, "Neural network output optimization using interval analysis," *IEEE Trans. Neural Netw.*, Vol. 20, pp. 638–653, 2009.
[42] H. Kara, M. A. Ali, H. Budak, "Hermite–Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals," *Math. Meth. Appl. Sci.*, Vol. 44, pp. 104–123, 2021.
[43] Z. B. Li, Kamran, M. S. Zahoor, H. Akhtar, "Hermite–Hadamard and fractional integral inequalities for interval-valued generalized p-convex function," *J. Math.*, Vol. 2020, Art. ID 4606439, 10 pages, 2020.
[44] K. Nikodem, T. Rajba, "On Hermite–Hadamard inequalities for (k, h)-convex set-valued maps," *Math. Inequal. Appl.*, Vol. 25, pp. 467–475, 2022.
[45] E. R. Nwaeze, "Hermite–Hadamard type inclusions for m-polynomial harmonically convex interval-valued functions," *Constructive Mathematical Analysis,* Vol. 4, pp. 260–273, 2021.
[46] M. B. Khan, J. E. Macías-Díaz, S. Treanţă, M. S. Soliman, H. G. Zaini, "Hermite–Hadamard inequalities in fractional calculus for left and right harmonically convex functions via interval-valued settings," *Fractal Fract.*, Vol. 6, Art. ID 178, 15 pages, 2022.
[47] H. Kalsoom, M. A. Latif, Z. A. Khan, M. Vivas-Cortez, "Some new Hermite–Hadamard–Fejér fractional type inequalities for h-convex and harmonically h-convex interval-valued functions," *Mathematics*, Vol. 10, Art. ID 74, 22 pages, 2022.
[48] H. Kara, M. A. Ali, H. Budak, "Hermite–Hadamard–Mercer type inclusions for interval-valued functions via Riemann–Liouville fractional integrals," *Turkish J. Math.*, Vol. 46, pp. 2193–2207, 2022.
[49] M. A. Ali, Z. Y. Zhang, H. Budak, M. Z. Sarikaya, "On Hermite–Hadamard type inequalities for interval-valued multiplicative integrals," *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, Vol. 69, pp. 1428–1448, 2020.
[50] Y. Chalco-Cano, W. A. Lodwick, W. Condori-Equice, "Ostrowski type inequalities and applications in numerical integration for interval-valued functions," *Soft Computing,* Vol. 19, pp. 3293–3300, 2015.
[51] T. M. Costa, Y. Chalco–Cano, H. Román–Flores, "Wirtinger-type integral inequalities for interval-valued functions," *Fuzzy Sets and Systems*, Vol. 396, pp. 102–114, 2020.
[52] M. A. Ali, H. Budak, G. Murtaza, Y. M. Chu, "Post-quantum Hermite–Hadamard type inequalities for interval-valued convex functions," *J. Inequal. Appl.*, Vol. 2021, Art. ID 84, 18 pages, 2021.
[53] V. Lupulescu, "Fractional calculus for interval-valued functions," *Fuzzy Sets and Systems*, Vol. 265, pp. 63–85, 2015.
[54] H. Román–Flores, Y. Chalco–Cano, W. A. Lodwick, "Some integral inequalities for interval-valued functions," *Comp. Appl. Math.*, Vol.37, pp. 1306–1318, 2018.
[55] Z. H. Sha, G. J. Ye, D. F. Zhao, W. Liu, "On interval-valued $\mathbb{K}$-Riemann integral and Hermite–Hadamard type inequalities," *AIMS Math.,* Vol. 6, pp. 1276–1295, 2020.
[56] D. F. Zhao, T. Q. An, G. J. Ye, D. F. M. Torres, "On Hermite–Hadamard type inequalities for harmonical h-convex interval-valued functions," *Math. Inequal. Appl.,* Vol. 23, pp. 95–105, 2020.
[57] H. Budak, T. Tunç, M. Z. Sarikaya, "Fractional Hermite–Hadamard-type inequalities for interval-valued functions," *Proc. Amer. Math. Soc.*, Vol. 148, pp. 705–718, 2020.
[58] Z. Dahmani, "On Minkowski and Hermite–Hadamard integral inequalities via fractional integration," *Ann. Funct. Anal.*, Vol. 1, pp. 51–58, 2010.
[59] L. Bougoffa, "On Minkowski and Hardy integral inequalities," *J. Inequal. Pure and Appl. Math.*, Vol. 7, Art. ID 60, 2006.
[60] D. F. Zhao, T. Q. An, G. J. Ye, W. Liu, "New Jensen and Hermite–Hadamard type inequalities for h-convex interval-valued functions," *J. Inequal. Appl.*, Vol. 2018, Art. ID 302, 14 pages, 2018.
[61] B. G. Pachpatte, "On some inequalities for convex functions," *RGMIA Research Report Collection E,* Vol. 6, 2003, Online: https://rgmia.org/papers/v6e/convex1.pdf.
[62] D. Singh, B. A. Dar, D. S. Kim, "KKT optimality conditions in interval valued multiobjective programming with generalized differentiable functions," *European J. Oper. Res.*, Vol. 254, pp. 29–39, 2016.
[63] A. Younus, O. Nisar, "Convex optimization of interval valued functions on mixed domains," *Filomat,* Vol. 33, pp. 1715–1725, 2019.