Solution of the Boundary Value Problems via Fixed Point Theorem on G-metric Space

R. Anna Thirumalai, and S. Thalapathiraj

***Abstract*—In this study, we demonstrate the fixed point theorem for rational contractive mapping on a complete $\mathtt{G}-$metric space. Furthermore, we provide an application of a first-order boundary value problem solution (abbreviated as $\mathcal{FOBVP}$ sol.) and an example of a binary relation in a Euclidean metric space. Additionally, we include a numerical case study to demonstrate the effectiveness of this new approach.**

***Index Terms*—rational contractive mapping, fixed point, G-Cauchy sequence, complete G-metric space.**

I. Introduction

THE classical Banach contraction principle [1] yielded various results in 1922, one of which was the existence of fixed points for contractive mappings. In various metric space settings, the Banach contraction principle has been extended and established. One of the expanded variants of the standard metric space is characterized by a proposed relationship between continuity, contraction and completeness, as suggested by Alam and Imdad [2], [3]. Furthermore, the fixed-point theorems of Ahmadullah et al. [4] and Boyd-Wong [5] have been extended to nonlinear contraction mappings. Additionally, several authors, including Senapati and Dey [6], have developed the concept of w-distance in relational metric spaces with arbitrary binary relations.

A new class of generalized metric spaces, known as G-metric spaces, was first established by Mustafa and Sims [7] in 2006 as a generalization of metric spaces. Subsequently, several fixed-point results on these spaces have emerged (as seen in [8]–[12]). Ali, Imdad and Sessa [13] demonstrated fixed-point theorems in Ⓡ-complete regular symmetric spaces. The concept of fixed-point theorems for nonexpansive mappings under binary relations was introduced and demonstrated by Alam, George, Imdad and Hasanuzzaman [14]. Fixed-point theorems on Ⓡ-complete metric spaces were proven by Javed, Arshad, Baazeem and Nabil [15]. Faruk et al. [16] demonstrated fixed-point theorems for generalized nonlinear contractions in a new metric space, utilizing a locally finitely T-transitive binary relation and auxiliary functions. The study by Samet et al. [17] focuses on metric fixed-point results in the context of Kannan contractions and introduces a novel concept called α-admissible mappings. Consequently, numerous authors have extended and refined a significant number of conclusions in metric fixed points for these mappings (as observed, for example, in [18]–[21]).

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Since then, numerous mathematicians have developed generalizations of the contraction mapping principle, leading to a wealth of fixed-point theorems in metric spaces that continue to be explored to this day. Gopi Prasad [22] investigated fixed points in relational metric spaces of Kannan contractive mappings. Gopi Prasad [23] discussed fixed-point theorems with applications to boundary value problems in relational metric spaces. Over the years, researchers have been exploring metric spaces to discover new perspectives and extensions of the extensively studied boundary value problem. Throughout several years, numerous researchers have focused on various metric spaces (as evident in [24]–[35]). The objective of this study is to establish a fixed-point theorem for rational contractive mappings in a complete G-metric spaces.

II. Preliminaries

Let us begin this article with a few fundamental definitions, propositions and relevant theorems on G-metric spaces. Mustafa and Sims introduced a new class of generalized metric spaces, called G-metric spaces, in 2006 (see [7]). These spaces extend the notion of standard metric spaces $(\mathtt{X}, \mathfrak{d})$ and since then, several fixed-point theorems have been established for them (refer to [8]–[12]). This paper presents the essential definitions and results of G-metric spaces that are relevant to the subsequent sections of the article. For more detailed information, we recommend referring to [7].

Definition 1. *[7] Let Λ be a non empty set and let $\mathtt{G} : \Lambda \times \Lambda \times \Lambda \to [0, \infty)$ be a function satisfying the following conditions:*

(1) $\mathtt{G}(\varpi, \upsilon, \varrho) = 0$ *if* $\varpi = \upsilon = \varrho$,
(2) $\mathtt{G}(\varpi, \upsilon, \upsilon) > 0$*; for all* $\varpi, \upsilon \in \Lambda$ *with* $\varpi \neq \upsilon$,
(3) $\mathtt{G}(\varpi, \upsilon, \upsilon) \leq \mathtt{G}(\varpi, \upsilon, \varrho)$ *for all* $\varpi, \upsilon, \varrho \in \Lambda$ *with* $\upsilon \neq \varrho$,
(4) $\mathtt{G}(\varpi, \upsilon, \varrho) = \mathtt{G}(\varpi, \varrho, \upsilon) = \mathtt{G}(\varrho, \varpi, \upsilon) = \ldots$ *(symmetry in all three variables),*
(5) $\mathtt{G}(\varpi, \upsilon, \varrho) \leq \mathtt{G}(\varpi, \vartheta, \vartheta) + \mathtt{G}(\vartheta, \upsilon, \varrho)$ *for all* $\varpi, \upsilon, \varrho, \vartheta \in \Lambda$ *(rectangle inequality).*

Then the function G is called a generalized metric or more specially, a G-metric on $\Lambda$and $(\Lambda, \mathtt{G})$ is called a G-metric space.

Definition 2. *[7] Let $(\Lambda, \mathtt{G})$ be a G-metric space.*

(i) *The sequence $\{\varpi_\sigma\}$ G-convergent to $\varpi \in \Lambda$ if and only if* $\lim\limits_{\sigma,\varrho\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi) = 0$.
(ii) *The sequence $\{\varpi_\sigma\}$ G-Cauchy sequence if and only if* $\lim\limits_{\varpi,\varrho,\zeta\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\zeta) = 0$.
(iii) $(\Lambda, \mathtt{G})$ *is G-complete if and only if every G-Cauchy sequence in Λ is G-convergent.*

Definition 3. *[7] A* $\mathtt{G}-$ *metric space* $(\Lambda,\mathtt{G})$ *is called symmetric if*

$\mathtt{G}(\varpi,\upsilon,\upsilon)=\mathtt{G}(\upsilon,\varpi,\varpi)$ *for all* $\varpi,\upsilon\in\Lambda$.

In this case, $\mathbb{N}$ represents the set of natural numbers, $\mathbb{N}_0$ represents the set of whole numbers and $\mathscr{G}$ denotes a non-empty binary relation ($\mathcal{BR}$, in short).

Definition 4. *Suppose* Λ *is a non-empty set under* $\mathscr{G}$*, defined as a subset of* $\Lambda\times\Lambda\times\Lambda$*. Consequently, we refer to* ϖ *as related to* υ *if and only if* $(\varpi,\upsilon,\upsilon)\in\mathscr{G}$ *under* $\mathscr{G}$*.*

Definition 5. *A binary relation* $\mathscr{G}$ *on a non-empty set* Λ *is defined such that two elements* $\varpi,\upsilon\in\Lambda$ *are* $\mathtt{G}$*-comparative if either* $(\varpi,\upsilon,\upsilon)\in\mathscr{G}$ *or* $(\upsilon,\varpi,\varpi)\in\mathscr{G}$*, which can be written as* $[\varpi,\upsilon,\upsilon]\in\mathscr{G}$*.*

Definition 6. *A binary relation* $\mathscr{G}$ *on a non-empty set* Λ *and two elements* $\varpi,\upsilon\in\Lambda$*, a sequence* $\{\varpi_\sigma\}$ *in* $\mathscr{G}$ *is* $\mathtt{G}$ *-preserving if* $(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})\in\mathscr{G}$ *for all* $\sigma\in\mathbb{N}_0$*.*

Definition 7. *A self-mapping* Δ *on a non-empty set* Λ *induces a* Δ*-closed binary relation* $\mathscr{G}$ *on* Λ *if for all* $\varpi,\upsilon\in\Lambda$ *such that* $(\varpi,\upsilon,\upsilon)\in\mathscr{G}$*, then* $(\Delta\varpi,\Delta\upsilon,\Delta\upsilon)\in\mathscr{G}$*.*

Definition 8. *Assume that* Λ *is a non-empty set under* $\mathscr{G}$ *on* Λ *and* Δ *is a self-mapping on* Λ*, if* $\mathscr{G}$ *is* Δ*-closed, then* $\mathscr{G}$ *is also* Δ^σ*-closed for all* $\sigma\in\mathbb{N}_0$*, where* Δ^σ *represents the* $\mathfrak{n}^{th}$ *iteration of* Δ*.*

Definition 9. *A* $\mathtt{G}$*-metric space* $(\Lambda,\mathtt{G})$ *with a binary relation* $\mathtt{G}$ *on* Λ *is said to be* $\mathtt{G}$*-complete if any* $\mathtt{G}$*-preserving* $\mathtt{G}$*-Cauchy sequence in* Λ *is* $\mathtt{G}$*-convergent.*

Definition 10. *Let* $(\Lambda,\mathtt{G})$ *be a* $\mathtt{G}$*-metric space,* $\mathscr{G}$ *a binary relation on* Λ *and* $\varpi\in\Lambda$*. A self mapping* Δ *on* Λ *is called* $\mathtt{G}$*-continuous at* ϖ *if for any* $\mathtt{G}$*-preserving sequence* $\{\varpi_\sigma\}$ *such that* $\varpi_\sigma\overset{\mathtt{G}}{\to}\varpi$*, we have* $\Delta(\varpi_\sigma)\overset{\mathtt{G}}{\to}\Delta(\varpi)$*. Moreover,* Δ *is called* $\mathtt{G}$*-continuous if it is* $\mathtt{G}$*-continuous at each point of* Λ*.*

Definition 11. *Assume that* $(\Lambda,\mathtt{G},\mathscr{G})$ *is a* $\mathtt{G}-$*metric space under* $\mathscr{G}$*, let* $\mathtt{E}$ *be a subset of* Λ*.* $\mathtt{E}$ *is said to be* $\mathtt{G}$*-connected if there is a path in* $\mathscr{G}$ *from* ϖ *to* υ *for every* $\varpi,\upsilon\in\mathtt{E}$*.*

III. Main Results

Theorem 1. *Let the mapping* $\Delta:\Lambda\to\Lambda$ *and* $(\Lambda,\mathtt{G})$ *be a complete* $\mathtt{G}-$ *metric space such that*

(a) $\Lambda(\Delta,\mathscr{G})$ *is non-empty set;*
(b) Δ *is* $\mathtt{G}$*-continuous;*
(c) $\mathtt{G}$ *is* Δ*-closed;*
(d) *There exists* $\mathfrak{p},\mathfrak{q},\mathfrak{r}\in[0,1)$ *such that*

$$\begin{aligned}\mathtt{G}(\Delta\varpi,\Delta\upsilon,\Delta\varpi)\le{}&\mathfrak{p}\mathtt{G}(\varpi,\upsilon,\varpi)\\&+\mathfrak{q}\frac{\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}\\&+\mathfrak{r}\frac{\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)},\end{aligned}\tag{1}$$

for all $\varpi,\upsilon\in\Lambda$ *with* $(\varpi,\upsilon,\varpi)\in\mathscr{G}$ *and* $0\le\mathfrak{p}+\mathfrak{q}+\mathfrak{r}<1$*. Then, there exists* $\varpi\in\Lambda$ *such that* $\varpi\in\Lambda\varpi$*.*

Proof: *Assume that* (a) *and let us take* ϖ_0 *as arbitrary element of* $\Lambda(\Delta,\mathscr{G})$*.*
Formulate a sequence $\{\varpi_\sigma\}$ *that is*

$$\varpi_\sigma=\Lambda^\sigma(\varpi_0)\ for\ all\ \sigma\in\mathbb{N}_0.$$

Because $(\varpi_0,\Delta\varpi_0,\Delta\varpi_0)\in\mathscr{G}$ *and* $\mathtt{G}$ *is* Λ*-closed, we can apply Theorem 1 to obtain*

$$(\Delta^1\varpi_0,\Delta^2\varpi_0,\Delta^2\varpi_0),(\Delta^2\varpi_0,\Delta^3\varpi_0,\Delta^3\varpi_0),\\ ....,(\Delta^\sigma\varpi_0,\Delta^{\sigma+1}\varpi_0,\Delta^{\sigma+1}\varpi_0)\in\mathscr{G}$$

so that

$$(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})\in\mathscr{G}\ for\ all\ \sigma\in\mathbb{N}_0.$$

Then, the sequence $\{\varpi_\sigma\}$ *is* $\mathtt{G}$*- preserving.*
By applying the contractive condition (d)*, we obtain*

$$\begin{aligned}\mathtt{G}(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})&=\mathtt{G}(\Delta\varpi_{\sigma-1},\Delta\varpi_\sigma,\Delta\varpi_\sigma)\\&\le\mathfrak{p}\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)\\&\quad+\mathfrak{q}\frac{\mathtt{G}(\varpi_{\sigma-1},\Delta\varpi_{\sigma-1},\Delta\varpi_{\sigma-1})\cdot\mathtt{G}(\varpi_\sigma,\Delta\varpi_\sigma,\Delta\varpi_\sigma)}{1+\mathtt{G}(\varpi_{\sigma-1},\Delta\varpi_{\sigma-1},\Delta\varpi_{\sigma-1})}\\&\quad+\mathfrak{r}\frac{\mathtt{G}(\varpi_\sigma,\Delta\varpi_{\sigma-1},\Delta\varpi_{\sigma-1})\cdot\mathtt{G}(\varpi_{\sigma-1},\Delta\varpi_\sigma,\Delta\varpi_\sigma)}{1+\mathtt{G}(\varpi_{\sigma-1},\Delta\varpi_{\sigma-1},\Delta\varpi_{\sigma-1})}\\&\le\mathfrak{p}\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)\\&\quad+\mathfrak{q}\frac{\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)\cdot\mathtt{G}(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})}{1+\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)}\\&\quad+\mathfrak{r}\frac{\mathtt{G}(\varpi_\sigma,\varpi_\sigma,\varpi_\sigma)\cdot\mathtt{G}(\varpi_{\sigma-1},\varpi_{\sigma+1},\varpi_{\sigma+1})}{1+\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)}\\&\le\mathfrak{p}\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)\\&\quad+\mathfrak{q}\frac{\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)\cdot\mathtt{G}(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})}{1+\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)}\\&\quad+\mathfrak{r}\frac{\mathtt{G}(\varpi_\sigma,\varpi_\sigma,\varpi_\sigma)\cdot\mathtt{G}(\varpi_{\sigma-1},\varpi_{\sigma+1},\varpi_{\sigma+1})}{1+\mathtt{G}(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)}\end{aligned}$$

$$G(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})\le\mathfrak{p}G(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)\\+\mathfrak{q}G(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)$$

$$G(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})\le(\mathfrak{p}+\mathfrak{q})G(\varpi_{\sigma-1},\varpi_\sigma,\varpi_\sigma)$$

By induction, we have

$$\mathtt{G}(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})\le\left(\mathfrak{p}+\mathfrak{q}\right)^\sigma\mathtt{G}(\varpi_0,\varpi_1,\varpi_1)$$

For any positive integers σ,ϱ *satisfying the condition* $\sigma<\varrho$*, we have*

$$\begin{aligned}&\mathtt{G}(\varpi_\sigma,\ \varpi_\varrho,\varpi_\varrho)\\&\quad\le\mathtt{G}(\varpi_\sigma,\varpi_{\sigma+1},\varpi_{\sigma+1})+....+\mathtt{G}(\varpi_{\varrho-1},\varpi_\varrho,\varpi_\varrho)\\&\quad\le(\delta^\sigma+....+\delta^{\varrho-1})\mathtt{G}(\varpi_0,\varpi_1,\varpi_1)\\&\quad\le\frac{\delta^\sigma}{1-\delta}\mathtt{G}(\varpi_0,\varpi_1,\varpi_1),\ where\ \delta=\mathfrak{p}+\mathfrak{q}\end{aligned}$$

Taking limit as $\sigma,\varrho\to\infty$*, we get*

$$\lim_{\sigma,\varrho\to\infty}\mathtt{G}(\varpi_\sigma,\varpi_\varrho,\varpi_\varrho)=0.\tag{2}$$

To prove: $\{\varpi_\sigma\}$ *is a* $\mathtt{G}$*-Cauchy sequence.*

$$\mathtt{G}(\varpi_\sigma,\varpi_\varrho,\varpi_\zeta)\le\mathtt{G}(\varpi_\sigma,\varpi_\varrho,\varpi_\varrho)+\mathtt{G}(\varpi_\varrho,\varpi_\varrho,\varpi_\zeta)$$

Taking limit as $\sigma,\varrho,\zeta\to\infty$*, we get*

$$\lim_{\sigma,\varrho,\zeta\to\infty}\mathtt{G}(\varpi_\sigma,\varpi_\varrho,\varpi_\zeta)=0.\tag{3}$$

Therefore, $\{\varpi_\sigma\}$ *is a* $\mathtt{G}$*-Cauchy sequence and since* $(\Lambda,\mathtt{G})$ *is a complete* $\mathtt{G}$*-metric space, there exists an element* $\varpi\in\Lambda$*. As a result, we have*

$$\lim_{\sigma\to\infty}\varpi_\sigma=\varpi.$$

Since Δ is $\mathtt{G}$-continuous, then $\varpi_{\sigma+1} = \Delta\varpi_{\sigma} \overset{\mathtt{G}}{\to} \Delta\varpi$.
Therefore, $\Delta\varpi = \varpi$.
Hence, ϖ is a fixed point of Δ.
Let ϖ and υ be two fixed points of Δ. Then, we obtain $(\varpi,\upsilon,\upsilon) \in \mathscr{G}$ (or) $(\upsilon,\varpi,\varpi) \in \mathscr{G}$.
For $(\varpi,\upsilon,\upsilon) \in \mathscr{G}$, we have

$$\begin{aligned}\mathtt{G}(\varpi,\upsilon,\upsilon) &= \mathtt{G}(\Delta\varpi,\Delta\upsilon,\Delta\upsilon)\\ &\leq \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\upsilon)\\ &\quad + \mathfrak{q}\frac{\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}\\ &\quad + \mathfrak{r}\frac{\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}\\ &\leq \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\upsilon) + \mathfrak{r}\mathtt{G}(\varpi,\upsilon,\upsilon)\\ &\leq (\mathfrak{p}+\mathfrak{r})\mathtt{G}(\varpi,\upsilon,\upsilon)\\ &< \mathtt{G}(\varpi,\upsilon,\upsilon)\end{aligned}$$

which is a contradiction. Hence, we must have $\varpi = \upsilon$. Similarly, for $(\upsilon,\varpi,\varpi) \in \mathscr{G}$, we have $\upsilon = \varpi$. Hence, Δ has a unique fixed point. ∎

Corollary 1. *Let the mapping $\Delta : \Lambda \to \Lambda$ and $(\Lambda, \mathtt{G})$ be a complete $\mathtt{G}-$ metric space such that*
(a) *$\Lambda(\Delta,\mathscr{G})$ is non-empty set;*
(b) *Δ is $\mathtt{G}$- continuous;*
(c) *$\mathtt{G}$ is Δ-closed;*
(d) *There exists $\mathfrak{p},\mathfrak{q},\mathfrak{r} \in [0,1)$ such that*

$$\begin{aligned}&\mathtt{G}(\Delta\varpi,\Delta\upsilon,\Delta\vartheta)\\ &\leq \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\vartheta)\\ &\quad + \mathfrak{q}\frac{\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\cdot\mathtt{G}(\vartheta,\Delta\vartheta,\Delta\vartheta)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}\\ &\quad + \mathfrak{r}\frac{\mathtt{G}(\upsilon,\Delta\varpi,\Delta\vartheta)\cdot\mathtt{G}(\varpi,\Delta\upsilon,\Delta\vartheta)\cdot\mathtt{G}(\vartheta,\Delta\varpi,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}, \quad (4)\end{aligned}$$

for all $\varpi,\upsilon,\vartheta \in \Lambda$ with $(\varpi,\upsilon,\vartheta) \in \mathscr{G}$ and $0 \leq \mathfrak{p}+\mathfrak{q}+\mathfrak{r} < 1$. Then, there exists $\varpi \in \Lambda$ such that $\varpi \in \Lambda\varpi$.

Proof: It follows from the fact that (4) implies (1). ∎

Example 1. *Let the binary relation be defined on the interval $\Lambda = [0,5]$, $\mathscr{G} = \Big\{(0,0,0),(0,\frac{1}{2},0),(\frac{1}{2},1,\frac{1}{2}),(1,1,1),(1,\frac{3}{2},1),(\frac{3}{2},2,\frac{3}{2}),(2,2,2),(2,\frac{5}{2},2),(\frac{5}{2},3,\frac{5}{2}),(3,3,3),(3,\frac{7}{2},3),(\frac{7}{2},4,\frac{7}{2}),(4,4,4),(4,\frac{9}{2},4),(\frac{9}{2},5,\frac{9}{2})\Big\}$ and Euclidean metric $\mathtt{G}_3$; defined by*

$$\begin{aligned}&\mathtt{G}\big((\varpi_1,\varpi_2,\varpi_3),(\upsilon_1,\upsilon_2,\upsilon_3),(\upsilon_1,\upsilon_2,\upsilon_3)\big)\\ &\qquad = \sqrt{(\varpi_1-\upsilon_1)^2+(\varpi_2-\upsilon_2)^2+(\varpi_3-\upsilon_3)^2}\end{aligned}$$

then Λ is a complete $\mathtt{G}$-metric space. Define the function $\Delta : \Lambda \to \Lambda$ as follows

$$\Delta(\varpi_1,\varpi_2,\varpi_3) = \begin{cases}(\varpi_1,0,\varpi_3) & \text{if } \varpi_3 \geq \varpi_1 \geq \varpi_2\\ (0,\varpi_2,0) & \text{if } \varpi_3 \leq \varpi_1 < \varpi_2\end{cases}$$

We notice that

$$\begin{aligned}\mathtt{G}(\Delta\varpi,\Delta\upsilon,\Delta\varpi) &\leq \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\varpi)\\ &\quad + \mathfrak{q}\frac{\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}\\ &\quad + \mathfrak{r}\frac{\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\cdot\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)}{1+\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)}\end{aligned}$$

is not valid if $(\varpi,\upsilon,\varpi) \in \Big\{(0,2,0),(1,0,1),(0,2,0)\Big\}$.
As any given $\mathfrak{p},\mathfrak{q},\mathfrak{r} \in [0,1)$, we have

$$\begin{aligned}&\mathtt{G}\Big(\Delta(\frac{3}{2},2,\frac{3}{2}),\Delta(1,1,1),\Delta(\frac{3}{2},2,\frac{3}{2})\Big)\\ &\quad < \mathfrak{p}\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),(1,1,1),(\frac{3}{2},2,\frac{3}{2})\Big)\\ &\quad + \mathfrak{q}\left\{\frac{\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2})\Big)}{1+\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2})\Big)}\right.\\ &\qquad \left.\cdot\,\mathtt{G}\Big((1,1,1),\Delta(1,1,1),\Delta(1,1,1)\Big)\right\}\\ &\quad + \mathfrak{r}\left\{\frac{\mathtt{G}\Big((1,1,1),\Delta(\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2})\Big)}{1+\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2}),\Delta(\frac{3}{2},2,\frac{3}{2})\Big)}\right.\\ &\qquad \left.\cdot\,\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),\Delta(1,1,1),\Delta(1,1,1)\Big)\right\}\end{aligned}$$

$$\begin{aligned}&\mathtt{G}\Big((0,2,0),(1,\ 0,1),(0,2,0)\Big)\\ &\quad < \mathfrak{p}\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),(1,1,1),(\frac{3}{2},2,\frac{3}{2})\Big)\\ &\quad + \mathfrak{q}\left\{\frac{\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),(0,2,0),(0,2,0)\Big)}{1+\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),(0,2,0),(0,2,0)\Big)}\right.\\ &\qquad \left.\cdot\,\mathtt{G}\Big((1,1,1),(1,0,1),(2,0,2)\Big)\right\}\\ &\quad + \mathfrak{r}\left\{\frac{\mathtt{G}\Big((1,1,1),(0,2,0),(0,2,0)\Big)}{1+\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),(0,2,0),(0,2,0)\Big)}\right.\\ &\qquad \left.\cdot\,\mathtt{G}\Big((\frac{3}{2},2,\frac{3}{2}),(1,0,1),(1,0,1)\Big)\right\}\end{aligned}$$

$$\sqrt{6} < \mathfrak{p}\frac{\sqrt{6}}{2} + \mathfrak{q}\frac{3}{\sqrt{2}+3} + \mathfrak{r}\frac{3}{\sqrt{2}+3}$$

Therefore, the map Δ does not possess a fixed point in Λ . Then, for all $\varpi,\upsilon \in \Lambda$, the triplet $(\varpi,\upsilon,\varpi) \in \mathscr{G}$ satisfies our contraction condition. Similarly, the $\mathtt{G}$-continuity of Δ is easily verifiable.

Thus, by satisfying all of the conditions of the above Theorem 1, Δ possesses a unique fixed point at $(0,0,0)$.

Theorem 2. *Let $\Delta : \Lambda \to \Lambda$ be a mapping and $(\Lambda, \mathtt{G})$ be a complete $\mathtt{G}-$metric space such that*
(a) *$\Lambda(\Delta,\mathscr{G})$ is non-empty set;*
(b) *Δ is $\mathtt{G}$-continuous;*
(c) *$\mathtt{G}$ is Δ-closed;*
(d) *There exists $\mathfrak{p},\mathfrak{q},\mathfrak{r} \in [0,1)$ such that*

$$\begin{aligned}\mathtt{G}(\Delta\varpi,\Delta\upsilon,\Delta\upsilon) &\leq \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\upsilon)\\ &\quad + \mathfrak{q}\Big[\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)+\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\Big]\\ &\quad + \mathfrak{r}\Big[\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)+\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\Big] \quad (5)\end{aligned}$$

for all $\varpi,\upsilon \in \Lambda$ with $(\varpi,\upsilon,\upsilon) \in \mathscr{G}$ and $0 \leq \mathfrak{p}+\mathfrak{q}+\mathfrak{r} < 1$. Then, Δ has a fixed point.

Proof: Assume that (a) *and let us take* ϖ_0 *as arbitrary element of* $\Lambda(\Delta, \mathscr{G})$.
Formulate a sequence $\{\varpi_\sigma\}$ *that is*

$$\varpi_\sigma = \Lambda^\sigma(\varpi_0) \ for \ all \ \sigma \in \mathbb{N}_0.$$

Because $(\varpi_0, \Delta\varpi_0, \Delta\varpi_0) \in \mathscr{G}$ *and* $\mathtt{G}$ *is* Λ*-closed, we can apply Theorem 1 to obtain*

$$(\Delta^1\varpi_0, \Delta^2\varpi_0, \Delta^2\varpi_0), (\Delta^2\varpi_0, \Delta^3\varpi_0, \Delta^3\varpi_0), \\ ...., (\Delta^\sigma\varpi_0, \Delta^{\sigma+1}\varpi_0, \Delta^{\sigma+1}\varpi_0) \in \mathscr{G}$$

so that

$$(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) \in \mathscr{G} \ for \ all \ \sigma \in \mathbb{N}_0.$$

Then, the sequence $\{\varpi_\sigma\}$ *is* $\mathtt{G}$*- preserving.*
By applying the contractive condition (d)*, we obtain*

$$\begin{aligned} &\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) = \mathtt{G}(\Delta\varpi_{\sigma-1}, \Delta\varpi_\sigma, \Delta\varpi_\sigma) \\ &\quad \le \mathfrak{p}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\ &\qquad + \mathfrak{q}\big[\mathtt{G}(\varpi_{\sigma-1}, \Delta\varpi_{\sigma-1}, \Delta\varpi_{\sigma-1}) + \mathtt{G}(\varpi_\sigma, \Delta\varpi_\sigma, \Delta\varpi_\sigma)\big] \\ &\qquad + \mathfrak{r}\big[\mathtt{G}(\varpi_\sigma, \Delta\varpi_{\sigma-1}, \Delta\varpi_{\sigma-1}) + \mathtt{G}(\varpi_{\sigma-1}, \Delta\varpi_\sigma, \Delta\varpi_\sigma)\big] \end{aligned}$$

$$\begin{aligned} \mathtt{G}(\varpi_\sigma, \ \varpi_{\sigma+1}, \varpi_{\sigma+1}) &\le \mathfrak{p}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\ &\quad + \mathfrak{q}\big[\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) + \mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1})\big] \\ &\quad + \mathfrak{r}\big[\mathtt{G}(\varpi_\sigma, \varpi_\sigma, \varpi_\sigma) + \mathtt{G}(\varpi_{\sigma-1}, \varpi_{\sigma+1}, \varpi_{\sigma+1})\big] \\ &\le \mathfrak{p}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) + \mathfrak{q}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\ &\quad + \mathfrak{q}\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) + \mathfrak{r}\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) \end{aligned}$$

$$\begin{aligned} (1 - \mathfrak{q} - \mathfrak{r})\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) &\le (\mathfrak{p} + \mathfrak{q})\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\ \mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) &\le \left(\frac{\mathfrak{p} + \mathfrak{q}}{1 - \mathfrak{q} - \mathfrak{r}}\right)\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \end{aligned}$$

By induction, we have

$$\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) \le \left(\frac{\mathfrak{p} + \mathfrak{q}}{1 - \mathfrak{q} - \mathfrak{r}}\right)^\sigma \mathtt{G}(\varpi_0, \varpi_1, \varpi_1)$$

for all $\sigma \in \mathbb{N}_0$*. For any positive integers* σ, ϱ *satisfying the condition* $\sigma < \varrho$*, we have*

$$\begin{aligned} &\mathtt{G}(\varpi_\sigma, \varpi_\varrho, \ \varpi_\varrho) \\ &\qquad \le \mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) + ... + \mathtt{G}(\varpi_{\varrho-1}, \varpi_\varrho, \varpi_\varrho) \\ &\qquad \le \big(\delta^\sigma + ... + \delta^{\varrho-1}\big)\mathtt{G}(\varpi_0, \varpi_1, \varpi_1) \\ &\qquad \le \frac{\delta^\sigma}{1 - \delta}\mathtt{G}(\varpi_0, \varpi_1, \varpi_1), \ where \ \delta = \frac{\mathfrak{p} + \mathfrak{q}}{1 - \mathfrak{q} - \mathfrak{r}} \end{aligned}$$

Taking limit as $\sigma, \varrho \to \infty$*, we get*

$$\lim_{\sigma,\varrho\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\varrho) = 0. \tag{6}$$

To prove: $\{\varpi_\sigma\}$ *is a* $\mathtt{G}$*-Cauchy sequence.*

$$\mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\zeta) \le \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\varrho) + \mathtt{G}(\varpi_\varrho, \varpi_\varrho, \varpi_\zeta)$$

Taking limit as $\sigma, \varrho, \zeta \to \infty$*, we get*

$$\lim_{\sigma,\varrho,\zeta\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\zeta) = 0. \tag{7}$$

Therefore, $\{\varpi_\sigma\}$ *is a* $\mathtt{G}$*-Cauchy sequence and since* $(\Lambda, \mathtt{G})$ *is a complete* $\mathtt{G}$*-metric space, there exists an element* $\varpi \in \Lambda$*. As a result, we have*

$$\lim_{\sigma\to\infty} \varpi_\sigma = \varpi.$$

Since Λ *is* $\mathtt{G}$*- continuous, then*

$$\varpi_{\sigma+1} = \Lambda\varpi_\sigma \ \xrightarrow{\mathtt{G}} \ \Delta\varpi.$$

Therefore, $\Delta\varpi = \varpi$*.*
Hence, ϖ *is a fixed point of* Δ*.* ■

Theorem 3. *If* $\Delta(\Lambda)$ *is* $\mathtt{G}^{\varkappa}$*-connected in addition to the conditions of Theorem 2, then there exists a unique fixed point of* Δ*.*

Proof: Suppose ϖ *and* μ *are fixed points of the function* Δ *and belong to the set* $\mathscr{F}(\Delta)$ *then for all* $\sigma \in \mathbb{N}_0$*, we have*

$$\Delta^\sigma\varpi = \varpi, \ \Delta^\sigma\mu = \mu. \tag{8}$$

Given our assumption, there is a finite length path l *(denoted by* $\kappa_0, \kappa_1, ..., \kappa_l$*) in the* $\mathtt{G}^{\varkappa}$ *such that it connects* ϖ *to* μ*.*

$$\kappa_0 = \varpi, \ \kappa_l = \mu \ and \ \big[\kappa_i, \kappa_{i+1}\big] \in \mathscr{G} \tag{9}$$

As a consequence of the mapping being Δ*-closed, it satisfies both the properties of* $\mathtt{G}$*-completeness and* $\mathtt{G}$*-continuity*

$$(\Delta^\sigma\kappa_i, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1}) \in \mathscr{G}, \ for \ all \ i \ (0 \le \ i \ \le l - 1) \tag{10}$$

By applying the contractive condition (d)*, we obtain*

$$\begin{aligned} &\mathtt{G}(\Delta^\sigma\kappa_i, \Delta^\sigma\kappa_{i+1}, \ \Delta^\sigma\kappa_{i+1}) \\ &\qquad \le \mathfrak{p}\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\ &\qquad\quad + \mathfrak{q}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^\sigma\kappa_i, \Delta^\sigma\kappa_i) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1})\Big] \\ &\qquad\quad + \mathfrak{r}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_i, \Delta^\sigma\kappa_i) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1})\Big] \end{aligned}$$

$$\begin{aligned} &\mathtt{G}(\Delta^\sigma\kappa_i, \Delta^\sigma\kappa_{i+1}, \ \Delta^\sigma\kappa_{i+1}) \\ &\qquad \le \mathfrak{p}\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\ &\qquad\quad + \mathfrak{q}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^\sigma\kappa_i, \Delta^\sigma\kappa_i) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1})\Big] \\ &\qquad\quad + \mathfrak{r}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_i, \Delta^\sigma\kappa_i) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1})\Big] \end{aligned}$$

$$\begin{aligned} &\mathtt{G}(\Delta^\sigma\kappa_i, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1}) \\ &\qquad \le \mathfrak{p}\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\ &\qquad\quad + \mathfrak{q}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_i, \Delta^\sigma\kappa_i) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1})\Big] \\ &\qquad\quad + \mathfrak{r}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_i, \Delta^\sigma\kappa_i) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\ &\qquad\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^\sigma\kappa_{i+1}, \Delta^\sigma\kappa_{i+1})\Big] \end{aligned}$$

$$\begin{aligned}
&\mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}) \\
&\quad \leq \mathfrak{p}\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\
&\quad + \mathfrak{q}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\
&\quad + \mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\
&\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1})\Big] \\
&\quad + \mathfrak{r}\Big[\mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\
&\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}) \\
&\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1})\Big]
\end{aligned}$$

$$\begin{aligned}
&\mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}) \\
&\quad \leq \mathfrak{p}\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\
&\quad + \mathfrak{q}\Big[\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1}) \\
&\quad + \mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1})\Big] \\
&\quad + \mathfrak{r}\Big[\mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}) \\
&\quad + \mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1})
\end{aligned}$$

$$\begin{aligned}
&(1-\mathfrak{q}-\mathfrak{r})\mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}) \\
&\leq (\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1})
\end{aligned}$$

$$\begin{aligned}
&\mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1}) \\
&\quad \leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)\mathtt{G}(\Delta^{\sigma-1}\kappa_i, \Delta^{\sigma-1}\kappa_{i+1}, \Delta^{\sigma-1}\kappa_{i+1})
\end{aligned}$$

We introduce the notation $\mathtt{G}^i_\sigma = \mathtt{G}(\Delta^{\sigma}\kappa_i, \Delta^{\sigma}\kappa_{i+1}, \Delta^{\sigma}\kappa_{i+1})$. *From this definition, we can deduce the following results:*

$$\mathtt{G}^i_\sigma \leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)\mathtt{G}^i_{\sigma-1}, \ \textit{for each } i \ (0 \leq i \leq l-1) \quad (11)$$

By induction, we have

$$\begin{aligned}
\mathtt{G}^i_\sigma &\leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)\mathtt{G}^i_{\sigma-1} \\
&\leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)^2\mathtt{G}^i_{\sigma-2} \\
&\leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)^3\mathtt{G}^i_{\sigma-3} \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&\leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)^\sigma\mathtt{G}^i_{0}
\end{aligned}$$

so that

$$\mathtt{G}^i_\sigma \leq \left(\frac{\mathfrak{p}+\mathfrak{q}+\mathfrak{r}}{1-\mathfrak{q}-\mathfrak{r}}\right)^\sigma\mathtt{G}^i_{0} \quad (12)$$

Taking limit as $\sigma \to \infty$ *in the above inequality, we have*

$$\lim_{\sigma\to\infty}\mathtt{G}^i_\sigma = 0 \ \ \textit{for each } i \ (0 \leq \ i \ \leq l-1) \quad (13)$$

From the definition of rectangular inequality in (11)*, we obtain*

$$\begin{aligned}
\mathtt{G}(\varpi,\upsilon,\upsilon) &= \mathtt{G}(\Delta^{\sigma}\kappa_0, \Delta^{\sigma}\kappa_l, \Delta^{\sigma}\kappa_l) \\
&\leq \mathtt{G}^0_\sigma + \mathtt{G}^1_\sigma + ... + \mathtt{G}^{l-1}_\sigma \to 0 \ as \ \sigma \to \infty.
\end{aligned}$$

Hence, Δ *has a unique fixed point.* ■

Corollary 2. *Let* $\Delta : \Lambda \to \Lambda$ *be a mapping and* $(\Lambda, \mathtt{G})$ *be a complete* $\mathtt{G}-$*metric space such that*

(a) $\Lambda(\Delta, \mathscr{G})$ *is non-empty set;*
(b) Δ *is* $\mathtt{G}$*-continuous;*
(c) $\mathtt{G}$ *is* Δ*-closed;*
(d) *There exists* $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in [0, 1)$ *such that*

$$\begin{aligned}
&\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\vartheta) \\
&\leq \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\vartheta) \\
&+ \mathfrak{q}\Big[\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi) + \mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon) + \mathtt{G}(\vartheta,\Delta\vartheta,\Delta\vartheta)\Big] \\
&+ \mathfrak{r}\Big[\mathtt{G}(\varpi,\Delta\upsilon,\Delta\vartheta) + \mathtt{G}(\upsilon,\Delta\varpi,\Delta\vartheta) + \mathtt{G}(\vartheta,\Delta\varpi,\Delta\upsilon)\Big]
\end{aligned} \quad (14)$$

for all $\varpi, \upsilon, \vartheta \in \Lambda$ *with* $(\varpi, \upsilon, \vartheta) \in \mathscr{G}$ *and* $0 \leq \mathfrak{p}+\mathfrak{q}+\mathfrak{r} < 1$. *Then,* Δ *has a fixed point.*

Proof: Assume that (a) and let us take ϖ_0 as arbitrary element of $\Lambda(\Delta, \mathscr{G})$.
Formulate a sequence $\{\varpi_\sigma\}$ that is

$$\varpi_\sigma = \Lambda^\sigma(\varpi_0) \ \ for \ all \ \sigma \in \mathbb{N}_0.$$

Because $(\varpi_0, \Delta\varpi_0, \Delta\varpi_0) \in \mathscr{G}$ and $\mathtt{G}$ is Λ-closed, we can apply Theorem 1 to obtain

$$\begin{aligned}
&(\Delta^1\varpi_0, \Delta^2\varpi_0, \ \Delta^2\varpi_0), (\Delta^2\varpi_0, \Delta^3\varpi_0, \Delta^3\varpi_0), \\
&\qquad ...., (\Delta^\sigma\varpi_0, \Delta^{\sigma+1}\varpi_0, \Delta^{\sigma+1}\varpi_0) \in \mathscr{G}
\end{aligned}$$

so that

$$(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) \in \mathscr{G} \ \ for \ all \ \sigma \in \mathbb{N}_0.$$

Then, the sequence $\{\varpi_\sigma\}$ is $\mathtt{G}$- preserving.
By applying the contractive condition (d), we obtain

$$\begin{aligned}
&\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) = \mathtt{G}(\Delta\varpi_{\sigma-1}, \Delta\varpi_\sigma, \Delta\varpi_\sigma) \\
&\leq \mathfrak{p}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\
&\quad + \mathfrak{q}\big[\mathtt{G}(\varpi_{\sigma-1}, \Delta\varpi_{\sigma-1}, \Delta\varpi_{\sigma-1}) + 2\mathtt{G}(\varpi_\sigma, \Delta\varpi_\sigma, \Delta\varpi_\sigma)\big] \\
&\quad + \mathfrak{r}\big[2\mathtt{G}(\varpi_\sigma, \Delta\varpi_{\sigma-1}, \Delta\varpi_{\sigma-1}) + \mathtt{G}(\varpi_{\sigma-1}, \Delta\varpi_\sigma, \Delta\varpi_\sigma)\big]
\end{aligned}$$

$$\begin{aligned}
\mathtt{G}(\varpi_\sigma, \ \varpi_{\sigma+1}, \varpi_{\sigma+1}) &\leq \mathfrak{p}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\
&\quad + \mathfrak{q}\big[\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) + 2\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1})\big] \\
&\quad + \mathfrak{r}\big[2\mathtt{G}(\varpi_\sigma, \varpi_\sigma, \varpi_\sigma) + \mathtt{G}(\varpi_{\sigma-1}, \varpi_{\sigma+1}, \varpi_{\sigma+1})\big] \\
&\leq \mathfrak{p}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) + \mathfrak{q}\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma) \\
&\quad + 2\mathfrak{q}\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) + \mathfrak{r}\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1})
\end{aligned}$$

$$\begin{aligned}
&(1-2\mathfrak{q}-\mathfrak{r})\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \ \varpi_{\sigma+1}) \\
&\qquad\qquad \leq (\mathfrak{p}+\mathfrak{q})\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma)
\end{aligned}$$

$$\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) \leq \left(\frac{\mathfrak{p}+\mathfrak{q}}{1-2\mathfrak{q}-\mathfrak{r}}\right)\mathtt{G}(\varpi_{\sigma-1}, \varpi_\sigma, \varpi_\sigma)$$

By induction, we have

$$\mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) \leq \left(\frac{\mathfrak{p}+\mathfrak{q}}{1-2\mathfrak{q}-\mathfrak{r}}\right)^{\sigma} \mathtt{G}(\varpi_0, \varpi_1, \varpi_1)$$

for all $\sigma \in \mathbb{N}_0$. For any positive integers σ, ϱ satisfying the condition $\sigma < \varrho$, we have

$$\begin{aligned}
\mathtt{G}(\varpi_\sigma, \varpi_\varrho,\ \varpi_\varrho) &\\
&\leq \mathtt{G}(\varpi_\sigma, \varpi_{\sigma+1}, \varpi_{\sigma+1}) + ... + \mathtt{G}(\varpi_{\varrho-1}, \varpi_\varrho, \varpi_\varrho)\\
&\leq \left(\delta^\sigma + ... + \delta^{\varrho-1}\right)\mathtt{G}(\varpi_0, \varpi_1, \varpi_1)\\
&\leq \frac{\delta^\sigma}{1-\delta}\mathtt{G}(\varpi_0, \varpi_1, \varpi_1),\ \textit{where}\ \delta = \frac{\mathfrak{p}+\mathfrak{q}}{1-2\mathfrak{q}-\mathfrak{r}}
\end{aligned}$$

Taking limit as $\sigma, \varrho \to \infty$, we get

$$\lim_{\sigma,\varrho\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\varrho) = 0. \tag{15}$$

To prove: $\{\varpi_\sigma\}$ is a G-Cauchy sequence.

$$\mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\zeta) \leq \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\varrho) + \mathtt{G}(\varpi_\varrho, \varpi_\varrho, \varpi_\zeta)$$

Taking limit as $\sigma, \varrho, \zeta \to \infty$, we get

$$\lim_{\sigma,\varrho,\zeta\to\infty} \mathtt{G}(\varpi_\sigma, \varpi_\varrho, \varpi_\zeta) = 0. \tag{16}$$

Therefore, $\{\varpi_\sigma\}$ is a G-Cauchy sequence and since $(\Lambda, \mathtt{G})$ is a complete G-metric space, there exists an element $\varpi \in \Lambda$. As a result, we have

$$\lim_{\sigma\to\infty} \varpi_\sigma = \varpi.$$

Since Λ is G- continuous, then

$$\varpi_{\sigma+1} = \Lambda\varpi_\sigma \xrightarrow{\mathtt{G}} \Delta\varpi.$$

Therefore, $\Delta\varpi = \varpi$.
Hence, ϖ is a fixed point of Δ.

■

Corollary 3. *If $\Delta(\Lambda)$ is $\mathtt{G}^{\varkappa}$-connected in addition to the conditions of Corollary 2, then there exists a unique fixed point of Δ.*

Proof: It follows from the fact that Corollary 3 implies Corollary 2. ■

Example 2. *Let the binary relation be defined on the interval* $\Lambda = [0,5]$, $\mathscr{G} = \Big\{(0,0,0), (0, \frac{1}{2}, 0), (\frac{1}{2}, 1, \frac{1}{2}), (1,1,1), (1, \frac{3}{2}, 1), (\frac{3}{2}, 2, \frac{3}{2}), (2,2,2), (2, \frac{5}{2}, 2), (\frac{5}{2}, 3, \frac{5}{2}), (3,3,3), (3, \frac{7}{2}, 3), (\frac{7}{2}, 4, \frac{7}{2}), (4,4,4), (4, \frac{9}{2}, 4), (\frac{9}{2}, 5, \frac{9}{2})\Big\}$ *and Euclidean metric* $\mathtt{G}_3$; *defined by*

$$\begin{aligned}
\mathtt{G}\big((\varpi_1, \varpi_2, \varpi_3), (\upsilon_1, \upsilon_2, \upsilon_3), (\upsilon_1, \upsilon_2, \upsilon_3)\big)&\\
&= \sqrt{(\varpi_1-\upsilon_1)^2 + (\varpi_2-\upsilon_2)^2 + (\varpi_3-\upsilon_3)^2}
\end{aligned}$$

then Λ is a complete G- metric space.
Define the function $\Delta : \Lambda \to \Lambda$ as follows

$$\Delta(\varpi_1, \varpi_2, \varpi_3) = \begin{cases} (\varpi_1, 0, \varpi_3) & \textit{if } \varpi_3 \geq \varpi_1 \geq \varpi_2 \\ (0, \varpi_2, 0) & \textit{if } \varpi_3 \leq \varpi_1 < \varpi_2 \end{cases}$$

We notice that

$$\begin{aligned}
\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \leq \mathfrak{p}\mathtt{G}(\varpi, \upsilon, \upsilon)&\\
&+ \mathfrak{q}\Big[\mathtt{G}(\varpi, \Delta\varpi, \Delta\varpi) + \mathtt{G}(\upsilon, \Delta\upsilon, \Delta\upsilon)\Big]\\
&+ \mathfrak{r}\Big[\mathtt{G}(\varpi, \Delta\upsilon, \Delta\upsilon) + \mathfrak{r}\mathtt{G}(\upsilon, \Delta\varpi, \Delta\varpi)\Big]
\end{aligned}$$

is not valid if $(\varpi, \upsilon, \varpi) \in \Big\{(0,2,0), (1,0,1), (1,0,1)\Big\}$.
As any given $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in [0,1)$, *we have*

$$\begin{aligned}
\mathtt{G}\Big(\Delta(\frac{3}{2}, 2, \frac{3}{2}), &\Delta(1,1,1), \Delta(1,1,1)\Big)\\
&< \mathfrak{p}\mathtt{G}\Big((\frac{3}{2}, 2, \frac{3}{2}), (1,1,1), (1,1,1)\Big)\\
&\quad + \mathfrak{q}\Bigg\{\mathtt{G}\Big((\frac{3}{2}, 2, \frac{3}{2}), \Delta(\frac{3}{2}, 2, \frac{3}{2}), \Delta(\frac{3}{2}, 2, \frac{3}{2})\Big)\\
&\qquad + \mathtt{G}\Big((1,1,1), \Delta(1,1,1), \Delta(1,1,1)\Big)\Bigg\}\\
&\quad + \mathfrak{r}\Bigg\{\mathtt{G}\Big((1,1,1), \Delta(\frac{3}{2}, 2, \frac{3}{2}), \Delta(\frac{3}{2}, 2, \frac{3}{2})\Big)\\
&\qquad + \mathtt{G}\Big((\frac{3}{2}, 2, \frac{3}{2}), \Delta(1,1,1), \Delta(1,1,1)\Big)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\mathtt{G}\Big((0,2,0), &(1,0,1), (1,0,1)\Big)\\
&< \mathfrak{p}\mathtt{G}\Big((\frac{3}{2}, 2, \frac{3}{2}), (1,1,1), (1,1,1)\Big)\\
&\quad + \mathfrak{q}\Bigg\{\mathtt{G}\Big((\frac{3}{2}, 2, \frac{3}{2}), (0,2,0), (0,2,0)\Big)\\
&\qquad + \mathtt{G}\Big((1,1,1), (1,0,1), (1,0,1)\Big)\Bigg\}\\
&\quad + \mathfrak{r}\Bigg\{\mathtt{G}\Big((1,1,1), (0,2,0), (0,2,0)\Big)\\
&\qquad + \mathtt{G}\Big((\frac{3}{2}, 2, \frac{3}{2}), (1,0,1), (1,0,1)\Big)\Bigg\}
\end{aligned}$$

$$\sqrt{6} < \mathfrak{p}\frac{\sqrt{6}}{2} + \mathfrak{q}\frac{3\sqrt{2}+2}{2} + \mathfrak{r}\frac{3\sqrt{2}+2\sqrt{3}}{2}$$

Therefore, the map Δ does not possess a fixed point in Λ . Then, for all $\varpi, \upsilon \in \Lambda$, the triplet $(\varpi, \upsilon, \varpi) \in \mathscr{G}$ satisfies our contraction condition. Similarly, the G-continuity and $\Delta(\Lambda) is$ $\mathtt{G}^{\varkappa}$-connected of Δ is easily verifiable.

Thus, by satisfying all of the conditions of the above Theorem 2 and 3, Δ possesses a unique fixed point at $(0,0,0)$.

IV. An Application

In order to apply our main findings, an example of the $\mathcal{FOBVP}$ sol. that incorporates a binary relation is given to illustrate its application. The problem is expressed as:

$$\varpi'(\theta) = \mathfrak{f}(\theta, \varpi(\theta));\ \theta \in \mathscr{I} = [0, \Delta];\ \varpi(0) = \varpi(\Delta). \tag{17}$$

Assuming $\Delta > 0$, the function $\mathfrak{f} : \Lambda \times \mathscr{G} \to \mathscr{G}$ is continuous.

Definition 12. *A function $\lambda \in \varpi^1(\Lambda)$ is referred to as a lower solution of* (17) *if the following holds true:*

$$\begin{aligned}
&\varpi(0) \leq \varpi(\Delta)\\
&\lambda'(\theta) \leq \mathfrak{f}(\theta, \lambda(\theta)), \quad \theta \in \Lambda.
\end{aligned} \tag{18}$$

Definition 13. *A function $\lambda \in \varpi^1(\Lambda)$ is referred to as a upper solution of* (17) *if the following holds true:*

$$\begin{aligned}
&\varpi(0) \geq \varpi(\Delta)\\
&\lambda'(\theta) \geq \mathfrak{f}(\theta, \lambda(\theta)), \quad \theta \in \Lambda.
\end{aligned} \tag{19}$$

Theorem 4. *Given the $\mathcal{FOBVP}$ of* (17), *there exists a constant $\varpi > 0$ that is applicable to all $\varpi, \upsilon \in \Lambda$ such that $\varpi \leq \upsilon$*

$$\begin{aligned} 0 &\leq \left[(\mathfrak{q}+\mathfrak{r})\mathfrak{f}(\theta,\upsilon)+\mathfrak{p}\upsilon\right]-\left[(\mathfrak{q}+\mathfrak{r})\mathfrak{f}(\theta,\varpi)+\mathfrak{p}\varpi\right] \\ &\leq \mathfrak{p}\left[\mathfrak{f}(\upsilon-\varpi)\right]+\mathfrak{q}\left[\mathfrak{f}(\Delta\varpi-\varpi)+\mathfrak{f}(\Delta\upsilon-\upsilon)\right] \\ &\quad +\mathfrak{r}\left[\mathfrak{f}(\Delta\varpi-\upsilon)+\mathfrak{f}(\Delta\upsilon-\varpi)\right] \end{aligned} \tag{20}$$

Then, Δ has a unique solution.

Proof: The equation can be seen as a result of a $\mathcal{FOBVP}$

$$\begin{aligned} \mathfrak{r}\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big)+\mathfrak{q}\varpi'(\theta)+\mathfrak{p}\varpi(\theta) &= \mathfrak{r}\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big) \\ &\quad +\mathfrak{q}\mathfrak{f}\big(\varpi(\theta),\varpi^{\sigma}(\theta)\big)+\mathfrak{p}\varpi(\theta) \end{aligned} \tag{21}$$

where, $\theta \in \mathfrak{I} = [0,\Delta]$ and $\varpi(0) = \varpi(\Delta)$.
The integral equation is equivalent to the equation, as derived from the above problem.

$$\begin{aligned} \varpi(\theta) &= \int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{r}\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\theta),\varpi^{\sigma}(\theta)\big)+\mathfrak{p}\varpi(\theta)\Big]\psi\varkappa \end{aligned}$$

where

$$\mathtt{G}(\theta,\varkappa) = \begin{cases} \dfrac{\mathfrak{e}^{\mathfrak{p}(\Delta+\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1} & \textit{if } 0 \leq \varkappa < \theta \leq \Delta \\ \dfrac{\mathfrak{e}^{\mathfrak{q}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1} & \textit{if } 0 \leq \theta < \varkappa \leq \Delta \\ \dfrac{\mathfrak{e}^{\mathfrak{r}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1} & \textit{if } 0 \leq \theta < \varkappa \leq \Delta \end{cases}$$

we can define a mapping $\Delta : \lambda(\Lambda) \to \lambda(\Lambda)$ and binary relation using the following expression

$$\begin{aligned} \varpi(\theta) &= \int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{r}\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\theta),\varpi^{\sigma}(\theta)\big)+\mathfrak{p}\varpi(\theta)\Big]\psi\varkappa \end{aligned}$$

$$\begin{aligned} \mathtt{G} = \Big\{&(\varpi,\upsilon,\upsilon) \in \lambda(\Lambda)\times\lambda(\Lambda)\times\lambda(\Lambda) \\ &: \varpi(\theta) \leq \upsilon(\theta) \textit{ for all } \theta \in \Lambda\Big\} \end{aligned}$$

(i) $\mathtt{G}(\varpi,\upsilon,\upsilon) = 2\sup_{\theta\in\Lambda}\left|\varpi(\theta)-\upsilon(\theta)\right|$ is the sup $\mathtt{G}-$metric with $\lambda(\Lambda)$ and the complete $\mathtt{G}-$metric space is $\varpi,\upsilon,\upsilon \in \lambda(\Lambda)$ and hence $(\lambda(\Lambda),\mathtt{G})$ is $\mathtt{G}$-complete.
(ii) By choosing a sequence $\{\varpi_\sigma\}$ that is $\mathtt{G}$-preserving such that $\varpi_\sigma \xrightarrow{\mathtt{G}} \varpi$, for all $\theta \in \Lambda$, then

$$\varpi_0(\theta) \leq \varpi_1(\theta) \leq .... \leq \varpi_\sigma(\theta) \leq \varpi_{\sigma+1} \leq ....$$

and convergent to $\varpi(\theta)$ which implies $\varpi_\sigma(\theta) \leq \upsilon(\theta)$ for all $\theta \in \Lambda, \sigma \in \mathbb{N}_0$, which implies $\left[\varpi_\sigma,\upsilon,\upsilon\right] \in \mathscr{G}$ for all $\sigma \in \mathbb{N}_0$.
Hence, $\mathtt{G}$-continuous.
(iii) Consider a lower solution $\alpha \in \lambda^1(\Lambda)$ of (18)*, then*

$$\begin{aligned} \mathfrak{r}\mathfrak{f}\big(\varpi(\theta),&\upsilon^{\sigma}(\theta)\big)+\mathfrak{q}\varpi'(\theta)+\mathfrak{p}\varpi(\theta) \\ &= \mathfrak{r}\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big)+\mathfrak{q}\mathfrak{f}\big(\varpi(\theta),\varpi^{\sigma}(\theta)\big)+\mathfrak{p}\varpi(\theta) \end{aligned}$$

for all $\theta \in \Lambda$.
Multiplying by $\mathfrak{e}^{\mathfrak{p}\theta+\mathfrak{q}\theta+\mathfrak{r}\theta}$, we have

$$\begin{aligned} \Big(\varpi(\theta)\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\theta}\Big)' &\leq \Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\theta),\varpi^{\sigma}(\theta)\big)+\mathfrak{p}\varpi(\theta)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\theta} \\ &\quad + \mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\theta} \ \forall\ \theta \in \Lambda, \end{aligned}$$

it follows that

$$\begin{aligned} \varpi(\theta)\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\theta} &\leq \varpi(0) \\ &\quad + \int_0^{\theta} \mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}\psi\varkappa \\ &\quad + \int_0^{\theta} \Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}\psi\varkappa \end{aligned}$$

As $\varpi(0) \leq \varpi(\Delta)$,

$$\begin{aligned} \varpi(0)\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})} &\leq \varpi(\Delta)\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})} \\ &\leq \varpi(0) + \int_0^{\Delta} \mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}\psi\varkappa \\ &\quad + \int_0^{\Delta} \Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}\psi\varkappa \end{aligned}$$

Thus,

$$\begin{aligned} \varpi(0) &\leq \int_0^{\Delta} \frac{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}}{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}}{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \end{aligned}$$

$$\begin{aligned} \varpi(\theta)\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\theta} &\leq \int_0^{\Delta} \frac{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}}{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}}{\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad + \int_0^{\theta} \mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}\psi\varkappa \\ &\quad + \int_0^{\theta} \Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\varkappa}\psi\varkappa \end{aligned}$$

$$\begin{aligned} \varpi(\theta)\mathfrak{e}^{(\mathfrak{p}+\mathfrak{q}+\mathfrak{r})\theta} &\leq \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{p}(\varkappa)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{p}(\varkappa)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{q}(\varkappa)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{q}(\varkappa)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{r}(\varkappa)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{r}(\varkappa)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \end{aligned}$$

so that,

$$\begin{aligned} \varpi(\theta) &\leq \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{p}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{p}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{q}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{q}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{r}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa),\upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad + \int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{r}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa),\varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \end{aligned}$$

$$\begin{aligned}\varpi(\theta) \leq &\int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{p}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{q}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{r}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{p}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{q}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \frac{\mathfrak{e}^{\mathfrak{r}(\varkappa-\theta)}}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa\end{aligned}$$

$$\begin{aligned}\varpi(\theta) \leq &\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ =&(\Delta\varpi)(\theta)\end{aligned}$$

that is $(\omega(\theta), \Delta\omega(\theta), \Delta\omega(\theta)) \in \mathscr{G}$ *for any* $\theta \in \Lambda$*, this means that* $\lambda(\Delta, \mathtt{G}) \neq \phi$.
(iv) For any $(\varpi, \upsilon, \upsilon) \in \mathscr{G}$*, that is* $\varpi(\theta) \leq \upsilon(\theta)$

$$\begin{aligned}&\mathfrak{r}\mathfrak{f}\big(\varpi(\theta),\upsilon^{\sigma}(\theta)\big)+\mathfrak{q}\mathfrak{f}\big(\varpi(\theta), \varpi^{\sigma}(\theta)\big)+\mathfrak{p}\varpi \\ &\quad\leq \mathfrak{r}\mathfrak{f}\big(\varpi(\theta), \upsilon^{\sigma}(\theta)\big)+\mathfrak{q}\mathfrak{f}\big(\upsilon(\theta), \upsilon^{\sigma}(\theta)\big)+\mathfrak{p}\varpi(\theta)\end{aligned}$$

and $\mathtt{G}(\theta,\varkappa) > 0$ *for* $(\theta,\varkappa) \in \Lambda \times \Lambda$,

$$\begin{aligned}(\Delta\varpi)(\theta) = &\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ \leq &\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &+\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\upsilon(\varkappa), \upsilon^{\sigma}(\varkappa)\big)+\mathfrak{p}\upsilon(\varkappa)\Big]\psi\varkappa \\ =&(\Delta\upsilon)(\theta) \quad for\ all\ \theta \in \Lambda,\end{aligned}$$

which implies that $(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \in \mathscr{G}$*, that is* $\mathtt{G}$ *is* Δ *closed.*
(v) For all $(\varpi, \upsilon, \upsilon) \in \mathscr{G}$,

$$\begin{aligned}&\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) = 2\sup_{\theta\,\in\,\Lambda}\big|(\Delta\varpi)(\theta)-(\Delta\upsilon)(\theta)\big| \\ &= 2\sup_{\theta\,\in\,\Lambda}(\Delta\upsilon)(\theta)-(\Delta\varpi)(\theta) \\ &\leq \sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad+2\sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)+\mathfrak{p}\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad-2\sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\mathfrak{r}\Big[\mathfrak{f}\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad-2\sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\Big[\mathfrak{q}\mathfrak{f}\big(\upsilon(\varkappa), \upsilon^{\sigma}(\varkappa)\big)+\mathfrak{p}\upsilon(\varkappa)\Big]\psi\varkappa\end{aligned}$$

$$\begin{aligned}&\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \\ &\leq 2\sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa)\mathfrak{p}^2\Big[\upsilon(\varkappa)-\varpi(\varkappa)\Big]\psi\varkappa \\ &\quad+2\sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa) \\ &\qquad\qquad \mathfrak{q}^2\Big[\big(\upsilon(\varkappa), \upsilon^{\sigma}(\varkappa)\big)-\big(\varpi(\varkappa), \varpi^{\sigma}(\varkappa)\big)\Big]\psi\varkappa \\ &\quad+2\sup_{\theta\,\in\,\Lambda}\int_0^{\Delta} \mathtt{G}(\theta,\varkappa) \\ &\qquad\qquad \mathfrak{r}^2\Big[\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)-\big(\varpi(\varkappa), \upsilon^{\sigma}(\varkappa)\big)\Big]\psi\varkappa\end{aligned}$$

$$\begin{aligned}&\leq \mathfrak{p}^2\mathtt{G}(\varpi,\upsilon,\upsilon)\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa \\ &\quad+\mathfrak{q}^2\Big[\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)+\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\Big]\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa \\ &\quad+\mathfrak{r}^2\Big[\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)+\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)\Big]\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa\end{aligned}$$

$$\begin{aligned}\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \leq &\,\mathfrak{p}^2\mathtt{G}(\varpi,\upsilon,\upsilon)\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa \\ &+\mathfrak{q}^2\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa \\ &+\mathfrak{q}^2\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa \\ &+\mathfrak{r}^2\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa \\ &+\mathfrak{r}^2\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)\int_0^{\Delta}\mathtt{G}(\theta,\varkappa)\psi\varkappa\end{aligned}$$

$$\begin{aligned}&\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \\ &\leq \mathfrak{p}^2\mathtt{G}(\varpi,\upsilon,\upsilon) \\ &\qquad \sup_{\theta\,\in\Lambda}\frac{1}{\mathfrak{e}^{-\mathfrak{p}\theta}-1}\left(\frac{1}{p}\mathfrak{e}^{p(\Delta+\varkappa-\theta)}\Big|_0^{\theta}+\frac{1}{p}\mathfrak{e}^{p(\varkappa-\theta)}\Big|_0^{\Delta}\right) \\ &\quad+\mathfrak{q}^2\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi) \\ &\qquad \sup_{\theta\,\in\Lambda}\frac{1}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\left(\frac{1}{q}\mathfrak{e}^{q(\varkappa-\theta)}\Big|_0^{\theta}+\frac{1}{q}\mathfrak{e}^{q(\varkappa-\theta)}\Big|_0^{\Delta}\right) \\ &\quad+\mathfrak{q}^2\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon) \\ &\qquad \sup_{\theta\,\in\Lambda}\frac{1}{\mathfrak{e}^{-\mathfrak{q}\theta}-1}\left(\frac{1}{q}\mathfrak{e}^{q(\varkappa-\theta)}\Big|_0^{\theta}+\frac{1}{q}\mathfrak{e}^{q(\varkappa-\theta)}\Big|_0^{\Delta}\right) \\ &\quad+\mathfrak{r}^2\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon) \\ &\qquad \sup_{\theta\,\in\,\Lambda}\frac{1}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\left(\frac{1}{r}\mathfrak{e}^{r(\varkappa-\theta)}\Big|_0^{\theta}+\frac{1}{r}\mathfrak{e}^{r(\varkappa-\theta)}\Big|_0^{\Delta}\right) \\ &\quad+\mathfrak{r}^2\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi) \\ &\qquad \sup_{\theta\,\in\,\Lambda}\frac{1}{\mathfrak{e}^{-\mathfrak{r}\theta}-1}\left(\frac{1}{r}\mathfrak{e}^{r(\varkappa-\theta)}\Big|_0^{\theta}+\frac{1}{r}\mathfrak{e}^{r(\varkappa-\theta)}\Big|_0^{\Delta}\right)\end{aligned}$$

$$\begin{aligned}
\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) &\le \mathfrak{p}^2\mathtt{G}(\varpi,\upsilon,\upsilon)\frac{1}{\mathfrak{p}(\mathfrak{e}^{-\mathfrak{p}\theta}-1)}\left(\mathfrak{e}^{-\mathfrak{p}\theta}-1\right)\\
&+ \mathfrak{q}^2\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)\frac{1}{\mathfrak{q}(\mathfrak{e}^{-\mathfrak{q}\theta}-1)}\left(\mathfrak{e}^{-\mathfrak{q}\theta}-1\right)\\
&+ \mathfrak{q}^2\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\frac{1}{\mathfrak{q}(\mathfrak{e}^{-\mathfrak{q}\theta}-1)}\left(\mathfrak{e}^{-\mathfrak{q}\theta}-1\right)\\
&+ \mathfrak{r}^2\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)\frac{1}{\mathfrak{r}(\mathfrak{e}^{-\mathfrak{r}\theta}-1)}\left(\mathfrak{e}^{-\mathfrak{r}\theta}-1\right)\\
&+ \mathfrak{r}^2\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\frac{1}{\mathfrak{r}(\mathfrak{e}^{-\mathfrak{r}\theta}-1)}\left(\mathfrak{e}^{-\mathfrak{r}\theta}-1\right)
\end{aligned}$$

$$\begin{aligned}
\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \le \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\upsilon) &+ \mathfrak{q}\Big[\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)+\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\Big]\\
&+ \mathfrak{r}\Big[\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)+\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\Big]
\end{aligned}$$

for all $\varpi, \upsilon \in \Lambda$. As a result of Theorem 2 and 3 as stated above, it follows that all necessary conditions have been fulfilled. Thus, Δ possesses a unique fixed point. ■

V. Numerical Example

To illustrate the importance of the obtained results, we provide a numerical example in this section.

Example 3. *Consider the following $\mathcal{FOBVP}$:*

$$\mathfrak{y}'(\mathfrak{t}) + \mathfrak{y}(\mathfrak{t}) = sin(\mathfrak{t}),\ 0 \le \mathfrak{t} \le 2\pi,\ \mathfrak{y}(0) = \mathfrak{y}(2\pi). \tag{22}$$

We define $\mathtt{S}$ as the set of all continuous real-valued functions on the closed interval $[0, 2\pi]$, i.e., $\mathtt{S} = \mathtt{C}\big([0, 2\pi], \mathtt{R}\big)$. Define $\mathtt{G} : \mathtt{S}\times\mathtt{S}\times\mathtt{S} \to [0,\infty]$ by

$$\mathtt{G}(\varpi,\upsilon,\upsilon) = 2\sup_{\mathfrak{t}\in[0,2\pi]}\big|\varpi(\mathfrak{t}) - \upsilon(\mathfrak{t})\big|. \tag{23}$$

Clearly, $(\mathtt{S}, \mathtt{G})$ is a complete $\mathtt{G}-$metric space.
Define a mapping $\Delta : \mathtt{S} \to \mathtt{S}$ by

$$\Delta\big(\mathfrak{f}(\mathfrak{t})\big) = \int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\mathfrak{f}(\mathfrak{t}))ds - \mathfrak{f}(\mathfrak{t}) + sin(\mathfrak{t}). \tag{24}$$

Assuming that the following conditions holds:

1. $\mathfrak{f}(\mathfrak{t})$ *is continuous.*
2. $\big|sin(\varpi(\mathfrak{t})) - sin(\upsilon(\mathfrak{t}))\big| \le \mathfrak{p}\big|\varpi(\mathfrak{t}) - \upsilon(\mathfrak{t})\big|$
3. $\big|sin(\varpi(\mathfrak{t})) - \Delta\big(\varpi(\mathfrak{t})\big)\big| \le \mathfrak{q}\big|\varpi(\mathfrak{t}) - \Delta\big(\varpi(\mathfrak{t})\big)\big|$
4. $\big|sin(\varpi(\mathfrak{t})) - \Delta\big(\upsilon(\mathfrak{t})\big)\big| \le \mathfrak{r}\big|\varpi(\mathfrak{t}) - \Delta\big(\upsilon(\mathfrak{t})\big)\big|$

Consider,

$$\begin{aligned}
\Big|\Delta\big(\varpi(\mathfrak{t})\big)-\Delta\big(\upsilon(\mathfrak{t})\big)\Big| &= \Big|\int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\varpi(\mathfrak{t}))ds - \mathfrak{f}(\mathfrak{t}) + sin(\mathfrak{t})\\
&- \int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\upsilon(\mathfrak{t}))ds + \mathfrak{f}(\mathfrak{t}) - sin(\mathfrak{t})\Big|
\end{aligned}$$

$$\begin{aligned}
\Big|\Delta\big(\varpi(\mathfrak{t})\big)-\Delta\big(\upsilon(\mathfrak{t})\big)\Big| &= \Big|\int_0^{\mathfrak{t}} \mathfrak{t}\big[sin(\varpi(\mathfrak{t})) - sin(\upsilon(\mathfrak{t}))\big]ds\\
&+ \int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\varpi(\mathfrak{t}))ds - \Delta\big(\varpi(\mathfrak{t})\big)\\
&+ \int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\upsilon(\mathfrak{t}))ds - \Delta\big(\upsilon(\mathfrak{t})\big)\\
&+ \Delta\big(\upsilon(\mathfrak{t})\big) - \int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\varpi(\mathfrak{t}))ds\\
&+ \Delta\big(\varpi(\mathfrak{t})\big) - \int_0^{\mathfrak{t}} \mathfrak{t}\ sin(\upsilon(\mathfrak{t}))ds\Big|
\end{aligned}$$

$$\begin{aligned}
\Big|\Delta\big(\varpi(\mathfrak{t})\big)-\Delta\big(\upsilon(\mathfrak{t})\big)\Big| < \Big|&sin(\varpi(\mathfrak{t})) - sin(\upsilon(\mathfrak{t}))\\
&+ sin(\varpi(\mathfrak{t})) - \Delta\big(\varpi(\mathfrak{t})\big)\\
&+ sin(\upsilon(\mathfrak{t})) - \Delta\big(\upsilon(\mathfrak{t})\big)\\
&+ \Delta\big(\upsilon(\mathfrak{t})\big) - sin(\varpi(\mathfrak{t}))\\
&+ \Delta\big(\varpi(\mathfrak{t})\big) - sin(\upsilon(\mathfrak{t}))\Big|
\end{aligned}$$

$$\begin{aligned}
\Big|\Delta\big(\varpi(\mathfrak{t})\big)-\Delta\big(\upsilon(\mathfrak{t})\big)\Big| \le &\Big|sin(\varpi(\mathfrak{t})) - sin(\upsilon(\mathfrak{t}))\Big|\\
&+ \Big|sin(\varpi(\mathfrak{t})) - \Delta\big(\varpi(\mathfrak{t})\big)\Big|\\
&+ \Big|sin(\upsilon(\mathfrak{t})) - \Delta\big(\upsilon(\mathfrak{t})\big)\Big|\\
&+ \Big|\Delta\big(\upsilon(\mathfrak{t})\big) - sin(\varpi(\mathfrak{t}))\Big|\\
&+ \Big|\Delta\big(\varpi(\mathfrak{t})\big) - sin(\upsilon(\mathfrak{t}))\Big|
\end{aligned}$$

$$\begin{aligned}
\big|\Delta\big(\varpi(\mathfrak{t})\big)-\Delta\big(\upsilon(\mathfrak{t})\big)\big| \le \mathfrak{p}\big|\varpi(\mathfrak{t}) - \upsilon(\mathfrak{t})\big|&\\
+ \mathfrak{q}\big|\varpi(\mathfrak{t}) - \Delta\big(\varpi(\mathfrak{t})\big)\big|&\\
+ \mathfrak{q}\big|\upsilon(\mathfrak{t}) - \Delta\big(\upsilon(\mathfrak{t})\big)\big|&\\
+ \mathfrak{r}\big|\varpi(\mathfrak{t}) - \Delta\big(\upsilon(\mathfrak{t})\big)\big|&\\
+ \mathfrak{r}\big|\upsilon(\mathfrak{t}) - \Delta\big(\varpi(\mathfrak{t})\big)\big|&
\end{aligned}$$

$$\begin{aligned}
2\sup_{\mathfrak{t}\in[0,2\pi]}&\big|\Delta\big(\varpi(\mathfrak{t})\big)-\Delta\big(\upsilon(\mathfrak{t})\big)\big|\\
&\le 2\sup_{\mathfrak{t}\in[0,2\pi]}\mathfrak{p}\big|\varpi(\mathfrak{t}) - \upsilon(\mathfrak{t})\big|\\
&+ 2\sup_{\mathfrak{t}\in[0,2\pi]}\mathfrak{q}\big|\varpi(\mathfrak{t}) - \Delta\big(\varpi(\mathfrak{t})\big)\big|\\
&+ 2\sup_{\mathfrak{t}\in[0,2\pi]}\mathfrak{q}\big|\upsilon(\mathfrak{t}) - \Delta\big(\upsilon(\mathfrak{t})\big)\big|\\
&+ 2\sup_{\mathfrak{t}\in[0,2\pi]}\mathfrak{r}\big|\varpi(\mathfrak{t}) - \Delta\big(\upsilon(\mathfrak{t})\big)\big|\\
&+ 2\sup_{\mathfrak{t}\in[0,2\pi]}\mathfrak{r}\big|\upsilon(\mathfrak{t}) - \Delta\big(\varpi(\mathfrak{t})\big)\big|
\end{aligned}$$

$$\begin{aligned}
\mathtt{G}(\Delta\varpi, \Delta\upsilon, \Delta\upsilon) \le \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\upsilon) &+ \mathfrak{q}\Big[\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)+\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\Big]\\
&+ \mathfrak{r}\Big[\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)+\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\Big]
\end{aligned}$$

This leads us to the conclusion that all axioms of Theorem 2 and 3 are validated and as a result, the $\mathcal{FOBVP}$ sol. (22) *has a unique.*

The validity of our approach can be demonstrated by utilizing the iteration method to confirm that the exact solution of Equation (22) *is indeed* $x(\mathrm{t}) = \mathrm{t}$.

$$\begin{aligned} x_{n+1}(\mathrm{t}) &= \Delta\big(x_n(\mathrm{t})\big) \\ &= \int_0^{\mathrm{t}} \mathrm{t}\ sin(x_n(\mathrm{t}))ds - x_n(\mathrm{t}) + sin(x_n(\mathrm{t})) \end{aligned} \quad (25)$$

The examples are presented in Table I, II, III and IV showcasing the convergence of the sequence (25) *towards the exact solutions of* $0.25, 1.13, -0.85$ *and* -0.98, *as depicted in Fig. 1, Fig.2, Fig.3 and Fig. 4, respectively.*

Let us consider the initial solution as $x_0(\mathrm{t}) = 0$ *to commence the iterative process.*

Table I
FOR $t = 0.25$, THE EXACT SOLUTION IS $x(0.25) = 0.25$

n	$x_{n+1}(t = 0.25)$	Approximate Solution	Absolute Error
0	$x_1(t = 0.25)$	0.000000	2.5×10^{-1}.
1	$x_2(t = 0.25)$	0.250767	7.67×10^{-4}
2	$x_3(t = 0.25)$	0.250487	4.87×10^{-4}
3	$x_4(t = 0.25)$	0.250488	4.88×10^{-4}
4	$x_5(t = 0.25)$	0.250488	4.88×10^{-4}

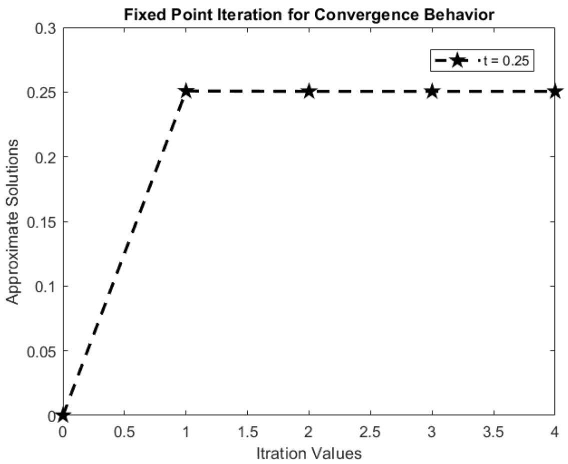

Figure 1. The graph shows that Eq. (25) converges to exact solution 0.25.

Table II
FOR $t = 1.13$, THE EXACT SOLUTION IS $x(1.13) = 1.13$

n	$x_{n+1}(t = 1.13)$	Approximate Solution	Absolute Error
0	$x_1(t = 1.13)$	1.195133	6.5133×10^{-2}.
1	$x_2(t = 1.13)$	1.130777	7.77×10^{-4}
2	$x_3(t = 1.13)$	1.130849	8.49×10^{-4}
3	$x_4(t = 1.13)$	1.130848	8.50×10^{-4}
4	$x_5(t = 1.13)$	1.130848	8.50×10^{-4}

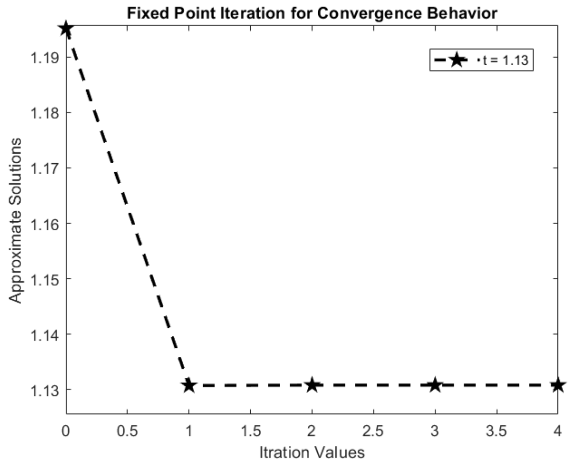

Figure 2. The graph shows that Eq. (25) converges to exact solution 1.13.

Table III
FOR $t = -0.85$, THE EXACT SOLUTION IS $x(-0.85) = -0.85$

n	$x_{n+1}(t = -0.85)$	Approximate Solution	Absolute Error
0	$x_1(t = -0.85)$	-0.849479	5.21×10^{-4}.
1	$x_2(t = -0.85)$	-0.853717	-3.717×10^{-3}
2	$x_3(t = -0.85)$	-0.853713	-3.713×10^{-3}
3	$x_4(t = -0.85)$	-0.853713	-3.713×10^{-3}
4	$x_5(t = -0.85)$	-0.853713	-3.713×10^{-3}

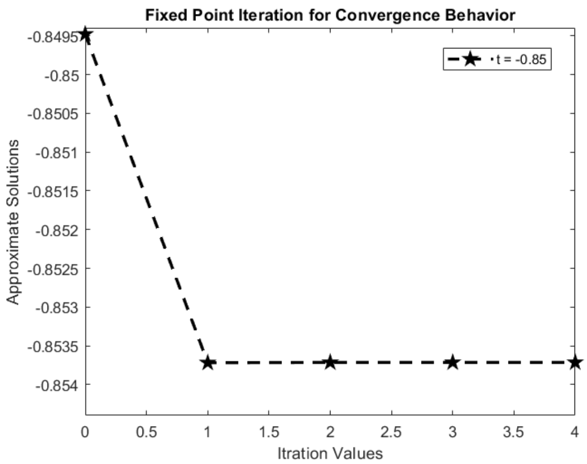

Figure 3. The graph shows that Eq. (25) converges to exact solution -0.85.

Table IV
FOR $t = -0.98$, THE EXACT SOLUTION IS $x(-0.98) = -0.98$

n	$x_{n+1}(t = -0.98)$	Approximate Solution	Absolute Error
0	$x_1(t = -0.98)$	-0.977125	2.875×10^{-3}.
1	$x_2(t = -0.98)$	-0.976037	3.963×10^{-3}
2	$x_3(t = -0.98)$	-0.976039	3.961×10^{-3}
3	$x_4(t = -0.98)$	-0.976039	3.961×10^{-3}
4	$x_5(t = -0.98)$	-0.976039	3.961×10^{-3}

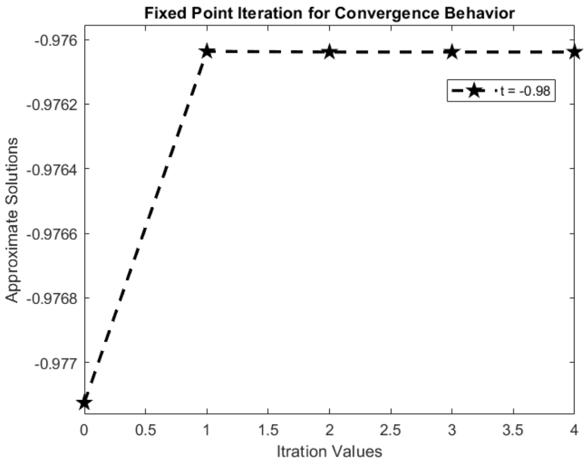

Figure 4. The graph shows that Eq. (25) converges to exact solution -0.98.

Example 4. *Consider the following $\mathcal{FOBVP}$:*

$$\mathfrak{y}'(\mathtt{t})+\mathfrak{y}(\mathtt{t})=cos(\mathtt{t}),\ 0\leq\mathtt{t}\leq2\pi,\ \mathfrak{y}(0)=\mathfrak{y}(2\pi). \tag{26}$$

We define $\mathtt{S}$ as the set of all continuous real-valued functions on the closed interval $[0,2\pi]$, i.e., $\mathtt{S}=\mathtt{C}\big([0,2\pi],\mathtt{R}\big)$. Define $\mathtt{G}:\mathtt{S}\times\mathtt{S}\times\mathtt{S}\to[0,\infty]$ by

$$\mathtt{G}(\varpi,\upsilon,\upsilon)=2\sup_{\mathtt{t}\in[0,2\pi]}\big|\varpi(\mathtt{t})-\upsilon(\mathtt{t})\big|. \tag{27}$$

Clearly, $(\mathtt{S},\mathtt{G})$ is a complete $\mathtt{G}-$metric space.
Define a mapping $\Delta:\mathtt{S}\to\mathtt{S}$ by

$$\Delta\big(\mathfrak{f}(\mathtt{t})\big)=\int_0^{\mathtt{t}}\mathtt{t}\ cos(\mathfrak{f}(\mathtt{t}))ds-\mathfrak{f}(\mathtt{t})+cos(\mathtt{t}). \tag{28}$$

Assuming that the following conditions holds:

1. $\mathfrak{f}(\mathtt{t})$ *is continuous.*
2. $\big|cos(\varpi(\mathtt{t}))-cos(\upsilon(\mathtt{t}))\big|\leq\mathfrak{p}\big|\varpi(\mathtt{t})-\upsilon(\mathtt{t})\big|$
3. $\big|cos(\varpi(\mathtt{t}))-\Delta\big(\varpi(\mathtt{t})\big)\big|\leq\mathfrak{q}\big|\varpi(\mathtt{t})-\Delta\big(\varpi(\mathtt{t})\big)\big|$
4. $\big|cos(\varpi(\mathtt{t}))-\Delta\big(\upsilon(\mathtt{t})\big)\big|\leq\mathfrak{r}\big|\varpi(\mathtt{t})-\Delta\big(\upsilon(\mathtt{t})\big)\big|$

Consider,

$$\begin{aligned}\Big|\Delta\big(\varpi(\mathtt{t})\big)-\Delta\big(\upsilon(\mathtt{t})\big)\Big| &=\Big|\int_0^{\mathtt{t}}\mathtt{t}\ cos(\varpi(\mathtt{t}))ds-\mathfrak{f}(\mathtt{t})+cos(\mathtt{t})\\&\quad-\int_0^{\mathtt{t}}\mathtt{t}\ cos(\upsilon(\mathtt{t}))ds+\mathfrak{f}(\mathtt{t})-cos(\mathtt{t})\Big|\end{aligned}$$

$$\begin{aligned}\Big|\Delta\big(\varpi(\mathtt{t})\big)-\Delta\big(\upsilon(\mathtt{t})\big)\Big| &=\Big|\int_0^{\mathtt{t}}\mathtt{t}\big[cos(\varpi(\mathtt{t}))-cos(\upsilon(\mathtt{t}))\big]ds\\&\quad+\int_0^{\mathtt{t}}\mathtt{t}\ cos(\varpi(\mathtt{t}))ds-\Delta\big(\varpi(\mathtt{t})\big)\\&\quad+\int_0^{\mathtt{t}}\mathtt{t}\ cos(\upsilon(\mathtt{t}))ds-\Delta\big(\upsilon(\mathtt{t})\big)\\&\quad+\Delta\big(\upsilon(\mathtt{t})\big)-\int_0^{\mathtt{t}}\mathtt{t}\ cos(\varpi(\mathtt{t}))ds\\&\quad+\Delta\big(\varpi(\mathtt{t})\big)-\int_0^{\mathtt{t}}\mathtt{t}\ cos(\upsilon(\mathtt{t}))ds\Big|\end{aligned}$$

$$\begin{aligned}\Big|\Delta\big(\varpi(\mathtt{t})\big)-\Delta\big(\upsilon(\mathtt{t})\big)\Big|<\Big|&cos(\varpi(\mathtt{t}))-cos(\upsilon(\mathtt{t}))\\&+cos(\varpi(\mathtt{t}))-\Delta\big(\varpi(\mathtt{t})\big)\\&+cos(\upsilon(\mathtt{t}))-\Delta\big(\upsilon(\mathtt{t})\big)\\&+\Delta\big(\upsilon(\mathtt{t})\big)-cos(\varpi(\mathtt{t}))\\&+\Delta\big(\varpi(\mathtt{t})\big)-cos(\upsilon(\mathtt{t}))\Big|\end{aligned}$$

$$\begin{aligned}\Big|\Delta\big(\varpi(\mathtt{t})\big)-\Delta\big(\upsilon(\mathtt{t})\big)\Big|\leq&\Big|cos(\varpi(\mathtt{t}))-cos(\upsilon(\mathtt{t}))\Big|\\&+\Big|cos(\varpi(\mathtt{t}))-\Delta\big(\varpi(\mathtt{t})\big)\Big|\\&+\Big|cos(\upsilon(\mathtt{t}))-\Delta\big(\upsilon(\mathtt{t})\big)\Big|\\&+\Big|\Delta\big(\upsilon(\mathtt{t})\big)-cos(\varpi(\mathtt{t}))\Big|\\&+\Big|\Delta\big(\varpi(\mathtt{t})\big)-cos(\upsilon(\mathtt{t}))\Big|\end{aligned}$$

$$\begin{aligned}\big|\Delta\big(\varpi(\mathtt{t})\big)-\Delta\big(\upsilon(\mathtt{t})\big)\big|\leq&\ \mathfrak{p}\big|\varpi(\mathtt{t})-\upsilon(\mathtt{t})\big|\\&+\mathfrak{q}\big|\varpi(\mathtt{t})-\Delta\big(\varpi(\mathtt{t})\big)\big|\\&+\mathfrak{q}\big|\upsilon(\mathtt{t})-\Delta\big(\upsilon(\mathtt{t})\big)\big|\\&+\mathfrak{r}\big|\varpi(\mathtt{t})-\Delta\big(\upsilon(\mathtt{t})\big)\big|\\&+\mathfrak{r}\big|\upsilon(\mathtt{t})-\Delta\big(\varpi(\mathtt{t})\big)\big|\end{aligned}$$

$$\begin{aligned}&2\sup_{\mathtt{t}\in[0,2\pi]}\big|\Delta\big(\varpi(\mathtt{t})\big)-\Delta\big(\upsilon(\mathtt{t})\big)\big|\\&\leq2\sup_{\mathtt{t}\in[0,2\pi]}\mathfrak{p}\big|\varpi(\mathtt{t})-\upsilon(\mathtt{t})\big|\\&\quad+2\sup_{\mathtt{t}\in[0,2\pi]}\mathfrak{q}\big|\varpi(\mathtt{t})-\Delta\big(\varpi(\mathtt{t})\big)\big|\\&\quad+2\sup_{\mathtt{t}\in[0,2\pi]}\mathfrak{q}\big|\upsilon(\mathtt{t})-\Delta\big(\upsilon(\mathtt{t})\big)\big|\\&\quad+2\sup_{\mathtt{t}\in[0,2\pi]}\mathfrak{r}\big|\varpi(\mathtt{t})-\Delta\big(\upsilon(\mathtt{t})\big)\big|\\&\quad+2\sup_{\mathtt{t}\in[0,2\pi]}\mathfrak{r}\big|\upsilon(\mathtt{t})-\Delta\big(\varpi(\mathtt{t})\big)\big|\end{aligned}$$

$$\begin{aligned}\mathtt{G}(\Delta\varpi,\Delta\upsilon,\Delta\upsilon)\leq&\ \mathfrak{p}\mathtt{G}(\varpi,\upsilon,\upsilon)\\&+\mathfrak{q}\Big[\mathtt{G}(\varpi,\Delta\varpi,\Delta\varpi)+\mathtt{G}(\upsilon,\Delta\upsilon,\Delta\upsilon)\Big]\\&+\mathfrak{r}\Big[\mathtt{G}(\varpi,\Delta\upsilon,\Delta\upsilon)+\mathtt{G}(\upsilon,\Delta\varpi,\Delta\varpi)\Big]\end{aligned}$$

This leads us to the conclusion that all axioms of Theorem 2 and 3 are validated and as a result, the $\mathcal{FOBVP}$ sol. (26) *has a unique.*

The validity of our approach can be demonstrated by utilizing the iteration method to confirm that the exact solution of Equation (26) *is indeed $x(\mathtt{t})=\mathtt{t}$.*

$$\begin{aligned}x_{n+1}(\mathtt{t})&=\Delta\big(x_n(\mathtt{t})\big)\\&=\int_0^{\mathtt{t}}\mathtt{t}\ cos(x_n(\mathtt{t}))ds-x_n(\mathtt{t})+cos(x_n(\mathtt{t}))\end{aligned} \tag{29}$$

The examples are presented in Table V, VI, VII and VIII showcasing the convergence of the sequence (29) *towards the exact solutions of* $1.11, 0.10, -0.75$ *and* 0.58, *as depicted in Fig. 5, Fig.6, Fig.7 and Fig. 8, respectively.*

Let us consider the initial solution as $x_0(\mathtt{t})=0$ to commence the iterative process.

Table V
FOR $t = 1.11$, THE EXACT SOLUTION IS $x(1.11) = 1.11$

n	$x_{n+1}(t = 1.11)$	Approximate Solution	Absolute Error
0	$x_1(t = 1.11)$	1.256637	1.46637×10^{-1}.
1	$x_2(t = 1.11)$	1.107490	2.510×10^{-3}
2	$x_3(t = 1.11)$	1.107656	2.344×10^{-3}
3	$x_4(t = 1.11)$	1.107656	2.344×10^{-3}
4	$x_5(t = 1.11)$	1.107656	2.344×10^{-3}

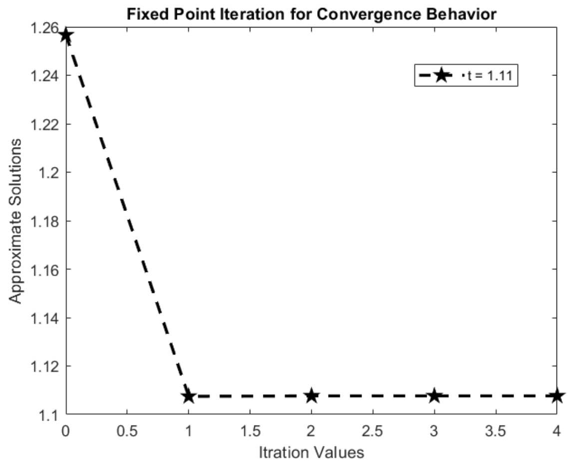

Figure 5. The graph shows that Eq. (29) converges to exact solution 1.11.

Table VI
FOR $t = 0.10$, THE EXACT SOLUTION IS $x(0.10) = 0.10$

n	$x_{n+1}(t = 1.10)$	Approximate Solution	Absolute Error
0	$x_1(t = 0.10)$	0.065823	3.4177×10^{-2}.
1	$x_2(t = 0.10)$	0.104099	4.099×10^{-3}
2	$x_3(t = 0.10)$	0.104057	4.057×10^{-3}
3	$x_4(t = 0.10)$	0.104057	4.057×10^{-3}
4	$x_5(t = 0.10)$	0.104057	4.057×10^{-3}

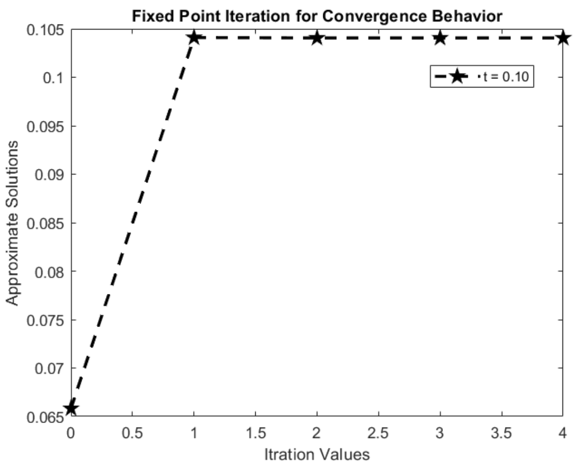

Figure 6. The graph shows that Eq. (29) converges to exact solution 0.10.

Table VII
FOR $t = -0.75$, THE EXACT SOLUTION IS $x(-0.75) = -0.75$

n	$x_{n+1}(t = -0.75)$	Approximate Solution	Absolute Error
0	$x_1(t = -0.75)$	-0.751398	-1.398×10^{-3}.
1	$x_2(t = -0.75)$	-0.748877	1.123×10^{-3}
2	$x_3(t = -0.75)$	-0.748880	1.120×10^{-3}
3	$x_4(t = -0.75)$	-0.748880	1.120×10^{-3}
4	$x_5(t = -0.75)$	-0.748880	1.120×10^{-3}

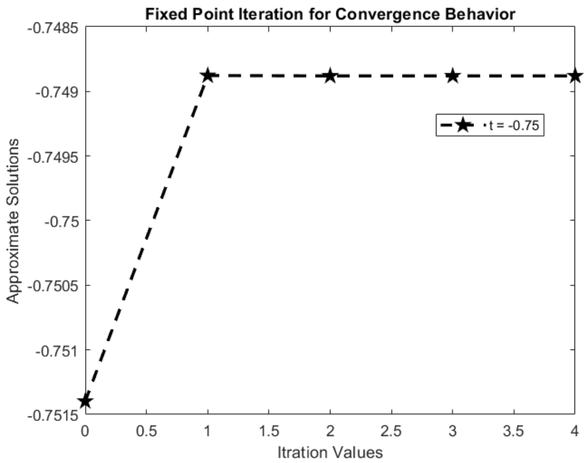

Figure 7. The graph shows that Eq. (29) converges to exact solution -0.75.

Table VIII
FOR $t = 0.58$, THE EXACT SOLUTION IS $x(0.58) = 0.58$

n	$x_{n+1}(t = 0.58)$	Approximate Solution	Absolute Error
0	$x_1(t = 0.58)$	0.581159	1.159×10^{-3}.
1	$x_2(t = 0.58)$	0.580512	5.12×10^{-4}
2	$x_3(t = 0.58)$	0.580512	5.12×10^{-4}
3	$x_4(t = 0.58)$	0.580512	5.12×10^{-4}
4	$x_5(t = 0.58)$	0.580512	5.12×10^{-4}

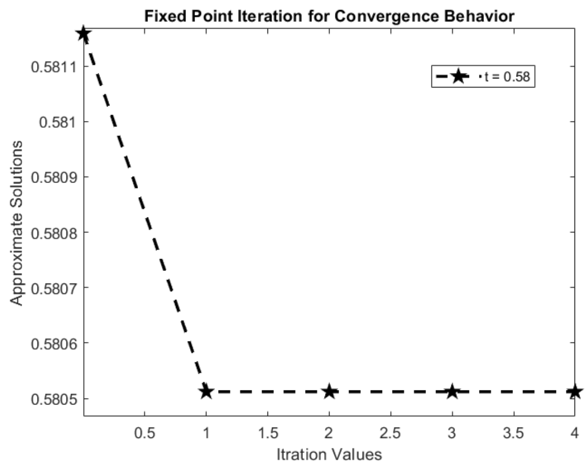

Figure 8. The graph shows that Eq. (29) converges to exact solution 0.58.

VI. Conclusion

The present study has demonstrated the fixed-point theorem for rational contractive mapping on G-metric space. Additionally, it has shown the application of a $\mathcal{FOBVP}$ sol. and presented an example of a binary relation in a Euclidean metric space. Subsequently, we proposed a simple $\mathcal{FOBVP}$ sol., employing the fixed point technique in G-metric space. To achieve this, we utilized an iterative method based on the fixed point approach, resulting in an approximate solution for Equations (22) and (26). The validity of this approach is confirmed by the numerical results.

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