Common Fixed Point Theorems for 2-Metric Space using Various E.A Properties

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Abstract—**In this article, some common fixed point theorems were proved in complete 2- metric space using weakly compatible mappings in which property (E.A), common property (E.A) and (E.A) like Property are involved.**

***Index Terms*—2-metric space, Fixed point, Common fixed point, Property (E.A), Common property (E.A), (E.A) like Property, Common (E.A) like Property, Compatible mapping, Weakly compatible mapping.**

I. Introduction

THE notion of 2-metric space was introduced by Gahler [1]. An extension of the metric space is the 2-metric space. Visually, the 2 metric function represents the area of a triangle and its metric function describes the length of the line segment. 2-metric space and metric space are not topologically identical. Numerous authors have proved fixed point theorems in metric spaces [3], [4], [6]. Contractive, non-expansive, and contraction mappings in 2 Metric space have also been explored by many authors [18] - [20]. Applications of fixed point theory were discussed in [2] and [5].

The idea of compatible maps was defined by Jungck [11], who showed that weakly commuting mappings are compatible but not vice versa. Many authors have proved the fixed point theorem for compatible maps in metric space. In particular, Rajesh Shrivastava proved compatible mappings in metric spaces and the common fixed point theorem [12].

Pant [17] first studied common fixed points of non-compatible mappings made in metric spaces. The property (E.A) always holds for pair of mappings since it was an extension of non-compatible mappings. This property was developed by Amir and El Moutaawaki in 2002 [9] along with several common fixed point theorems. Fixed point theorems have been shown by Vildan Ozturk [10], Bonuga Vijayabaskerreddy [7] and Rakesh Tiwari [8] using property (E.A). [13], [14], and [15] demonstrated the common fixed point theorem.

Our aim is to prove some common fixed point theorems for weakly compatible maps using property (E.A), common property (E.A) and (E.A) like Property, common (E.A) like Property.

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II. Preliminaries

Definition 1. *Let X be a non-empty set and $\sigma : X \times X \times X \longrightarrow [0, \infty)$ be a function satisfying the following condition:*

1) *for distinct point x, y with $x \neq y$ from X, there exists a point z in X such that $\sigma(x, y, z) \neq 0$.*
2) *$\sigma(x, y, z) = 0$ when two or more variable are same.*
3) *$\sigma(x, y, z) = \sigma(y, z, x) = \sigma(x, z, y) = \ldots$ for all $x, y, z \in X$.*
4) *$\sigma(x, y, w)+\sigma(x, w, z)+\sigma(w, y, z) \geq \sigma(x, y, z)$ for all $x, y, z, w \in X$.*

Then the function σ is called a 2-metric space and (X, σ) is called a 2-metric space. Visually, a 2-metric $\sigma(x, y, z)$ defines the area of a triangle, where x, y and z notate the vertices of the triangle. Throughout this paper, let (X, σ) be 2-metric space.

Definition 2. *Suppose a sequence $\{x_n\}$ in a 2-metric space (X, σ) and if $\lim\limits_{n\longrightarrow\infty} \sigma(x_n, x, z) = 0$ for all z in X, then we say the sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (i.e) $\lim\limits_{n\longrightarrow\infty} \{x_n\} = x$.*

Definition 3. *Suppose a sequence $\{x_n\}$ in a 2-metric space (X, σ) and if $\lim_{n,m} \longrightarrow \infty\, \sigma(x_n, x_m, z) = 0$ for all $z \in X$, then we say the sequence $\{x_n\}$ is said to be Cauchy sequence.*

Definition 4. *Let (X, σ) be 2-metric space is said to be complete if every Cauchy sequence in X is convergent.*

Definition 5. *Given $\mathcal{E}$ and $\mathcal{F}$ be pair of self mappings in 2-metric space (X, σ). Then the pair of mappings $\mathcal{E}$ and $\mathcal{F}$ are said to be compatible if and only if*

$$\lim_{n\to\infty} \sigma(\mathcal{E}\mathcal{F}x_n, \mathcal{F}\mathcal{E}x_n, z) = 0,$$

whenever a sequence $\{x_n\}$ in X such that

$$\lim_{n\longrightarrow\infty} \{\mathcal{E}x_n\} = \lim_{n\longrightarrow\infty} \{\mathcal{F}x_n\} = t$$

for some $t \in X$.

Definition 6. *Let $\mathcal{E}$ and $\mathcal{F}$ be two self mappings of a 2-metric space (X, σ). Then the pair of mappings $\mathcal{E}$ and $\mathcal{F}$ are said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\longrightarrow\infty} \{\mathcal{E}x_n\} = \lim_{n\longrightarrow\infty} \{\mathcal{F}x_n\} = t$$

for some $t \in X$, but $\lim_{n\to\infty} \sigma(\mathcal{E}\mathcal{F}x_n, \mathcal{F}\mathcal{E}x_n, z)$ is either non-zero or does not exist.

Definition 7. *Given $\mathcal{E}$ and $\mathcal{F}$ be pair of self mappings of a 2-metric space (X, σ). Suppose $\mathcal{E}$ and $\mathcal{F}$ commute at their coincidence points, (i.e) $\mathcal{E}\ \mathcal{F}x = \mathcal{F}\ \mathcal{E}x$ for some $x \in X$ whenever*

$\mathcal{E}x = \mathcal{F}x$, then the pair of mappings $\mathcal{E}$ and $\mathcal{F}$ are said to be weakly compatible.

Definition 8. *Let $\mathcal{E}$ and $\mathcal{F}$ be pair of self mappings of a 2-metric space (X, σ). Then the pair of mapping $\mathcal{E}$ and F are satisfying the Property (E.A) if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n \longrightarrow \infty} \mathcal{E}x_n = \lim_{n \longrightarrow \infty} \mathcal{F}x_n = t$$

for some $t \in X$.
Remark: Non-compatibility implies property (E.A).

Definition 9. *Suppose taking the pair of mappings $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{F}, \mathcal{N})$ in 2-metric space (X, σ) and if $\{l_n\}$ and $\{t_n\}$ are the sequences in X satisfying*

$$\lim_{n \longrightarrow \infty} \mathcal{E}l_n = \lim_{n \longrightarrow \infty} \mathcal{M}l_n = \lim_{n \longrightarrow \infty} \mathcal{F}t_n = \lim_{n \longrightarrow \infty} Nt_n = \xi$$

for some point $\xi \in X$, then we say that these pair of mappings hold the common property (E.A).

Definition 10. *Let $\mathcal{E}$ and $\mathcal{F}$ be pair of self mappings of a 2-metric space (X, σ). Then the pair of mapping $\mathcal{E}$ and F satisfy the (E.A) like Property if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n \longrightarrow \infty} \mathcal{E}x_n = \lim_{n \longrightarrow \infty} \mathcal{F}x_n = t$$

for some $t \in \mathcal{E}(X) \cup \mathcal{F}(X)$.

Definition 11. *Two Pairs $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{F}, \mathcal{N})$ of four self mappings of 2-metric space (X, σ) are said to satisfy common E.A like property if*

$$\lim_{n \longrightarrow \infty} \mathcal{E}l_n = \lim_{n \longrightarrow \infty} \mathcal{M}l_n = \lim_{n \longrightarrow \infty} \mathcal{F}t_n = \lim_{n \longrightarrow \infty} Nt_n = \xi$$

Whenever there exists two sequence $\{l_n\}$ and $\{t_n\}$ in X such that $\xi \in \mathcal{E}(X) \cap \mathcal{F}(X)$ or $\xi \in \mathcal{M}(X) \cap \mathcal{N}(X)$.

III. Main Results

Theorem 1. *Let (X, σ) be a complete 2-metric space and $\mathcal{E}, \mathcal{F}, \mathcal{M}$ and $\mathcal{N}$ be mappings from (X, σ) into itself that satisfy the requirement*

1) *$\mathcal{E}(X) \subset \mathcal{N}(X)$ and $\mathcal{F}(X) \subset \mathcal{M}(X)$.*
2) *one of the pair $(\mathcal{E}, \mathcal{M})$ or $(\mathcal{F}, N)$ satisfies property E.A.*
3) *the pair $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{F}, \mathcal{N})$ are weakly compatible.*
4) *if $\mathcal{M}(x), \mathcal{N}(x), \mathcal{E}(x)$ and $\mathcal{F}(x)$ is a complete subspace of X.*
5)

$$\begin{aligned}\sigma(\mathcal{E}x, \mathcal{F}y, z) \leq & \lambda_1[\sigma(\mathcal{M}x, \mathcal{N}y, z) + \sigma(\mathcal{E}x, \mathcal{M}x, z)] \\ & + \lambda_2[\sigma(\mathcal{M}x, \mathcal{N}y, z) + \sigma(\mathcal{N}y, \mathcal{F}y, z)] \\ & + \lambda_3\left[\frac{\sigma(\mathcal{M}x, \mathcal{F}y, z) + \sigma(\mathcal{E}x, \mathcal{N}y, z)}{2}\right] \\ & + \lambda_3\left[\sigma(\mathcal{M}x, \mathcal{N}y, z)\right]\end{aligned}$$

for all $x, y, z \in X$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 < \frac{1}{2}$then the mappings $\mathcal{E}, \mathcal{F}, \mathcal{M}$ and $\mathcal{N}$ have a unique common fixed point in X.

Proof: Assume that the pair $(\mathcal{F}, \mathcal{N})$ satisfies property E.A, as a result, a sequence $\{l_n\}_{n \in \mathbb{N}}$ exists in X, which satisfies $\mathcal{F}l_n \longrightarrow \eta, \mathcal{N}l_n \longrightarrow \eta$ for some $\eta \in X$ as $n \longrightarrow \infty, n \in \mathbb{N}$.

As $\mathcal{F}(X) \subset \mathcal{M}(X)$ a sequence $\{t_n\}_{n \in \mathbb{N}}$ exists in X thus $\mathcal{F}l_n = \mathcal{M}t_n$ for all $n \in \mathbb{N}$. Hence, $Mt_n \longrightarrow \eta$ as $n \longrightarrow \infty$. To prove $\mathcal{E}t_n \longrightarrow \eta$. Suppose not, then by using (5) with $x = t_n$ and $y = l_n$, we have

$$\begin{aligned}\sigma(\mathcal{E}t_n, \mathcal{F}l_n, z) \leq & \lambda_1[\sigma(\mathcal{M}t_n, \mathcal{N}l_n, z) + \sigma(\mathcal{E}t_n, \mathcal{M}t_n, z)] \\ & + \lambda_2[\sigma(\mathcal{M}t_n, \mathcal{N}l_n, z) + \sigma(\mathcal{N}l_n, \mathcal{F}l_n, z)] \\ & + \lambda_3\left[\frac{\sigma(\mathcal{M}t_n, \mathcal{F}l_n, z) + \sigma(\mathcal{E}t_n, \mathcal{N}l_n, z)}{2}\right] \\ & + \lambda_3\left[\sigma(\mathcal{M}t_n, \mathcal{N}l_n, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$,
$\sigma(\mathcal{E}t_n, \eta, z) \leq \left(\lambda_1 + \frac{\lambda_3}{2}\right) \sigma(\mathcal{E}t_n, \eta, z)$
which is a contradiction. Therefore $\mathcal{E}t_n = \mathcal{F}l_n$ for all $n \in \mathbb{N}$.
Now, using condition (4) from hypothesis, we obtain $\eta = \mathcal{M}\mu$ for some $\mu \in X$, and therefore the subsequence $\mathcal{F}l_n, \mathcal{M}t_n, \mathcal{N}l_n, \mathcal{E}t_n \longrightarrow \eta (= \mathcal{M}\mu)$ as $n \longrightarrow \infty$.
Now claim that $E\mu = \mathcal{M}\mu$. Suppose $E\mu \neq \mathcal{M}\mu$
we put $x = \mu$ and $y = l_n$ in (5), we have

$$\begin{aligned}\sigma(\mathcal{E}\mu, \mathcal{F}l_n, z) \leq & \lambda_1[\sigma(\mathcal{M}\mu, \mathcal{N}l_n, z) + \sigma(\mathcal{E}\mu, \mathcal{M}\mu, z)] \\ & + \lambda_2[\sigma(\mathcal{M}\mu, \mathcal{N}l_n, z) + \sigma(\mathcal{N}l_n, \mathcal{F}l_n, z)] \\ & + \lambda_3\left[\frac{\sigma(\mathcal{M}\mu, \mathcal{F}l_n, z) + \sigma(\mathcal{E}\mu, \mathcal{N}l_n, z)}{2}\right] \\ & + \lambda_3\left[\sigma(\mathcal{M}\mu, \mathcal{N}l_n, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$,
$\sigma(\mathcal{E}\mu, \eta, z) \leq \left(\lambda_1 + \frac{\lambda_3}{2}\right) \sigma(\mathcal{E}\mu, \eta, z)$
which is a contradiction, this implies $\mathcal{E}\mu \longrightarrow \eta$.
Thus $\mathcal{E}\mu = \mathcal{M}\mu$.
Hence, $(\mathcal{E}, \mathcal{M})$ has a coincidence point μ
It follows that either $\mathcal{E}\mathcal{M}\mu = \mathcal{M}\mathcal{E}\mu$ or $\mathcal{E}\eta = \mathcal{M}\eta$ given the pair $(\mathcal{E}, \mathcal{M})$ is weakly compatible.
since $\mathcal{E}(X) \subset \mathcal{N}(X)$, there exists a point $v \in X$ such that $\mathcal{E}\mu = \mathcal{N}v$.
Consequently, $\mathcal{E}\mu = \mathcal{N}v = \mathcal{M}\mu = \eta$.
To claim v must be a coincidence point of $(\mathcal{F}, \mathcal{N})$.
(i.e.), $Nv = Fv = \eta$. If not, then substitute $x = \mu$ and $y = v$ in (5), we obtain

$$\begin{aligned}\sigma(\mathcal{E}\mu, \mathcal{F}v, z) \leq & \lambda_1[\sigma(\mathcal{M}\mu, \mathcal{N}v, z) + \sigma(\mathcal{E}\mu, \mathcal{M}\mu, z)] \\ & + \lambda_2[\sigma(\mathcal{M}\mu, \mathcal{N}v, z) + \sigma(\mathcal{N}v, \mathcal{F}v, z)] \\ & + \lambda_3\left[\frac{\sigma(\mathcal{M}\mu, \mathcal{F}v, z) + \sigma(\mathcal{E}\mu, \mathcal{N}v, z)}{2}\right] \\ & + \lambda_3\left[\sigma(\mathcal{M}\mu, \mathcal{N}v, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$,
$\sigma(\mathcal{F}v, \eta, z) \leq \left(\lambda_2 + \frac{\lambda_3}{2}\right) \sigma(\mathcal{F}v, \eta, z)$
which is a contradiction, Therefore $\mathcal{N}v = \mathcal{F}v$.
Hence, v is a coincidence point of $(\mathcal{F}, \mathcal{N})$. Also by using condition (3) of the hypothesis, we have $\mathcal{F}\mathcal{N}v = \mathcal{N}\mathcal{F}v$ or $N\eta = F\eta$.
Thus $\mathcal{E}, \mathcal{F}, \mathcal{M}$ and $\mathcal{N}$ have a common coincidence point η.
we claim that η is a common fixed point. If not, then taking $x = \mu$ and $y = \eta$ in (5)

$$\begin{aligned}\sigma(\mathcal{E}\mu, \mathcal{F}\eta, z) \leq & \lambda_1[\sigma(\mathcal{M}\mu, \mathcal{N}\eta, z) + \sigma(\mathcal{E}\mu, \mathcal{M}\mu, z)] \\ & + \lambda_2[\sigma(\mathcal{M}\mu, \mathcal{N}\eta, z) + \sigma(\mathcal{N}\eta, \mathcal{F}\eta, z)] \\ & + \lambda_3\left[\frac{\sigma(\mathcal{M}\mu, \mathcal{F}\eta, z) + \sigma(\mathcal{E}\mu, \mathcal{N}\eta, z)}{2}\right] \\ & + \lambda_3\left[\sigma(\mathcal{M}\mu, \mathcal{N}\eta, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$,
$\sigma(\mathcal{F}\eta,\eta,z) \le (\lambda_1+\lambda_2+2\lambda_3)\,\sigma(\mathcal{F}\eta,\eta,z)$
which is a contradiction. Since $\lambda_1+\lambda_2+\lambda_3<1$
Therefore $\mathcal{F}\eta \longrightarrow \eta$.
Hence $\mathcal{F}\eta=\mathcal{N}\eta=\eta$ *and consequently* η *is common fixed of* $\mathcal{F}$ *and* $\mathcal{N}$.
Suppose $\mathcal{N}(X)$ *is a complete subspace of* X *and the pair* $(\mathcal{E},\mathcal{M})$ *satisfy property E.A will provide similar outcomes.*
To prove the uniqueness of common fixed point. If substituting $x=w$ *and* $y=\eta$ *in (5)*

$$\begin{aligned}\sigma(\mathcal{E}w,\mathcal{F}\eta,z) \le{}& \lambda_1[\sigma(\mathcal{M}w,\mathcal{N}\eta,z)+\sigma(\mathcal{E}w,\mathcal{M}w,z)]\\ &+\lambda_2[\sigma(\mathcal{M}w,\mathcal{N}\eta,z)+\sigma(\mathcal{N}\eta,\mathcal{F}\eta,z)]\\ &+\lambda_3\left[\frac{\sigma(\mathcal{M}w,\mathcal{F}\eta,z)+\sigma(\mathcal{E}w,\mathcal{N}\eta,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\mathcal{M}w,\mathcal{N}\eta,z)\right]\end{aligned}$$

$\sigma(w,\eta,z) \le \lambda_1+\lambda_2+\lambda_3\sigma(w,\eta,z)$
which is a contradiction. Therefore $w=\eta$
Thus η *is the unique common fixed point of* $\mathcal{E},\mathcal{F},\mathcal{M}$ *and* $\mathcal{N}$ *in* X. ■

Theorem 2. *Let* (X,σ) *be a complete 2-metric space and* $\mathcal{E},\mathcal{F},\mathcal{M}$ *and* $\mathcal{N}$ *be mappings from* (X,σ) *into itself that satisfy the requirement*

1) $\mathcal{F}(X)\subset\mathcal{M}(X)$ *and* $\mathcal{E}(X)\subset\mathcal{N}(X)$
2) *if* $\mathcal{M}(x),\mathcal{N}(x),\mathcal{E}(x)$ *and* $\mathcal{F}(x)$ *is a complete subspace of* X
3) *one of the pair* $(\mathcal{E},\mathcal{M})$ *or* $(\mathcal{F},N)$ *satisfies property E.A.*
4) *the pair* $(\mathcal{E},\mathcal{M})$ *and* $(\mathcal{F},\mathcal{N})$ *weakly compatible.*
5)
$$\begin{aligned}\sigma^2(\mathcal{E}x,\mathcal{F}y,z) \le{}& \sigma(\mathcal{E}x,\mathcal{F}y,z)\big[\lambda_1\sigma(\mathcal{M}x,\mathcal{N}y,z)\\ &+\lambda_2\sigma(\mathcal{E}x,\mathcal{M}x,z)+\lambda_3\sigma(\mathcal{N}y,\mathcal{F}y,z)\big]\\ &+\lambda_4[\sigma(\mathcal{M}x,\mathcal{F}y,z)\sigma(\mathcal{E}x,\mathcal{N}y,z)]\end{aligned}$$

for all $x,y,z\in X$, *where* $\lambda_1,\lambda_2,\lambda_3,\lambda_4\ge 0$, $\lambda_1+\lambda_2+\lambda_3<1$ *and* $\lambda_1+\lambda_4<1$
then the self mappings $\mathcal{E},\mathcal{F},\mathcal{M}$ *and* $\mathcal{N}$ *have a unique common fixed point in* X.

Proof: Assume that $(\mathcal{F},\mathcal{N})$ *satisfies property E.A, as a result, a sequence* $\{l_n\}_{n\in\mathbb{N}}$ *exists in* X *which satisfies*

$$\mathcal{F}l_n \longrightarrow \eta, \mathcal{N}l_n \longrightarrow \eta$$

for some $\eta\in X$ *as* $n\longrightarrow\infty, n\in\mathbb{N}$.
As $\mathcal{F}(X)\subset\mathcal{M}(X)$, *a sequence* $\{t_n\}_{n\in\mathbb{N}}$ *exists in* X *thus* $\mathcal{F}l_n=\mathcal{M}t_n$ *for all* $n\in\mathbb{N}$. *Hence,* $\mathcal{M}t_n\longrightarrow\eta$ *as* $n\longrightarrow\infty$.
Now we show that $\mathcal{E}t_n\longrightarrow\eta$. *If not, then by using (5) with* $x=t_n$ *and* $y=l_n$, *we have*

$$\begin{aligned}\sigma^2(\mathcal{E}t_n,\mathcal{F}l_n,z) \le{}&\\ &\sigma(\mathcal{E}t_n,\mathcal{F}l_n,z)[\lambda_1\sigma(\mathcal{M}t_n,\mathcal{N}l_n,z)\\ &+\lambda_2\sigma(\mathcal{E}t_n,\mathcal{M}t_n,z)+\lambda_3\sigma(\mathcal{N}l_n,\mathcal{F}l_n,z)]\\ &+\lambda_4[\sigma(\mathcal{M}t_n,\mathcal{F}l_n,z)\sigma(\mathcal{E}t_n,\mathcal{N}l_n,z)]\end{aligned}$$

Taking limits on both sides, we get

$$\begin{aligned}\sigma^2(\mathcal{E}t_n,\eta,z) \le{}& \sigma(\mathcal{E}t_n,\eta,z)[\lambda_1\sigma(\eta,\eta,z)\\ &+\lambda_2\sigma(\mathcal{E}t_n,\eta,z)+\lambda_3\sigma(\eta,\eta,z)]\\ &+\lambda_4[\sigma(\eta,\eta,z)\sigma(\mathcal{E}t_n,\eta,z)]\end{aligned}$$

$\sigma^2(\mathcal{E}t_n,\eta,z) \le \lambda_2\sigma^2(\mathcal{E}t_n,\eta,z)$ *which is a contradiction,* $\mathcal{E}t_n\longrightarrow\eta$.
Thus, $\mathcal{E}t_n=\mathcal{F}l_n$ *for all* $n\in\mathbb{N}$.
Now, using condition (2) from hypothesis, we obtain $\eta=\mathcal{M}\mu$
Therefore the subsequence
$\mathcal{F}l_n,\mathcal{M}t_n,\mathcal{N}l_n,\mathcal{E}t_n\longrightarrow\eta(=M\mu)$ *as* $n\longrightarrow\infty$.
Now claim that $\mathcal{E}\mu=\mathcal{M}\mu$.
we put $x=\mu$ *and* $y=l_n$ *in (5), we get*

$$\begin{aligned}\sigma^2(\mathcal{E}\mu,\mathcal{F}l_n,z) \le{}& \sigma(\mathcal{E}\mu,\mathcal{F}l_n,z)[\lambda_1\sigma(\mathcal{M}\mu,\mathcal{N}l_n,z)\\ &+\lambda_2\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)+\lambda_3\sigma(\mathcal{N}l_n,\mathcal{F}l_n,z)]\\ &+\lambda_4[\sigma(\mathcal{M}\mu,\mathcal{F}l_n,z)\sigma(\mathcal{E}\mu,\mathcal{N}l_n,z)]\end{aligned}$$

Taking limits on both sides, we get

$$\begin{aligned}\sigma^2(\mathcal{E}\mu,\eta,z) \le{}& \sigma(\mathcal{E}\mu,\eta,z)[\lambda_1\sigma(\eta,\eta,z)\\ &+\lambda_2\sigma(\mathcal{E}\mu,\eta,z)+\lambda_3\sigma(\eta,\eta,z)]\\ &+\lambda_4[\sigma(\eta,\eta,z)\sigma(\mathcal{E}\mu,\eta,z)]\end{aligned}$$

$\sigma^2(\mathcal{E}\mu,\eta,z) \le \lambda_2\sigma^2(\mathcal{E}\mu,\eta,z)$
which is a contradiction, $E\mu\longrightarrow\eta$. *Thus* $\mathcal{E}\mu=\mathcal{M}\mu$.
Hence, $(\mathcal{E},\mathcal{M})$ *has a coincidence point* μ
It follows that either $\mathcal{E}\mathcal{M}\mu=\mathcal{M}\mathcal{E}\mu$ *or* $\mathcal{E}\eta=\mathcal{M}\eta$ *given the pair* $(\mathcal{E},\mathcal{M})$ *is weakly compatible.*
given $\mathcal{E}(X)\subset\mathcal{N}(X)$, *there exists a point* $v\in X$ *such that* $\mathcal{E}\mu=\mathcal{N}v$.
Consequently, $\mathcal{E}\mu=\mathcal{N}v=\mathcal{M}\mu=\eta$.
To claim v *must be a coincidence point of* $(\mathcal{F},\mathcal{N})$
(i.e)., $\mathcal{N}v=\mathcal{F}v=\eta$. *If not, then taking* $x=\mu$ *and* $y=v$ *in (5), we obtain*

$$\begin{aligned}\sigma^2(\mathcal{E}\mu,\mathcal{F}v,z) \le{}& \sigma(\mathcal{E}\mu,\mathcal{F}v,z)[\lambda_1\sigma(\mathcal{M}\mu,\mathcal{N}v,z)\\ &+\lambda_2\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)+\lambda_3\sigma(Nv,\mathcal{F}v,z)]\\ &+\lambda_4[\sigma(\mathcal{M}\mu,\mathcal{F}v,z)\sigma(\mathcal{E}\mu,\mathcal{N}v,z)]\\ \sigma^2(\eta,\mathcal{F}v,z) \le{}& \sigma(\eta,\mathcal{F}v,z)[\lambda_1\sigma(\eta,\eta,z)\\ &+\lambda_2\sigma(\eta,\eta,z)+\lambda_3\sigma(\eta,\mathcal{F}v,z)]\\ &+\lambda_4[\sigma(\eta,\mathcal{F}v,z)\sigma(\eta,\eta,z)]\end{aligned}$$

$\sigma^2(\mathcal{F}v,\eta,z) \le \lambda_3\sigma^2(\mathcal{F}v,\eta,z)$
which is a contradiction, $\mathcal{F}v\longrightarrow\eta$. *Thus* $\mathcal{N}v=\mathcal{F}v$.
Hence, v *is coincidence point of* $(\mathcal{F},\mathcal{N})$. *Further, the weakly compatible of the pair* $(\mathcal{F},\mathcal{N})$ *implies that* $\mathcal{F}\mathcal{N}v=\mathcal{N}\mathcal{F}v$ *or* $\mathcal{N}\eta=\mathcal{F}\eta$.
Thus $\mathcal{E},\mathcal{F},\mathcal{M}$ *and* $\mathcal{N}$ *have a common coincidence point* η.
To show that η *is a common fixed point. Suppose not, then substituting* $x=\mu$ *and* $y=\eta$ *in (5)*

$$\begin{aligned}\sigma^2(\mathcal{E}\mu,\mathcal{F}\eta,z) \le{}& \sigma(\mathcal{E}\mu,\mathcal{F}\eta,z)[\lambda_1\sigma(\mathcal{M}\mu,\mathcal{N}\eta,z)\\ &+\lambda_2\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)+\lambda_3\sigma(\mathcal{N}\eta,\mathcal{F}\eta,z)]\\ &+\lambda_4[\sigma(\mathcal{M}\mu,\mathcal{F}\eta,z)\sigma(\mathcal{E}\mu,\mathcal{N}\eta,z)]\\ \sigma^2(\eta,\mathcal{F}\eta,z) \le{}& \sigma(\eta,\mathcal{F}\eta,z)[\lambda_1\sigma(\eta,\mathcal{N}\eta,z)\\ &+\lambda_2\sigma(\eta,\eta,z)+\lambda_3\sigma(\mathcal{N}\eta,\mathcal{F}\eta,z)]\\ &\lambda_4[\sigma(\eta,\mathcal{F}\eta,z)\sigma(\eta,\mathcal{N}\eta,z)]\end{aligned}$$

$\sigma^2(\mathcal{F}\eta,\eta,z) \le (\lambda_1+\lambda_4)\sigma^2(\mathcal{F}\eta,\eta,z)$
which is a contradiction.
Thus $F\eta\longrightarrow\eta$.
Hence $\mathcal{F}\eta=\mathcal{N}\eta=\eta$ *and consequently* η *is common fixed of* $\mathcal{F}$ *and* $\mathcal{N}$.

Suppose $\mathcal{N}(X)$ is a complete subspace of X and the property E.A of the pair $(\mathcal{E},\mathcal{M})$ will provide similar outcomes.
Next, to prove the uniqueness of common fixed point. Suppose not, then put $x=w$ and $y=\eta$ in (5)

$$\begin{aligned}\sigma^2(\mathcal{E}w,\mathcal{F}\eta,z) \le &\sigma(\mathcal{E}w,\mathcal{F}\eta,z)[\lambda_1\sigma(\mathcal{M}w,\mathcal{N}\eta,z)\\ &+\lambda_2\sigma(\mathcal{E}w,\mathcal{M}w,z)+\lambda_3\sigma(\mathcal{N}\eta,\mathcal{F}\eta,z)]\\ &+\lambda_4[\sigma(\mathcal{M}w,\mathcal{F}\eta,z)\sigma(\mathcal{E}w,\mathcal{N}\eta,z)]\end{aligned}$$

taking $n\longrightarrow\infty$
$\sigma^2(w,\eta,z)\le(\lambda_1+\lambda_4)\sigma^2(w,\eta,z)$
which is a contradiction.
Therefore $w=\eta$ and η is the unique common fixed point of $\mathcal{E},\mathcal{F},\mathcal{M}$ and $\mathcal{N}$ in X. ■

Theorem 3. *Let $\mathcal{E},\mathcal{F},\mathcal{M}$ and $\mathcal{N}$ be self-mapping on 2-metric space (X,σ) satisfying the condition (1), (5) of theorem 1. Moreover, if*
(i). the pair $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},\mathcal{N})$ satisfies the common property (E.A),
(ii). $\mathcal{M}(X)$ and $\mathcal{N}(X)$ are closed subset of X.
Then the pair $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},\mathcal{N})$ have a coincidence point and hence $\mathcal{E},\mathcal{F},\mathcal{M}$,$\mathcal{N}$ all have a unique common fixed point.

Proof: If the pair $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},N)$ satisfy the condition (i), the sequences $\{l_n\}$ and $\{t_n\}$ exists in X thus

$$\lim_{n\longrightarrow\infty}\mathcal{E}l_n=\lim_{n\longrightarrow\infty}\mathcal{M}l_n=\lim_{n\longrightarrow\infty}Ft_n=\lim_{n\longrightarrow\infty}\mathcal{N}t_n=\xi$$

for some $\xi\in X$.
using condition (ii) from the hypothesis, we obtain $\lim_{n\longrightarrow\infty}\mathcal{M}x_n=\xi\in\mathcal{M}(X)$ and then there exists a point $\mu\in X$ such that $\mathcal{M}\mu=\xi$.
To prove $\mathcal{E}\mu=\mathcal{M}\mu$, to prove this, keep $x=\mu$ and $y=t_n$ in (5) have

$$\begin{aligned}\sigma(\mathcal{E}\mu,\mathcal{F}t_n,z)\le&\lambda_1[\sigma(\mathcal{M}\mu,\mathcal{N}t_n,z)+\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)]\\ &+\lambda_2[\sigma(\mathcal{M}\mu,\mathcal{N}t_n,z)+\sigma(\mathcal{N}t_n,\mathcal{F}t_n,z)]\\ &+\lambda_3\left[\frac{\sigma(\mathcal{M}\mu,\mathcal{F}t_n,z)+\sigma(\mathcal{E}\mu,\mathcal{N}t_n,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\mathcal{M}\mu,\mathcal{N}t_n,z)\right]\end{aligned}$$

Letting $n\longrightarrow\infty$,
$\sigma(E\mu,\xi,z)\le\left(\lambda_1+\frac{\lambda_3}{2}\right)\sigma(\mathcal{E}\mu,\xi,z)$
since $\lambda_1+\lambda_2+\lambda_3<\frac{1}{2}$ gives $\mathcal{E}\mu=\xi$. Hence $\mathcal{E}\mu=\mathcal{M}\mu$
Thus, the point μ is a coincidence point of the pair $(\mathcal{E},\mathcal{M})$.
As $\mathcal{N}(X)$ is a closed subset of X, this gives $\lim_{n\longrightarrow\infty}\mathcal{N}t_n=\xi\in\mathcal{N}(X)$ which means there exists a point $w\in X$ such that $Nw=\xi$.
Next, to prove $\mathcal{F}w=\mathcal{N}w$.
using contraction condition (5) with $x=\mu$ and $y=w$

$$\begin{aligned}\sigma(\mathcal{E}\mu,\mathcal{F}w,z)\le&\lambda_1[\sigma(\mathcal{M}\mu,\mathcal{N}w,z)+\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)]\\ &+\lambda_2[\sigma(\mathcal{M}\mu,\mathcal{N}w,z)+\sigma(\mathcal{N}w,\mathcal{F}w,z)]\\ &+\lambda_3\left[\frac{\sigma(\mathcal{M}\mu,\mathcal{F}w,z)+\sigma(\mathcal{E}\mu,\mathcal{N}w,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\mathcal{M}\mu,\mathcal{N}w,z)\right]\end{aligned}$$

Letting $n\longrightarrow\infty$,
$\sigma(\mathcal{F}w,\xi,z)\le\left(\lambda_2+\frac{\lambda_3}{2}\right)\sigma(\mathcal{F}w,\xi,z)$
since $\lambda_1+\lambda_2+\lambda_3<\frac{1}{2}$ gives
$\mathcal{F}w=\xi$. Hence $\mathcal{N}w=\mathcal{F}w$. Therefore w coincidence point of the pair (F,N). given that, the pair (F,N) is weakly compatible mappings then $\mathcal{E}\xi=\mathcal{E}\mathcal{M}\mu=\mathcal{M}\mathcal{E}\mu=\mathcal{M}\xi$ this implies
$\mathcal{E}\xi=\mathcal{M}\xi$
To prove ξ a fixed point of the pair $(\mathcal{E},\mathcal{M})$. Suppose not, then substituting $x=\xi$ and $y=w$ in (5), we obtain

$$\begin{aligned}\sigma(\mathcal{E}\xi,\mathcal{F}w,z)\le&\lambda_1[\sigma(\mathcal{M}\xi,\mathcal{N}w,z)+\sigma(\mathcal{E}\xi,\mathcal{M}\xi,z)]\\ &+\lambda_2[\sigma(\mathcal{M}\xi,\mathcal{N}w,z)+\sigma(\mathcal{N}w,\mathcal{F}w,z)]\\ &+\lambda_3\left[\frac{\sigma(\mathcal{M}\xi,\mathcal{F}w,z)+\sigma(\mathcal{E}\xi,\mathcal{N}w,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\mathcal{M}\xi,\mathcal{N}w,z)\right]\end{aligned}$$

by using $\mathcal{F}w=\mathcal{N}w=\xi$ and $\mathcal{E}\xi=\xi$ implies
$\sigma(\mathcal{E}\xi,\xi,z)\le\left(\lambda_1+\frac{\lambda_3}{2}\right)\sigma(\mathcal{E}\xi,\xi,z)$
since $\lambda_1+\lambda_2+\lambda_3<\frac{1}{2}$ gives
$\mathcal{E}\xi=\mathcal{M}\xi=\xi$.
Thus the pair $(\mathcal{E},\mathcal{M})$ is having ξ as fixed point.
Since $(\mathcal{F},\mathcal{N})$ is weakly compatible mappings then
$\mathcal{F}\xi=\mathcal{F}\mathcal{N}w=\mathcal{N}\mathcal{F}w=\mathcal{N}\xi$.
This gives
$\mathcal{F}\xi=\mathcal{N}\xi$
Next, to demonstrate that ξ is also common fixed point of $(\mathcal{F},\mathcal{N})$.
Suppose ξ is not a common fixed point of $(\mathcal{F},\mathcal{N})$, then taking $x=\mu$ and $y=\xi$ in (5), we get

$$\begin{aligned}\sigma(\mathcal{E}\mu,\mathcal{F}\xi,z)\le&\lambda_1[\sigma(\mathcal{M}\mu,\mathcal{N}\xi,z)+\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)]\\ &+\lambda_2[\sigma(\mathcal{M}\mu,\mathcal{N}\xi,z)+\sigma(\mathcal{N}\xi,\mathcal{F}\xi,z)]\\ &+\lambda_3\left[\frac{\sigma(\mathcal{M}\mu,\mathcal{F}\xi,z)+\sigma(\mathcal{E}\mu,\mathcal{N}\xi,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\mathcal{M}\mu,\mathcal{N}\xi,z)\right]\end{aligned}$$

letting $\lim_{n\longrightarrow\infty}$,

$$\begin{aligned}\sigma(\xi,\mathcal{F}\xi,z)\le&\lambda_1[\sigma(\xi,\mathcal{N}\xi,z)+\sigma(\xi,\xi,z)]\\ &+\lambda_2[\sigma(\xi,\mathcal{N}\xi,z)+\sigma(\mathcal{N}\xi,\mathcal{F}\xi,z)]\\ &+\lambda_3\left[\frac{\sigma(\xi,\mathcal{F}\xi,z)+\sigma(\xi,\mathcal{N}\xi,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\xi,\mathcal{N}\xi,z)\right]\end{aligned}$$

$\sigma(\xi,\mathcal{F}\xi,z)\le(\lambda_1+\lambda_2+\lambda_3)\sigma(\mathcal{F}\xi,\xi,z)$
since $\lambda_1+\lambda_2+\lambda_3<\frac{1}{2}$ gives $\mathcal{F}\xi=\xi$
$\mathcal{F}\xi=N\xi=\xi$
Thus, the common fixed point for the four maps $\mathcal{E},\mathcal{F},\mathcal{M}$ and $\mathcal{N}$ is ξ.
The uniqueness of the fixed point can be easily proved. ■

Theorem 4. *Let $\mathcal{E},\mathcal{F},\mathcal{M}$ and $\mathcal{N}$ be four self-mapping in complete 2-metric space (X,σ) that satisfy the following requirements.*

1) *$\mathcal{E}(X)\subset\mathcal{N}(X)$ and $\mathcal{F}(X)\subset\mathcal{M}(X)$.*
2) *the pair $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},\mathcal{N})$ weakly compatible.*
3) *the pair $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},\mathcal{N})$ satisfy E.A like property.*
4)

$$\begin{aligned}\sigma(\mathcal{E}x,\mathcal{F}y,z)\le&\lambda_1[\sigma(\mathcal{M}x,\mathcal{N}y,z)+\sigma(\mathcal{E}x,\mathcal{M}x,z)]\\ &+\lambda_2[\sigma(\mathcal{M}x,\mathcal{N}y,z)+\sigma(\mathcal{N}y,\mathcal{F}y,z)]\\ &+\lambda_3\left[\frac{\sigma(\mathcal{M}x,\mathcal{F}y,z)+\sigma(\mathcal{E}x,\mathcal{N}y,z)}{2}\right]\\ &+\lambda_3\left[\sigma(\mathcal{M}x,\mathcal{N}y,z)\right]\end{aligned}$$

for all $x, y, z \in X,\ \lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 < \frac{1}{2}$ then the mappings $\mathcal{E}, \mathcal{F}, \mathcal{M}$ and $\mathcal{N}$ have a unique fixed point in X.

Proof: If the pair $(\mathcal{E}, \mathcal{M})$ satisfies E.A like property then there exists a sequence $\{l_n\}$ in X such that

$$\lim_{n\longrightarrow\infty} \mathcal{E}l_n = \lim_{n\longrightarrow\infty} \mathcal{M}l_n = \eta$$

for some $\eta \in \mathcal{E}(x) \cup \mathcal{M}(x)$. From (1), that is, $\mathcal{E}(X) \subset \mathcal{N}(X)$, for each sequence $\{l_n\} \subset X$ there corresponds sequence $\{t_n\} \subset X$ such that $\mathcal{E}l_n = \mathcal{N}t_n$, Hence

$$\lim_{n\longrightarrow\infty} \mathcal{E}l_n = \lim_{n\longrightarrow\infty} \mathcal{N}t_n = \eta.$$

Thus, $\mathcal{E}l_n \longrightarrow \eta, \mathcal{M}l_n \longrightarrow \eta$ and $\mathcal{N}t_n \longrightarrow \eta$.
To prove $\mathcal{F}t_n \longrightarrow \eta$ put $x = l_n$and $y = t_n$ in contraction condition (4) then we get

$$\begin{aligned}\sigma(\mathcal{E}l_n, \mathcal{F}t_n, z) \leq &\lambda_1[\sigma(\mathcal{M}l_n, \mathcal{N}t_n, z) + \sigma(\mathcal{E}l_n, \mathcal{M}l_n, z)] \\ &+ \lambda_2[\sigma(\mathcal{M}l_n, \mathcal{N}t_n, z) + \sigma(\mathcal{N}t_n, \mathcal{F}t_n, z)] \\ &+ \lambda_3\left[\frac{\sigma(\mathcal{M}l_n, \mathcal{F}t_n, z) + \sigma(\mathcal{E}l_n, \mathcal{N}t_n, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\mathcal{M}l_n, \mathcal{N}t_n, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$

$$\begin{aligned}\sigma(\eta, \mathcal{F}t_n, z) \leq &\lambda_1[\sigma(\eta, \eta, z) + \sigma(\eta, \eta, z)] \\ &+ \lambda_2[\sigma(\eta, \eta, z) + \sigma(\mathcal{N}t_n, \mathcal{F}t_n, z)] \\ &+ \lambda_3\left[\frac{\sigma(\eta, \mathcal{F}t_n, z) + \sigma(\eta, \eta, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\eta, \eta, z)\right] \\ \leq &\Big(\lambda_2 + \frac{\lambda_3}{2}\Big)\sigma(\eta, \mathcal{F}t_n, z)\end{aligned}$$

since $\lambda_1 + \lambda_2 + \lambda_3 < 1$ this implies $\mathcal{F}t_n \longrightarrow \eta$. Suppose $\eta \in \mathcal{M}$ then there exists a $\mu \in X$ such that $\eta = \mathcal{M}\mu$ put $x = \mu$ and $y = t_n$ in (4) then we get

$$\begin{aligned}\sigma(\mathcal{E}\mu, \mathcal{F}t_n, z) \leq &\lambda_1[\sigma(\mathcal{M}\mu, \mathcal{N}t_n, z) + \sigma(\mathcal{E}\mu, \mathcal{M}\mu, z)] \\ &+ \lambda_2[\sigma(\mathcal{M}\mu, \mathcal{N}t_n, z) + \sigma(\mathcal{N}t_n, \mathcal{F}t_n, z)] \\ &+ \lambda_3\left[\frac{\sigma(\mathcal{M}\mu, \mathcal{F}t_n, z) + \sigma(\mathcal{E}\mu, \mathcal{N}t_n, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\mathcal{M}\eta, \mathcal{N}t_n, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$

$$\begin{aligned}\sigma(\mathcal{E}\mu, \eta, z) \leq &\lambda_1[\sigma(\eta, \eta, z) + \sigma(\mathcal{E}\mu, \eta, z)] \\ &+ \lambda_2[\sigma\eta, \eta, z) + \sigma(\eta, \eta, z)] \\ &+ \lambda_3\left[\frac{\sigma(\eta, \eta, z) + \sigma(\mathcal{E}\mu, \eta, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\eta, \eta, z)\right] \\ \leq &\Big(\lambda_1 + \frac{\lambda_3}{2}\Big)(\mathcal{E}\mu, \eta, z)\end{aligned}$$

This gives $\mathcal{E}\mu \longrightarrow \eta$. Therefore $\mathcal{E}\mu = \eta = \mathcal{M}\mu$
Since the pair $(\mathcal{E}, \mathcal{M})$ is weakly compatible mapping which implies

$$\mathcal{E}\eta = \mathcal{E}\mathcal{M}\mu = \mathcal{M}\mathcal{E}\mu = \mathcal{M}\eta$$

since the pair $(\mathcal{F}, \mathcal{N})$ satisfy property E.A like then there exists a sequence $\{t_n\} \in X$ such that
$\lim_{n\longrightarrow\infty} \mathcal{F}t_n = \lim_{n\longrightarrow\infty} \mathcal{N}t_n = \eta$ for some $\eta \in \mathcal{F}(X) \cup \mathcal{N}(X)$
If $\eta \in \mathcal{N}(X)$ then there exists $w \in X$ such that $\mathcal{N}w = \eta$

Now we prove $\mathcal{F}w = \eta$, put $x = l_n$ and $y = w$ in (4) then we get

$$\begin{aligned}\sigma(\mathcal{E}l_n, \mathcal{F}w, z) \leq &\lambda_1[\sigma(\mathcal{M}l_n, \mathcal{N}w, z) + \sigma(\mathcal{E}l_n, \mathcal{M}l_n, z)] \\ &+ \lambda_2[\sigma(\mathcal{M}l_n, \mathcal{N}w, z) + \sigma(\mathcal{N}w, \mathcal{F}w, z)] \\ &+ \lambda_3\left[\frac{\sigma(\mathcal{M}l_n, \mathcal{F}w, z) + \sigma(\mathcal{E}l_n, \mathcal{N}w, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\mathcal{M}l_n, \mathcal{N}w, z)\right] \\ \sigma(\eta, \mathcal{F}w, z) \leq &\lambda_1[\sigma(\eta, \eta, z) + \sigma(\eta, \eta, z)] \\ &+ \lambda_2[\sigma(\eta, \eta, z) + \sigma(\eta, \mathcal{F}w, z)] \\ &+ \lambda_3\left[\frac{\sigma(\eta, \mathcal{F}w, z) + \sigma(\eta, \eta, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\eta, \eta, z)\right] \\ &\leq \Big(\lambda_2 + \frac{\lambda_3}{2}\Big)\sigma(\eta, \mathcal{F}w, z)\end{aligned}$$

which gives $\mathcal{F}w = \eta$. Since the pair $(\mathcal{F}, \mathcal{N})$ is weakly compatible mapping, then we have

$$\mathcal{N}\eta = \mathcal{N}\mathcal{F}w = \mathcal{F}\mathcal{N}w = \mathcal{F}\eta$$

Now we prove $\mathcal{F}\eta = \eta$. For this put $x = l_n$ and $y = \eta$ in (4), we get

$$\begin{aligned}\sigma(\eta, \mathcal{F}\eta, z) \leq &\lambda_1[\sigma(\eta, \mathcal{N}\eta, z) + \sigma(\eta, \eta, z)] \\ &+ \lambda_2[\sigma(\eta, \mathcal{N}\eta, z) + \sigma(\mathcal{N}\eta, \mathcal{F}\eta, z)] \\ &+ \lambda_3\left[\frac{\sigma(\eta, \mathcal{F}\eta, z) + \sigma(\eta, \mathcal{N}\eta, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\eta, \mathcal{N}\eta, z)\right]\end{aligned}$$

Taking $n \longrightarrow \infty$ and $\mathcal{N}\eta = \mathcal{F}\eta$, we get

$$\sigma(\eta, \mathcal{F}\eta, z) \leq \Big(\lambda_1 + \lambda_2 + 2\lambda_3\Big)\sigma(\eta, \mathcal{F}\eta, z)$$

since $\lambda_1 + \lambda_2 + 2\lambda_3 < 1$, which implies $\mathcal{F}\eta = \eta$ this gives $\mathcal{N}\eta = \mathcal{F}\eta = \eta$.
New to prove $\mathcal{F}\eta = \mathcal{E}\eta$, put $x = \eta$ and $y = \eta$ in inequality (4), then we get

$$\begin{aligned}\sigma(\mathcal{E}\eta, \mathcal{F}\eta, z) \leq &\lambda_1[\sigma(\mathcal{M}\eta, \mathcal{N}\eta, z) + \sigma(\mathcal{E}\eta, \mathcal{M}\eta, z)] \\ &+ \lambda_2[\sigma(\mathcal{M}\eta, \mathcal{N}\eta, z) + \sigma(\mathcal{N}\eta, \mathcal{F}\eta, z)] \\ &+ \lambda_3\left[\frac{\sigma(\mathcal{M}\eta, \mathcal{F}\eta, z) + \sigma(\mathcal{E}\eta, \mathcal{N}\eta, z)}{2}\right] \\ &+ \lambda_3\left[\sigma(\mathcal{M}\eta, \mathcal{N}\eta, z)\right] \\ \sigma(\mathcal{E}\eta, \eta, z) \leq &(\lambda_1 + \lambda_2 + 2\lambda_3)\sigma(\mathcal{E}\eta, \eta, z)\end{aligned}$$

which gives $\mathcal{E}\eta = \eta$ and $\mathcal{F}\eta = \mathcal{E}\eta = \eta$. This implies

$$\mathcal{F}\eta = \mathcal{E}\eta = \eta = \mathcal{N}\eta = \mathcal{M}\eta$$

Therefore η is a common fixed point of the maps $\mathcal{M}(x), \mathcal{N}(x), \mathcal{E}(x)$ and $\mathcal{F}(x)$.
The fixed point's uniqueness is easily proved. ■

Theorem 5. *Let $\mathcal{E}, \mathcal{F}, \mathcal{M}$ and $\mathcal{N}$ be self-maps of a 2-metric space (X, σ) satisfying the conditions (3), (5) of theorem 1 and assuming*
(i). the pairs the pair $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{F}, \mathcal{N})$ satisfies the common (E.A) like property.
Then the pairs $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{F}, \mathcal{N})$ have a coincidence point each and hence the maps $\mathcal{E}, \mathcal{F}, \mathcal{M}$ and $\mathcal{N}$ have unique common fixed point.

Proof: *Since the two pairs of mappings $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},\mathcal{N})$ satisfy the common E.A like property, we find two sequence $\{l_n\}$ and $\{t_n\}$ in X such that*

$$\lim_{n\longrightarrow\infty}\mathcal{E}l_n=\lim_{n\longrightarrow\infty}\mathcal{M}l_n=\lim_{n\longrightarrow\infty}\mathcal{F}t_n=\lim_{n\longrightarrow\infty}Nt_n=\xi$$

where $\xi\in\mathcal{E}(X)\cap\mathcal{F}(X)$ or $\xi\in\mathcal{M}(X)\cap\mathcal{N}(X)$.
Suppose $\xi\in\mathcal{M}(X)\cap\mathcal{N}(X)$.
Now $\xi\in\mathcal{M}(X)$ there exists $\mu\in X$ such that $\mathcal{M}\mu=\xi$.
Now, we prove that $\mathcal{M}\mu=\mathcal{E}\mu$, using inequality (5) with $x=\mu,y=t_n$, we get

$$\begin{aligned}\sigma(\mathcal{E}\mu,\mathcal{F}t_n,z)\leq&\lambda_1[\sigma(\mathcal{M}\mu,\mathcal{N}t_n,z)+\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)]\\&+\lambda_2[\sigma(\mathcal{M}\mu,\mathcal{N}t_n,z)+\sigma(\mathcal{N}t_n,\mathcal{F}t_n,z)]\\&+\lambda_3\left[\frac{\sigma(\mathcal{M}\mu,\mathcal{F}t_n,z)+\sigma(\mathcal{E}\mu,\mathcal{N}t_n,z)}{2}\right]\\&+\lambda_3\left[\sigma(\mathcal{M}\mu,\mathcal{N}t_n,z)\right]\end{aligned}$$

Letting $n\longrightarrow\infty$,

$$\begin{aligned}\sigma(\mathcal{E}\mu,\xi,z)\leq&\lambda_1[\sigma(\xi,\xi,z)+\sigma(\mathcal{E}\mu,\xi,z)]\\&+\lambda_2[\sigma\xi,\xi,z)+\sigma(\xi,\xi,z)]\\&+\lambda_3\left[\frac{\sigma(\xi,\xi,z)+\sigma(\mathcal{E}\mu,\xi,z)}{2}\right]\\&+\lambda_3\left[\sigma(\xi,\xi,z)\right]\\\sigma(\mathcal{E}\mu,\xi,z)\leq&\left(\lambda_1+\frac{\lambda_3}{2}\right)\sigma(\mathcal{E}\mu,\xi,z)\end{aligned}$$

this implies $\mathcal{E}\mu=\xi$. Hence $\mathcal{E}\mu=\xi=\mathcal{M}\mu$, this gives μ is a coincidence point of the pair of mappings $(\mathcal{E},\mathcal{M})$.
Again $\xi\in\mathcal{N}(X)$, we have $\xi=\mathcal{N}\nu$ for some $\nu\in X$.
we show that $\mathcal{N}\nu=\mathcal{F}\nu$, using the contraction condition (5) with $x=l_n,y=\nu$

$$\begin{aligned}\sigma(\mathcal{E}l_n,\mathcal{F}\nu,z)\leq&\lambda_1[\sigma(\mathcal{M}l_n,\mathcal{N}\nu,z)+\sigma(\mathcal{E}l_n,\mathcal{M}l_n,z)]\\&+\lambda_2[\sigma(\mathcal{M}l_n,\mathcal{N}\nu,z)+\sigma(\mathcal{N}\nu,\mathcal{F}\nu,z)]\\&+\lambda_3\left[\frac{\sigma(\mathcal{M}l_n,\mathcal{F}\nu,z)+\sigma(\mathcal{E}l_n,\mathcal{N}\nu,z)}{2}\right]\\&+\lambda_3\left[\sigma(\mathcal{M}l_n,\mathcal{N}\nu,z)\right]\end{aligned}$$

letting $n\longrightarrow\infty$,

$$\begin{aligned}\sigma(\xi,\mathcal{F}\nu,z)\leq&\lambda_1[\sigma(\xi,\mathcal{N}\nu,z)+\sigma(\xi,\xi,z)]\\&+\lambda_2[\sigma(\xi,\mathcal{N}\nu,z)+\sigma(\mathcal{N}\nu,\mathcal{F}\nu,z)]\\&+\lambda_3\left[\frac{\sigma(\xi,\mathcal{F}\nu,z)+\sigma(\xi,\mathcal{N}\nu,z)}{2}\right]\\&+\lambda_3\left[\sigma(\xi,\xi,z)\right]\\\leq&\lambda_1[\sigma(\xi,\xi,z)+\sigma(\xi,\xi,z)]\\&+\lambda_2[\sigma(\xi,\xi,z)+\sigma(\xi,\mathcal{F}\nu,z)]\\&+\lambda_3\left[\frac{\sigma(\xi,\mathcal{F}\nu,z)+\sigma(\xi,\xi,z)}{2}\right]\\&+\lambda_3\left[\sigma(\xi,\xi,z)\right]\\\leq&\left(\lambda_2+\frac{\lambda_3}{2}\right)\sigma(\xi,\mathcal{F}\nu,z)\end{aligned}$$

implies $\mathcal{F}\nu=\xi$ is gives $\mathcal{F}\nu=\xi=\mathcal{N}\nu$.
By using the weakly compatible mappings of the pairs $(\mathcal{E},\mathcal{M})$ and $(\mathcal{F},\mathcal{N})$ and $\mathcal{E}\mu=\mathcal{M}\mu,\mathcal{F}\nu=\mathcal{N}\nu$, therefore $\mathcal{E}\mu=\mathcal{E}\mathcal{M}\mu=\mathcal{M}\mathcal{E}\mu=\mathcal{M}\mu$ and $\mathcal{F}\nu=\mathcal{F}\mathcal{N}\nu=\mathcal{N}\mathcal{F}\nu=\mathcal{N}\nu$.
Now, we establish that ξ is a common fixed point of $\mathcal{E}$ and $\mathcal{M}$.
On using contraction condition (5) with $x=\xi,y=\nu$ implies

$$\begin{aligned}\sigma(\mathcal{E}\xi,\mathcal{F}\nu,z)\leq&\lambda_1[\sigma(\mathcal{M}\xi,\mathcal{N}\nu,z)+\sigma(\mathcal{E}\xi,\mathcal{M}\xi,z)]\\&+\lambda_2[\sigma(\mathcal{M}\xi,\mathcal{N}\nu,z)+\sigma(\mathcal{N}\nu,\mathcal{F}\nu,z)]\\&+\lambda_3\left[\frac{\sigma(\mathcal{M}\xi,\mathcal{F}\nu,z)+\sigma(\mathcal{E}\xi,\mathcal{N}\nu,z)}{2}\right]\\&+\lambda_3\left[\sigma(\mathcal{M}\xi,\mathcal{N}\nu,z)\right]\\\sigma(\mathcal{E}\xi,\xi,z)\leq&\lambda_1[\sigma(\mathcal{M}\xi,\xi,z)+\sigma(\mathcal{E}\xi,\mathcal{M}\xi,z)]\\&+\lambda_2[\sigma(\mathcal{M}\xi,\xi,z)+\sigma(\xi,\xi,z)]\\&+\lambda_3\left[\frac{\sigma(\mathcal{M}\xi,\xi,z)+\sigma(\mathcal{E}\xi,\xi,z)}{2}\right]\\&+\lambda_3\left[\sigma(\mathcal{M}\xi,\xi,z)\right]\\\leq&(\lambda_1+\lambda_2+2\lambda_3)\,\sigma(\mathcal{E}\xi,\xi,z)\end{aligned}$$

implies $\mathcal{E}\xi=\xi$. Therefore $\mathcal{E}\xi=\xi=\mathcal{M}\xi$. Now, we show that the pair $(\mathcal{F},\mathcal{N})$ has common fixed point ξ.
Using contraction condition (5) with $x=\mu,y=\xi$, we get

$$\begin{aligned}\sigma(\xi,\mathcal{F}\xi,z)\leq&\lambda_1[\sigma(\mathcal{M}\mu,\mathcal{N}\xi,z)+\sigma(\mathcal{E}\mu,\mathcal{M}\mu,z)]\\&+\lambda_2[\sigma(\mathcal{M}\mu,\mathcal{N}\xi,z)+\sigma(\mathcal{N}\xi,\mathcal{F}\xi,z)]\\&+\lambda_3\left[\frac{\sigma(\mathcal{M}\mu,\mathcal{F}\xi,z)+\sigma(\mathcal{E}\mu,\mathcal{N}\xi,z)}{2}\right]\\&+\lambda_3\left[\sigma(\mathcal{M}\mu,\mathcal{N}\xi,z)\right]\\\sigma(\xi,\mathcal{F}\xi,z)\leq&(\lambda_1+\lambda_2+2\lambda_3)\,\sigma(\xi,\mathcal{F}\xi,z)\end{aligned}$$

implies $\xi=\mathcal{F}\xi$. Therefore $\xi=\mathcal{F}\xi=\mathcal{N}\xi$. Thus $\mathcal{E},\mathcal{F},\mathcal{M}$ and $\mathcal{N}$ have ξ as a common fixed point.
Similarly, the theorem holds $\xi\in\mathcal{E}(X)\cap\mathcal{F}(X)$.
The uniqueness of a common fixed point can be easily verified. ■

IV. Conclusion

This work demonstrates the existence of common fixed points for weakly compatible mapping in 2-metric space. Moreover, the fixed point theorem in 2-metric space is proved using the property (E.A), common property (E.A) and (E.A) like property, common (E.A) like Property.

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