On Tripled Fixed Points Via Altering Distance Functions In G-Metric Spaces With Applications

N.Mangapathi, B.Srinuvasa Rao, K.R.K.Rao, and MD Imam Pasha

***Abstract*—We investigate the present applications of homotopy theory and integral equations in a complete G-metric space using general tripled fixed point theorems. The distance function is modified in these applications. Furthermore, this study provides an example that aids in explaining the critical discovery. The findings supplement, align, and broaden the scope of previous discoveries stated in the literature.**

***Index Terms*—Common tripled fixed point; altering distance function; ω-compatible and G-completeness.**

I. Introduction

FIXED point theory is one of the most prolific positions in nonlinear analysis due to its vast applications in approximation theory, homotopy theory, integral, integrodifferential, and impulsive differential equations, which have been explored in numerous metric spaces.

Berinde and Borcut [1] present the concept of tripled fixed points, as well as certain tripled fixed point results for contractive type mappings with mixed monotone features in partially ordered metric spaces. Borcut et al. [2] also introduced the concept of a tripled coincidence point for a pair of nonlinear contractive mappings. Aydi et al. [3] investigated the common tripled fixed point theorem for ω-compatible mappings in abstract metric spaces. Numerous academics have established triple fixed point results for different spaces; see ([4]- [11]) for more information.

Mustafa and Sims [12] introduced the notion of G-metric spaces in 2006, in addition to providing variant-related fixed point results. Since then, several fixed point results on the formation of G-metric spaces have been published ([13]-[23]).

Khan et al. [24] in 1984 pioneered the concept of altering distance function for self mapping on a metric space. Guttia and Kumssa [25] in their research on fixed point theory, where they generalised the notion of altering distance function and dubbed them control function. Under implicit relations, Pupa and Mocanu [26] introducing altering distance and common fixed points. Many authors extended the Banach Contraction Principle by using control functions, see ([27]-[30]).

In this study, two mappings meeting generalised contractive requirements in G-metric space are demonstrated. These mappings involve altering distance function and also demonstrate the existence of a singular common tripled fixed point. Examples of applications are also given for homotopy theory and integral equations. These results expand and generalise a number of well-known, pertinent, recent findings in the literature.

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II. Preliminaries

In order to obtain our results we need to consider the followings.

Definition II.1:([12]) Let $G : \Im \times \Im \times \Im \to [0, \infty)$ be a function defined on a non-empty set $\Im$ is said to be a G-metric on $\Im$ if satisfying the conditions specified below :

($\mathcal{G}_0$) $G(\partial_1, \partial_2, \partial_3) = 0$ if $\partial_1 = \partial_2 = \partial_3$;

($\mathcal{G}_1$) $0 < G(\partial_1, \partial_1, \partial_2)$ for any $\partial_1, \partial_2 \in \Im$ with $\partial_1 \neq \partial_2$;

($\mathcal{G}_2$) if $G(\partial_1, \partial_1, \partial_2) \leq G(\partial_1, \partial_2, \partial_3)$ for all $\partial_1, \partial_2, \partial_3 \in \Im$ with $\partial_2 \neq \partial_3$;

($\mathcal{G}_3$) $G(\partial_1, \partial_2, \partial_3) = G(P[\partial_1, \partial_2, \partial_3])$, where P is a permutation of $\partial_1, \partial_2, \partial_3$ (symmetry);

($\mathcal{G}_4$) $G(\partial_1, \partial_2, \partial_3) \leq G(\partial_1, \ell, \ell) + G(\ell, \partial_2, \partial_3)$ for all $\partial_1, \partial_2, \partial_3, \ell \in \Im$ (rectangle inequality).

Here the pair $(\Im, G)$ is called a G-metric space.

Definition II.2:([12]) A G- metric space $(\Im, G)$ is said to be symmetric if
$G(\partial_1, \partial_2, \partial_2) = G(\partial_2, \partial_1, \partial_1)$ for all $\partial_1, \partial_2 \in \Im$.

Definition II.3:([12]) Let $\Im$ be a G-metric space. A sequence $\{\ell_i\}$ in $\Im$ is called:

(a) If an integer i_0 exists in $\mathbf{Z}^+$ such that $\forall\ i, j, k \geq i_0$, $G(\ell_i, \ell_j, \ell_k) < \epsilon$. is true for any $\epsilon > 0$, then the sequence is said to be a G-Cauchy sequence.

(b) If an integer i_0 exists in $\mathbf{Z}^+$ such that $\forall\ i, j \geq i_0$, $G(\ell_i, \ell_j, \ell) < \epsilon$, then G is convergent to a point $\ell \in \Im$.

If every G-Cauchy sequence in $\Im$ is G-convergent in $\Im$, then a G-metric space on $\Im$ is said to be G-complete.

We direct the reader to ([12]) for a list of other qualities of a G-metric.

Definition II.4:([1]) Let $\Im$ be a nonempty set. An element $(\partial_1, \partial_2, \partial_3) \in \Im$ is called a tripled fixed point of a given mapping $\mathcal{H} : \Im^3 \to \Im$ if $\mathcal{H}(\partial_1, \partial_2, \partial_3) = \partial_1$,
$\mathcal{H}(\partial_2, \partial_3, \partial_1) = \partial_2$ and $\mathcal{H}(\partial_3, \partial_1, \partial_2) = \partial_3$.

Definition II.5: ([2]) Let $\mathcal{H} : \Im^3 \to \Im$ and $\mathcal{V} : \Im \to \Im$ be two mappings. An element $(\partial_1, \partial_2, \partial_3)$ is said to be a tripled coincident point of $\mathcal{H}$ and $\mathcal{V}$ if $\mathcal{H}(\partial_1, \partial_2, \partial_3) = \mathcal{V}\partial_1$,
$\mathcal{H}(\partial_2, \partial_3, \partial_1) = \mathcal{V}\partial_2$ and $\mathcal{H}(\partial_3, \partial_1, \partial_2) = \mathcal{V}\partial_3$.

Definition II.6:([2]) Let $\mathcal{H} : \Im^3 \to \Im$ and $\mathcal{V} : \Im \to \Im$ be two mappings. An element $(\partial_1, \partial_2, \partial_3)$ is said to be a tripled common point of $\mathcal{H}$ and $\mathcal{V}$ if
$\mathcal{H}(\partial_1, \partial_2, \partial_3) = \mathcal{V}\partial_1 = \partial_1$, $\mathcal{H}(\partial_2, \partial_3, \partial_1) = \mathcal{V}\partial_2 = \partial_2$ and
$\mathcal{H}(\partial_3, \partial_1, \partial_2) = \mathcal{V}\partial_3 = \partial_3$.

Definition II.7:([3]) Let $(\Im, G)$ be a G metric space. A pair $(\mathcal{H}, \mathcal{V})$ is called weakly compatible if for all $\partial_1, \partial_2, \partial_3 \in \Im$,

$\mathcal{V}(\mathcal{H}(\partial_1,\partial_2,\partial_3)) = \mathcal{H}(\mathcal{V}\partial_1,\mathcal{V}\partial_2,\mathcal{V}\partial_3)$ whenever $\mathcal{H}(\partial_1,\partial_2,\partial_3) = \mathcal{V}\partial_1, \mathcal{H}(\partial_2,\partial_3,\partial_1) = \mathcal{V}\partial_2$, and $\mathcal{H}(\partial_3,\partial_1,\partial_2) = \mathcal{V}\partial_3$.

Definition II.8:[24] The function $\zeta : [0,\infty) \to [0,\infty)$ is called an altering distance function if

(i) ζ is continuous and nondecreasing,
(ii) $\zeta(t) = 0 \iff t = 0$,
(iii) $\zeta(t+s) \leq \zeta(t) + \zeta(s) \ \ \forall\ t,s \in [0,\infty)$.

III. Main Results

Theorem III.1: Let$(\Im, G)$ be a G-metric space. Assume that $\chi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\chi(t) = 0 \iff t = 0$ and $\zeta : [0,\infty) \to [0,\infty)$ is an altering distance function. Additionally, let's assume that $T : \Im^3 \to \Im$ and $f : \Im \to \Im$ are two mappings that meet the criteria listed below:

$$\begin{aligned} & \zeta\left(G\left(T(\imath,\jmath,\ell), T(\aleph,\wp,\varpi), T\left(\rho,\varrho,\varsigma\right)\right)\right) \\ \leq\ & \zeta\left(\lambda M(\imath,\jmath,\ell,\aleph,\wp,\varpi,\rho,\varrho,\varsigma)\right) \\ & -\chi\left(\lambda M(\imath,\jmath,\ell,\aleph,\wp,\varpi,\rho,\varrho,\varsigma)\right) \\ & \qquad +LN(\imath,\jmath,\ell,\aleph,\wp,\varpi,\rho,\varrho,\varsigma). \end{aligned} \tag{1}$$

a) $T(\Im^3) \subseteq f(\Im)$,
b) (T,f) are ω-compatible,
c) $f(\Im)$ is complete.

where $M(\imath,\jmath,\ell,\aleph,\wp,\varpi,\rho,\varrho,\varsigma)$

$$= \max\left\{\begin{array}{c} \max\left\{\begin{array}{c} G(f\imath, f\aleph, f\rho), \\ G\left(T(\imath,\jmath,\ell), T(\rho,\varrho,\varsigma), f\rho\right), \\ G\left(T(\imath,\jmath,\ell), T(\aleph,\wp,\varpi), f\rho\right), \\ G\left(T(\imath,\jmath,\ell), f\imath, f\rho\right), \\ G\left(T(\aleph,\wp,\varpi), f\rho, T(\rho,\varrho,\varsigma)\right), \\ G\left(f\rho, T(\aleph,\wp,\varpi), f\aleph\right), \\ G\left(T(\aleph,\wp,\varpi), T(\imath,\jmath,\ell), f\aleph\right), \\ G\left(T(\rho,\varrho,\varsigma), T(\rho,\varrho,\varsigma), f\rho\right) \end{array}\right\}, \\ \max\left\{\begin{array}{c} G(f\jmath, f\wp, f\varrho), \\ G\left(T(\jmath,\ell,\imath), T(\varrho,\varsigma,\rho), fq\right), \\ G\left(T(\jmath,\ell,\imath), T(\wp,\varpi,\aleph), f\varrho\right), \\ G\left(T(\jmath,\ell,\imath), f\jmath, f\varrho\right), \\ G\left(T(\wp,\varpi,\aleph), f\varrho, T(\varrho,\varsigma,\rho)\right), \\ G\left(f\varrho, T(\wp,\varpi,\aleph), f\wp\right), \\ G\left(T(\wp,\varpi,\aleph), T(\jmath,\ell,\imath), f\wp\right), \\ G\left(T(\varrho,\varsigma,\rho), T(\varrho,\varsigma,\rho), f\varrho\right) \end{array}\right\}, \\ \max\left\{\begin{array}{c} G(f\ell, f\varpi, f\varsigma), \\ G\left(T(\ell,\imath,\jmath), T(\varsigma,\rho,\varrho), f\varsigma\right), \\ G\left(T(\ell,\imath,\jmath), T(\varpi,\aleph,\wp), f\varsigma\right), \\ G\left(T(\ell,\imath,\jmath), f\ell, f\varsigma\right), \\ G\left(T(\varpi,\aleph,\wp), f\varsigma, T(\varsigma,\rho,\varrho)\right), \\ G\left(f\varsigma, T(\varpi,\aleph,\wp), f\varpi\right), \\ G\left(T(\varpi,\aleph,\wp), T(\ell,\imath,\jmath), f\varpi\right), \\ G\left(T(\varsigma,\rho,\varrho), T(\varsigma,\rho,\varrho), f\varsigma\right) \end{array}\right\}, \end{array}\right\}$$

and
$N(\imath,\jmath,\ell,\aleph,\wp,\varpi,\rho,\varrho,\varsigma)$

$$= \min\left\{\begin{array}{c} \min\left\{\begin{array}{c} G(f\imath, f\aleph, f\rho), \\ G\left(T(\imath,\jmath,\ell), T(\rho,\varrho,\varsigma), f\rho\right), \\ G\left(T(\imath,\jmath,\ell), T(\aleph,\wp,\varpi), f\rho\right), \\ G\left(T(\imath,\jmath,\ell), f\imath, f\rho\right), \\ G\left(T(\aleph,\wp,\varpi), f\rho, T(\rho,\varrho,\varsigma)\right), \\ G\left(f\rho, T(\aleph,\wp,\varpi), f\aleph\right), \\ G\left(T(\aleph,\wp,\varpi), T(\imath,\jmath,\ell), f\aleph\right), \\ G\left(T(\rho,\varrho,\varsigma), T(\rho,\varrho,\varsigma), f\rho\right) \end{array}\right\}, \\ \min\left\{\begin{array}{c} G(f\jmath, f\wp, f\varrho), \\ G\left(T(\jmath,\ell,\imath), T(\varrho,\varsigma,\rho), fq\right), \\ G\left(T(\jmath,\ell,\imath), T(\wp,\varpi,\aleph), f\varrho\right), \\ G\left(T(\jmath,\ell,\imath), f\jmath, f\varrho\right), \\ G\left(T(\wp,\varpi,\aleph), f\varrho, T(\varrho,\varsigma,\rho)\right), \\ G\left(f\varrho, T(\wp,\varpi,\aleph), f\wp\right), \\ G\left(T(\wp,\varpi,\aleph), T(\jmath,\ell,\imath), f\wp\right), \\ G\left(T(\varrho,\varsigma,\rho), T(\varrho,\varsigma,\rho), f\varrho\right) \end{array}\right\}, \\ \min\left\{\begin{array}{c} G(f\ell, f\varpi, f\varsigma), \\ G\left(T(\ell,\imath,\jmath), T(\varsigma,\rho,\varrho), f\varsigma\right), \\ G\left(T(\ell,\imath,\jmath), T(\varpi,\aleph,\wp), f\varsigma\right), \\ G\left(T(\ell,\imath,\jmath), f\ell, f\varsigma\right), \\ G\left(T(\varpi,\aleph,\wp), f\varsigma, T(\varsigma,\rho,\varrho)\right), \\ G\left(f\varsigma, T(\varpi,\aleph,\wp), f\varpi\right), \\ G\left(T(\varpi,\aleph,\wp), T(\ell,\imath,\jmath), f\varpi\right), \\ G\left(T(\varsigma,\rho,\varrho), T(\varsigma,\rho,\varrho), f\varsigma\right) \end{array}\right\}, \end{array}\right\}$$

with $L \geq 0$ and $0 < \lambda < 1$.Then, T, f has a unique common tripled fixed point in $\Im$.

Proof: Let $\imath,\jmath,\ell \in \Im$ be arbitrary, from (a), we build the sequences $\{\imath_n\}, \{\jmath_n\}, \{\ell_n\}$, $\{\aleph_n\}, \{\wp_n\}, \{\varpi_n\}$, in $\Im$ as

$$\begin{aligned} T\left(\imath_n,\jmath_n,\ell_n\right) &= f\imath_{n+1} = \aleph_n, \\ T\left(\jmath_n,\ell_n,\imath_n\right) &= f\jmath_{n+1} = \wp_n, \\ T\left(\ell_n,\imath_n,\jmath_n\right) &= f\ell_{n+1} = \varpi_n \end{aligned}$$

where $n = 0,1,2,\ldots$.
Now we show that T and f have unique common tripled fixed point in $\Im$. Assume that $G(\aleph_n,\aleph_n,\aleph_{n+1}) > 0$, $G(\wp_n,\wp_n,\wp_{n+1}) > 0$ and $G(\varpi_n,\varpi_n,\varpi_{n+1}) > 0\ \forall\ \ n$. Otherwise, there exists some positive integer n such that $\aleph_n = \aleph_{n+1}$, $\wp_n = \wp_{n+1}$ and $\varpi_n = \varpi_{n+1}$ and so $(\aleph_n,\wp_n,\varpi_n)$ is a tripled fixed point of T, f, and the proof is complete.
By using (1), for each $n \in N$, we have

$$\begin{aligned} & \zeta\left(G\left(\aleph_n,\aleph_n,\aleph_{n+1}\right)\right) \\ =\ & \zeta\left[G\left(\begin{array}{c} T\left(\imath_n,\jmath_n,\ell_n\right), T\left(\imath_n,\jmath_n,\ell_n\right), \\ T\left(\imath_{n+1},\jmath_{n+1},\ell_{n+1}\right) \end{array}\right)\right] \\ \leq\ & \zeta\left(\lambda M(\imath_n,\jmath_n,\ell_n,\imath_n,\jmath_n,\ell_n,\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\ & -\chi\left(\lambda M(\imath_n,\jmath_n,\ell_n,\imath_n,\jmath_n,\ell_n,\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\ & +LN(\imath_n,\jmath_n,\ell_n,\imath_n,\jmath_n,\ell_n,\imath_{n+1},\jmath_{n+1},\ell_{n+1}). \end{aligned} \tag{2}$$

where
$M(\imath_n, \jmath_n, \ell_n, \imath_n, \jmath_n, \ell_n, \imath_{n+1}, \jmath_{n+1}, \ell_{n+1})$

$$= \max \left\{ \begin{array}{c} \max \left\{ \begin{array}{c} G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n), \\ G(\aleph_n, \aleph_{n+1}, \aleph_n), \\ G(\aleph_n, \aleph_n, \aleph_n), \\ G(\aleph_n, \aleph_{n-1}, \aleph_n), \\ G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\aleph_n, \aleph_n, \aleph_{n-1}), \\ G(\aleph_n, \aleph_n, \aleph_{n-1}), \\ G(\aleph_{n+1}, \aleph_{n+1}, \aleph_n) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\wp_{n-1}, \wp_{n-1}, \wp_n), \\ G(\wp_n, \wp_{n+1}, \wp_n), \\ G(\wp_n, \wp_n, \wp_n), \\ G(\wp_n, \wp_{n-1}, \wp_n), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n-1}), \\ G(\wp_n, \wp_n, \wp_{n-1}), \\ G(\wp_{n+1}, \wp_{n+1}, \wp_n) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n), \\ G(\varpi_n, \varpi_{n+1}, \varpi_n), \\ G(\varpi_n, \varpi_n, \varpi_n), \\ G(\varpi_n, \varpi_{n-1}, \varpi_n), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n-1}), \\ G(\varpi_n, \varpi_n, \varpi_{n-1}), \\ G(\varpi_{n+1}, \varpi_{n+1}, \varpi_n) \end{array} \right\} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ \begin{array}{c} G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n), \\ G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\aleph_n, \aleph_n, \aleph_n), \\ G(\aleph_n, \aleph_{n-1}, \aleph_{n-1}), \\ G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\wp_{n-1}, \wp_{n-1}, \wp_n), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\wp_n, \wp_n, \wp_n), \\ G(\wp_n, \wp_{n-1}, \wp_{n-1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\wp_{n-1}, \wp_{n-1}, \wp_n) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_n), \\ G(\varpi_n, \varpi_{n-1}, \varpi_{n-1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}), \\ G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n) \end{array} \right\} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ \begin{array}{c} G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n), 0, \\ G(\aleph_n, \aleph_n, \aleph_{n+1}) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\wp_{n-1}, \wp_{n-1}, \wp_n), 0, \\ G(\wp_n, \wp_n, \wp_{n+1}) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n), 0, \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \max \left\{ \begin{array}{c} G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n), \\ G(\aleph_n, \aleph_n, \aleph_{n+1}) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\wp_{n-1}, \wp_{n-1}, \wp_n), \\ G(\wp_n, \wp_n, \wp_{n+1}) \end{array} \right\}, \\ \max \left\{ \begin{array}{c} G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\} \end{array} \right\}.$$

By similar arguments we obtain

$$N(\imath_n, \jmath_n, \ell_n, \imath_n, \jmath_n, \ell_n, \imath_{n+1}, \jmath_{n+1}, \ell_{n+1})$$
$$= \min \left\{ \begin{array}{c} \min \left\{ \begin{array}{c} G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n), 0, \\ G(\aleph_n, \aleph_n, \aleph_{n+1}) \end{array} \right\}, \\ \min \left\{ \begin{array}{c} G(\wp_{n-1}, \wp_{n-1}, \wp_n), 0, \\ G(\wp_n, \wp_n, \wp_{n+1}) \end{array} \right\}, \\ \min \left\{ \begin{array}{c} G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n), 0, \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\} \end{array} \right\}$$
$$= \min \left\{ \begin{array}{c} 0, 0, 0 \end{array} \right\} = 0.$$

We show that $G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n) \geq G(\aleph_n, \aleph_n, \aleph_{n+1})$, $G(\wp_{n-1}, \wp_{n-1}, \wp_n) \geq G(\wp_n, \wp_n, \wp_{n+1})$
and $G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n) \geq G(\varpi_n, \varpi_n, \varpi_{n+1})$ $\forall n \in N$.
Assume that $G(\aleph_{n-1}, \aleph_{n-1}, \aleph_n) < G(\aleph_n, \aleph_n, \aleph_{n+1})$, $G(\wp_{n-1}, \wp_{n-1}, \wp_n) < G(\wp_n, \wp_n, \wp_{n+1})$ and $G(\varpi_{n-1}, \varpi_{n-1}, \varpi_n) < G(\varpi_n, \varpi_n, \varpi_{n+1})$, $n \in N$,
then we have,

$$M(\imath_n, \jmath_n, \ell_n, \imath_n, \jmath_n, \ell_n, \imath_{n+1}, \jmath_{n+1}, \ell_{n+1})$$
$$= \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}.$$

Then from (2), we can get

$$\begin{aligned} & \zeta(G(\aleph_n, \aleph_n, \aleph_{n+1})) \\ \leq\ & \zeta\left(\lambda \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}\right) \\ & -\chi\left(\lambda \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}\right) \\ \leq\ & \zeta\left(\lambda \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}\right). \end{aligned}$$

Since ζ is increasing, we get

$$G(\aleph_n, \aleph_n, \aleph_{n+1}) \leq \lambda \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}. \tag{3}$$

By similar arguments we obtain

$$G(\wp_n, \wp_n, \wp_{n+1}) \leq \lambda \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}. \tag{4}$$

Also, we have

$$G(\varpi_n, \varpi_n, \varpi_{n+1}) \leq \lambda \max \left\{ \begin{array}{c} G(\aleph_n, \aleph_n, \aleph_{n+1}), \\ G(\wp_n, \wp_n, \wp_{n+1}), \\ G(\varpi_n, \varpi_n, \varpi_{n+1}) \end{array} \right\}. \tag{5}$$

Combining (3)- (5), we can get

$$\begin{aligned} &\max\left\{\begin{array}{c} G\left(\aleph_n,\aleph_n,\aleph_{n+1}\right), \\ G\left(\wp_n,\wp_n,\wp_{n+1}\right), \\ G\left(\varpi_n,\varpi_n,\varpi_{n+1}\right) \end{array}\right\} \\ \leq\ &\lambda\max\left\{\begin{array}{c} G\left(\aleph_n,\aleph_n,\aleph_{n+1}\right), \\ G\left(\wp_n,\wp_n,\wp_{n+1}\right), \\ G\left(\varpi_n,\varpi_n,\varpi_{n+1}\right) \end{array}\right\} \end{aligned}$$

which is a contradiction, because $0<\lambda<1$.
Thus, $G(\aleph_{n-1},\aleph_{n-1},\aleph_n)\geq G\left(\aleph_n,\aleph_n,\aleph_{n+1}\right)$, $G(\wp_{n-1},\wp_{n-1},\wp_n)\geq G\left(\wp_n,\wp_n,\wp_{n+1}\right)$ and $G(\varpi_{n-1},\varpi_{n-1},\varpi_n)\geq G\left(\varpi_n,\varpi_n,\varpi_{n+1}\right)$.
Therefore by above inequality we get

$$\begin{aligned} &\max\left\{\begin{array}{c} G\left(\aleph_n,\aleph_n,\aleph_{n+1}\right), \\ G\left(\wp_n,\wp_n,\wp_{n+1}\right), \\ G\left(\varpi_n,\varpi_n,\varpi_{n+1}\right) \end{array}\right\} \\ \leq\ &\lambda\max\left\{\begin{array}{c} G\left(\aleph_{n-1},\aleph_{n-1},\aleph_{n}\right), \\ G\left(\wp_{n-1},\wp_{n-1},\wp_{n}\right), \\ G\left(\varpi_{n-1},\varpi_{n-1},\varpi_{n}\right) \end{array}\right\} \\ \leq\ &\lambda^2\max\left\{\begin{array}{c} G\left(\aleph_{n-2},\aleph_{n-2},\aleph_{n-1}\right), \\ G\left(\wp_{n-2},\wp_{n-2},\wp_{n-1}\right), \\ G\left(\varpi_{n-2},\varpi_{n-2},\varpi_{n-1}\right) \end{array}\right\} \\ &\vdots \\ \leq\ &\lambda^n\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\}. \end{aligned}$$

Thus, we have

$$G\left(\aleph_n,\aleph_n,\aleph_{n+1}\right)\leq\lambda^n\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\},$$

$$G\left(\wp_n,\wp_n,\wp_{n+1}\right)\leq\lambda^n\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\}$$

and $G\left(\varpi_n,\varpi_n,\varpi_{n+1}\right)\leq\lambda^n\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\}$.

By use of the rectangle inequality, for $n>m$, we get

$$\begin{aligned} &G\left(\aleph_n,\aleph_n,\aleph_m\right) \\ \leq\ &G\left(\aleph_m,\aleph_{m+1},\aleph_{m+1}\right)+G\left(\aleph_{m+1},\aleph_n,\aleph_n\right) \\ \leq\ &G\left(\aleph_m,\aleph_{m+1},\aleph_{m+1}\right)+G\left(\aleph_{m+2},\aleph_{m+2},\aleph_{m+1}\right) \\ &\quad+G\left(\aleph_{m+2},\aleph_n,\aleph_n\right) \\ \leq\ &G\left(\aleph_m,\aleph_{m+1},\aleph_{m+1}\right)+G\left(\aleph_{m+2},\aleph_{m+2},\aleph_{m+1}\right) \\ &+\cdots+G\left(\aleph_{n-1},\aleph_n,\aleph_n\right) \\ \leq\ &\left(\lambda^m+\lambda^{m+1}+\cdots+\lambda^{n-1}\right)\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\} \\ \leq\ &\left(\lambda^m+\lambda^{m+1}+\cdots\right)\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\} \\ \leq\ &\frac{\lambda^m}{1-\lambda}\max\left\{\begin{array}{c} G\left(\aleph_0,\aleph_0,\aleph_1\right), \\ G\left(\wp_0,\wp_0,\wp_1\right), \\ G\left(\varpi_0,\varpi_0,\varpi_1\right) \end{array}\right\}\to 0 \text{ as } m\to\infty. \end{aligned}$$

By similar arguments, we obtain $G\left(\wp_n,\wp_n,\wp_m\right)\to 0$ as $n,m\to\infty$, $G\left(\varpi_n,\varpi_n,\varpi_m\right)\to 0$ as $n,m\to\infty$. This demonstrates that in the G-metric space $(\Im,G)$, $\{\aleph_n\},\{\wp_n\}$ and $\{\varpi_n\}$are Cauchy sequences. Assuming that $f(\Im)$ is complete subspace of $(\Im,G)$, then the sequences $\{\aleph_n\},\{\wp_n\}$ and $\{\varpi_n\}$ are convergence to x,y,z respectively in $f(\Im)$. Thus, there exist $\ddot{u},\ddot{v},\ddot{w}\in f(\Im)$ such that

$$\begin{aligned} &\lim_{n\to\infty}\aleph_n=x=f\ddot{u}\qquad\lim_{n\to\infty}\wp_n=y=f\ddot{v} \\ &\lim_{n\to\infty}\varpi_n=z=f\ddot{w} \end{aligned}\tag{6}$$

We claim that $T(\ddot{u},\ddot{v},\ddot{w})=x$, $T(\ddot{v},\ddot{w},\ddot{u})=y$ and $T(\ddot{w},\ddot{u},\ddot{v})=z$.
By using (1), we have

$$\begin{aligned} &\zeta\left(G\left(T(\ddot{u},\ddot{v},\ddot{w}),T(\ddot{u},\ddot{v},\ddot{w}),\aleph_{n+1}\right)\right) \\ =\ &\zeta\left[G\left(\ T(\ddot{u},\ddot{v},\ddot{w}),T(\ddot{u},\ddot{v},\ddot{w}),T\left(\imath_{n+1},\jmath_{n+1},\ell_{n+1}\right)\ \right)\right] \\ \leq\ &\zeta\left(\lambda M(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\ &-\chi\left(\lambda M(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\ &\qquad+LN(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1}) \end{aligned}\tag{7}$$

where $\lim_{n\to\infty}M(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1})=\lim_{n\to\infty}$

$$\max\left\{\begin{array}{c} \max\left\{\begin{array}{c} G(f\ddot{u},f\ddot{u},\aleph_n), \\ G\left(T(\ddot{u},\ddot{v},\ddot{w}),\aleph_{n+1},\aleph_n\right), \\ G\left(T(\ddot{u},\ddot{v},\ddot{w}),T(\ddot{u},\ddot{v},\ddot{w}),\aleph_n\right), \\ G\left(T(\ddot{u},\ddot{v},\ddot{w}),f\ddot{u},\aleph_n\right), \\ G\left(T(\ddot{u},\ddot{v},\ddot{w}),\aleph_n,\aleph_{n+1}\right), \\ G\left(\aleph_n,T(\ddot{u},\ddot{v},\ddot{w}),\aleph_n\right), \\ G\left(T(\ddot{u},\ddot{v},\ddot{w}),T(\ddot{u},\ddot{v},\ddot{w}),f\ddot{u}\right), \\ G\left(\aleph_{n+1},\aleph_{n+1},\aleph_n\right) \end{array}\right\}, \\ \max\left\{\begin{array}{c} G(f\ddot{v},f\ddot{v},\wp_n), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),\wp_{n+1},\wp_n\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),T(\ddot{v},\ddot{w},\ddot{u}),\wp_n\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),f\ddot{v},\wp_n\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),\wp_n,\wp_{n+1}\right), \\ G\left(\wp_n,T(\ddot{v},\ddot{w},\ddot{u}),\wp_n\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),T(\ddot{v},\ddot{w},\ddot{u}),f\ddot{v}\right), \\ G\left(\wp_{n+1},\wp_{n+1},\wp_n\right) \end{array}\right\}, \\ \max\left\{\begin{array}{c} G(f\ddot{w},f\ddot{w},\varpi_n), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),\varpi_{n+1},\varpi_n\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),T(\ddot{w},\ddot{u},\ddot{v}),\varpi_n\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),f\ddot{w},\varpi_n\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),\varpi_n,\varpi_{n+1}\right), \\ G\left(\varpi_n,T(\ddot{w},\ddot{u},\ddot{v}),\varpi_n\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),T(\ddot{w},\ddot{u},\ddot{v}),f\ddot{w}\right), \\ G\left(\varpi_{n+1},\varpi_{n+1},\varpi_n\right) \end{array}\right\} \end{array}\right\}$$

$$=\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}$$

and

$$\begin{aligned} &\lim_{n\to\infty}N(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1}) \\ &=\min\left\{\begin{array}{c} \min\left\{\ 0,G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right)\ \right\}, \\ \min\left\{\ 0,G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right)\ \right\}, \\ \min\left\{\ 0,G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right)\ \right\} \end{array}\right\}=0. \end{aligned}$$

By taking the upper limit when $n \to \infty$ in (7), we obtain

$$\begin{aligned}
& \zeta\left(G\left(T(\ddot{u},\ddot{v},\ddot{w}),T(\ddot{u},\ddot{v},\ddot{w}),x\right)\right) \\
= & \lim_{n\to\infty}\zeta\left(G\left(T(\ddot{u},\ddot{v},\ddot{w}),T(\ddot{u},\ddot{v},\ddot{w}),\aleph_{n+1}\right)\right) \\
\leq & \zeta\left(\lambda\lim_{n\to\infty}M(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\
& -\chi\left(\lambda\lim_{n\to\infty}M(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\
& +L\lim_{n\to\infty}N(\ddot{u},\ddot{v},\ddot{w},\ddot{u},\ddot{v},\ddot{w},\imath_{n+1},\jmath_{n+1},\ell_{n+1}) \\
\leq & \zeta\left(\lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}\right) \\
& -\chi\left(\lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}\right)+L(0) \\
\leq & \zeta\left(\lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}\right)
\end{aligned}$$

which implies that

$$G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right)\leq\lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}.$$

Similarly, we can prove that

$$G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right)\leq\lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right), \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}$$

and

$$G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right)\leq\lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right) \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}$$

therefore, we have

$$\begin{aligned}
& \max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right) \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\} \\
\leq & \lambda\max\left\{\begin{array}{c} G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right), \\ G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right) \\ G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right) \end{array}\right\}
\end{aligned}$$

which is impossible. Hence $G\left(T(\ddot{u},\ddot{v},\ddot{w}),x,x\right)=0$, $G\left(T(\ddot{v},\ddot{w},\ddot{u}),y,y\right)=0$ and $G\left(T(\ddot{w},\ddot{u},\ddot{v}),z,z\right)=0$ which implies that $T(\ddot{u},\ddot{v},\ddot{w})=x$, $T(\ddot{v},\ddot{w},\ddot{u})=y$ and $T(\ddot{w},\ddot{u},\ddot{v})=z$. It follows that $T(\ddot{u},\ddot{v},\ddot{w})=x=f\ddot{u}$, $T(\ddot{v},\ddot{w},\ddot{u})=y=f\ddot{v}$ and $T(\ddot{w},\ddot{u},\ddot{v})=z=f\ddot{w}$. Since $\{T,f\}$ is weakly compatible pair, we have $T(x,y,z)=fx$, $T(y,z,x)=fy$ and $T(z,x,y)=fz$. Now we prove that $fx=x$ and $fy=y$ and $fz=z$.

By using (1) and taking the upper limit when $n\to\infty$, we have

$$\begin{aligned}
& \zeta\left(G\left(fx,fx,x\right)\right)=\lim_{n\to\infty}\zeta\left(G\left(fx,fx,\aleph_{n+1}\right)\right) \\
= & \lim_{n\to\infty}\zeta\left(G\left(T(x,y,z),T(x,y,z),T(\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right)\right) \\
\leq & \zeta\left(\lambda\lim_{n\to\infty}M(x,y,z,x,y,z,\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\
& -\chi\left(\lambda\lim_{n\to\infty}M(x,y,z,x,y,z,\imath_{n+1},\jmath_{n+1},\ell_{n+1})\right) \\
& +L\lim_{n\to\infty}N(x,y,z,x,y,z,\imath_{n+1},\jmath_{n+1},\ell_{n+1}) \qquad (8)
\end{aligned}$$

where

$$\begin{aligned}
& \lim_{n\to\infty}M(x,y,z,x,y,z,\imath_{n+1},\jmath_{n+1},\ell_{n+1}) \\
& =\max\left\{\ G\left(fx,x,x\right),G\left(fy,y,y\right),G\left(fz,z,z\right)\ \right\}
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{n\to\infty}N(x,y,z,x,y,z,\imath_{n+1},\jmath_{n+1},\ell_{n+1}) \\
& =\min\left\{\begin{array}{c} \min\left\{\ 0,G\left(fx,x,x\right)\ \right\}, \\ \min\left\{\ 0,G\left(fy,y,y\right)\ \right\}, \\ \min\left\{\ 0,G\left(fz,z,z\right)\ \right\} \end{array}\right\}=0.
\end{aligned}$$

From (8), we have

$$\begin{aligned}
& \zeta\left(G\left(fx,x,x\right)\right) \\
\leq & \zeta\left(\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right), \\ G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}\right) \\
& -\chi\left(\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right), \\ G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}\right)+L(0) \\
\leq & \zeta\left(\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right), \\ G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}\right)
\end{aligned}$$

which implies that

$$G\left(fx,x,x\right)\leq\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right),G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}.$$

Similarly, we can prove that

$$G\left(fy,y,y\right)\leq\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right),G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}$$

and

$$G\left(fz,z,z\right)\leq\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right),G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}$$

therefore, we have

$$\max\left\{\begin{array}{c} G\left(fx,x,x\right), \\ G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}\leq\lambda\max\left\{\begin{array}{c} G\left(fx,x,x\right), \\ G\left(fy,y,y\right), \\ G\left(fz,z,z\right) \end{array}\right\}.$$

Hence $G\left(fx,x,x\right)=0$, $G\left(fy,y,y\right)=0$, $G\left(fz,z,z\right)=0$ which implies that $fx=x$, $fy=y$ and $fz=z$. It follows that $T(x,y,z)=x=fx$, $T(y,z,x)=y=fy$. and $T(z,x,y)=z=fz$. As a result, the tripled common fixed point of T and f is (x,y,z). The following section will demonstrate the common tripled fixed point in $\Im$'s uniqueness. Assume that there is a another tripled fixed point of type (x',y',z') for this purpose. We have

$$\begin{aligned}
& \zeta\left(G\left(x,x,x'\right)\right) \\
= & \zeta\left(G\left(T(x,y,z),T(x,z,y),T(x',y',z')\right)\right) \\
\leq & \zeta\left(\lambda M(x,y,z,x,y,z,x',y',z')\right) \\
& -\chi\left(\lambda M(x,y,z,x,y,z,x',y',z')\right) \\
& +LN(x,y,z,x,y,z,x',y',z') \qquad (9)
\end{aligned}$$

where

$$\begin{aligned}
& M(x,y,z,x,y,z,x',y',z') \\
& =\max\left\{\begin{array}{c} G\left(x,x,x'\right),G\left(y,y,y'\right), \\ G\left(z,z,z'\right) \end{array}\right\}
\end{aligned}$$

and $N(x,y,z,x,y,z,x',y',z')=0.$

From (9), we have

$$\begin{aligned}&\zeta\left(G\left(x,x,x'\right)\right)\\ \leq\ &\zeta\left(\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}\right)\\ &-\chi\left(\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),\\ G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}\right)+L(0)\\ \leq\ &\zeta\left(\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}\right)\end{aligned}$$

which implies that

$G\left(x,x,x'\right)\leq\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}.$

Similarly, we can prove that

$G\left(y,y,y'\right)\leq\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}$

and

$G\left(z,z,z'\right)\leq\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}$

therefore, we have

$$\max\left\{\begin{array}{c}G\left(x,x,x'\right),\\ G\left(z,z,z'\right)\\ G\left(z,z,z'\right)\end{array}\right\}\leq\lambda\max\left\{\begin{array}{c}G\left(x,x,x'\right),\\ G\left(y,y,y'\right),\\ G\left(z,z,z'\right)\end{array}\right\}$$

which is impossible. Hence $G\left(x,x,x'\right)=0$, $G\left(y,y,y'\right)=0$ and $G\left(z,z,z'\right)=0$ which implies that $x=x'$, $y=y'$ and $z=z'$. Therefore, (x,y,z) is uniqueness of common tripled fixed point of T and f. Following, we'll demonstrate the one and only fixed point in $\Im$.

Now,

$$\begin{aligned}\zeta\left(G\left(x,y,z\right)\right)&=\zeta\left(G\left(T(x,y,z),T(y,z,x),T(z,x,y)\right)\right)\\ &\leq\zeta\left(\lambda M(x,y,z,y,z,x,z,x,y)\right)\\ &\quad-\chi\left(\lambda M(x,y,z,y,z,x,z,x,y)\right)\\ &\quad\quad+LN(x,y,z,y,z,x,z,x,y)\end{aligned}\tag{10}$$

$$M(x,y,z,y,z,x,z,x,y)=\max\left\{\begin{array}{c}G\left(x,y,z\right),\\ G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}$$

and $N(x,y,z,y,z,x,z,x,y)=0.$

From (10), we have

$$\begin{aligned}\zeta\left(G\left(x,y,z\right)\right)&\leq\zeta\left(\lambda\max\left\{\begin{array}{c}G\left(x,y,z\right),\\ G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}\right)\\ &-\chi\left(\lambda\max\left\{\begin{array}{c}G\left(x,y,z\right),\\ G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}\right)+L(0)\\ &\leq\zeta\left(\lambda\max\left\{\begin{array}{c}G\left(x,y,z\right),\\ G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}\right)\end{aligned}$$

which implies that

$G\left(x,y,z\right)\leq\lambda\max\left\{\begin{array}{c}G\left(x,y,z\right),G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}.$

Similarly, we can prove that

$G\left(y,z,x\right)\leq\lambda\max\left\{\ G\left(x,y,z\right),G\left(y,z,x\right),G\left(z,x,y\right)\ \right\}$

and

$G\left(z,x,y\right)\leq\lambda\max\left\{\ G\left(x,y,z\right),G\left(y,z,x\right),G\left(z,x,y\right)\ \right\}.$

Therefore, we have

$$\max\left\{\begin{array}{c}G\left(x,y,z\right),\\ G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}\leq\lambda\max\left\{\begin{array}{c}G\left(x,y,z\right),\\ G\left(y,z,x\right),\\ G\left(z,x,y\right)\end{array}\right\}$$

which is impossible. Hence $G\left(x,y,z\right)=0$, $G\left(y,z,x\right)=0$ and $G\left(z,x,y\right)=0$, hence, we get $x=y=z$. Which means that T and f have a unique common fixed point.

Corollary III.2: In reduced to the hypotheses of Theorem III.1, assuming $L=0$ we deduce that T and f have common tripled fixed point in $\Im$.

Corollary III.3: Let $(\Im,G)$ be a complete G-metric space. Suppose that $T:\Im^3\to\Im$ be a mapping such that

$$\begin{aligned}&G\left(T(\imath,\jmath,\ell),T(\aleph,\wp,\varpi),T\left(\rho,\varrho,\varsigma\right)\right)\\ &\quad\leq\lambda\max\left\{G(\imath,\aleph,\rho),G(\jmath,\wp,\varrho),G(\ell,\varpi,\varsigma)\right\}\end{aligned}$$

for all $\imath,\jmath,\ell,\aleph,\wp,\varpi,\rho,\varrho,\varsigma\in\Im$ with $0<\lambda<1$, then there is a unique tripled fixed point of T in $\Im$.

Example III.4: Let $\Im=[0,\infty)$ and $G(\aleph,\wp,\varpi)=\max\{|\aleph-\wp|,|\wp-\varpi|,|\varpi-\aleph|\}$. In this case $(\Im,G)$ is a complete G-metric spaces. Let $T:\Im^3\to\Im$ and $f:\Im\to\Im$ be given by $f(\aleph)=\frac{\aleph}{3}$ and $T(\aleph,\wp,\varpi)=\frac{\aleph-\wp+\varpi}{24}$, also, $\zeta(t)=\frac{2t}{7}$ and $\chi(t)=\frac{t}{7}$ for all $t\in[0,\infty)$. Then obviously, $T(\Im^3)\subseteq f(\Im)$, and the pair (T,f) is ω-compatible. Now we have

$$\begin{aligned}&\zeta\left(G\left(T(\alpha,\beta,\gamma),T(\alpha,\beta,\gamma),T(\aleph,\wp,\varpi)\right)\right)\\ =\ &\frac{2}{7}G\left(T(\alpha,\beta,\gamma),T(\alpha,\beta,\gamma),T(\aleph,\wp,\varpi)\right)\\ \leq\ &\frac{1}{2}\max\left\{|T(\alpha,\beta,\gamma)-T(\aleph,\wp,\varpi)|\right\}\\ =\ &\frac{1}{2}\max\left\{|\frac{\alpha-\beta+\gamma}{24}-\frac{\aleph-\wp+\varpi}{24}|\right\}\\ =\ &\frac{1}{48}\max\left\{|\alpha-\beta+\gamma-\aleph+\wp-\varpi|\right\}\\ \leq\ &\frac{1}{14}\max\left\{\begin{array}{c}\max\left\{\ |\frac{\alpha}{3}-\frac{\aleph}{3}|,0,\ \right\}\\ \max\left\{\ |\frac{\beta}{3}-\frac{\wp}{3}|,0,\ \right\}\\ \max\left\{\ |\frac{\gamma}{3}-\frac{\varpi}{3}|,0,\ \right\}\end{array}\right\}\\ \leq\ &\frac{1}{7}\left(\frac{1}{2}\max\left\{\begin{array}{c}\max\left\{\ G(f\alpha,f\alpha,f\aleph),0,\ \right\}\\ \max\left\{\ G(f\beta,f\beta,f\wp),0,\ \right\}\\ \max\left\{\ G(f\gamma,f\gamma,f\varpi),0,\ \right\}\end{array}\right\}\right).\\ \leq\ &\frac{2}{7}\left(\frac{1}{2}\max\left\{\begin{array}{c}\max\left\{\ G(f\alpha,f\alpha,f\aleph),0,\ \right\}\\ \max\left\{\ G(f\beta,f\beta,f\wp),0,\ \right\}\\ \max\left\{\ G(f\gamma,f\gamma,f\varpi),0,\ \right\}\end{array}\right\}\right)\\ &-\frac{1}{7}\left(\frac{1}{2}\max\left\{\begin{array}{c}\max\left\{\ G(f\alpha,f\alpha,f\aleph),0,\ \right\}\\ \max\left\{\ G(f\beta,f\beta,f\wp),0,\ \right\}\\ \max\left\{\ G(f\gamma,f\gamma,f\varpi),0,\ \right\}\end{array}\right\}\right).\\ \leq\ &\zeta\left(\lambda M(\alpha,\beta,\gamma,\alpha,\beta,\gamma,\aleph,\wp,\varpi)\right)\\ &-\chi\left(\lambda M(\alpha,\beta,\gamma,\alpha,\beta,\gamma,\aleph,\wp,\varpi)\right)\\ &+LN(\alpha,\beta,\gamma,\alpha,\beta,\gamma,\aleph,\wp,\varpi).\end{aligned}$$

Thus all the conditions of the Theorem $III.1$ are satisfied and $(0,0,0)$ is unique common tripled fixed point of T and f.

A. APPLICATION TO INTEGRAL EQUATIONS

In this section, we study the existence of an unique solution to an initial value problem, as an application to Corollary III.2

Theorem III.5: Consider the initial value problem

$$\frac{d\aleph}{dt} = S(t, (\aleph(t), \aleph(t), \aleph(t)),\ t \in I = [0,1], \aleph(0) = \aleph_0 \tag{11}$$

where $S : I \times \mathcal{R}^2 \to \mathcal{R}$ and $\aleph_0 \in \mathcal{R}$ with $\int_0^t S(s, \aleph(s), \wp(s), \varpi(s))ds =$
$\max\left\{ \begin{array}{c} \int_0^t S(s, \aleph(s), \aleph(s))ds, \\ \int_0^t S(s, \wp(s), \wp(s), \wp(s))ds \\ \int_0^t S(s, \varpi(s), \varpi(s), \varpi(s))ds \end{array} \right\}$. Then the initial value problem (11) has a unique solution in $C(I, \mathcal{R})$.

Proof: Initial Value Problem (11)'s equivalent integral equation is

$$\aleph(t) = \aleph_0 + \int_0^t S(s, \aleph(s), \aleph(s), \aleph(s))ds.$$

Let $\Im = C(I, \mathcal{R})$ and
$G(\aleph, \wp, \varpi) = |\aleph - \wp| + |\wp - \varpi| + |\varpi - \aleph|\ \forall\ \aleph, \wp, \varpi \in \Im$.
Define $\zeta, \chi : [0, \infty) \to [0, \infty)$ by $\zeta(t) = \frac{2t}{5}$, $\chi(t) = \frac{t}{6}$.
Define $T : \Im^3 \to \Im$ and $f : \Im \to \Im$ by

$$T(\aleph, \wp, \varpi)(t) = \frac{\aleph_0}{16} + \int_0^t S(s, \aleph(s), \wp(s), \varpi(s))ds,$$

$$f(\aleph)(t) = \aleph_0 + 16 \int_0^t S(s, \aleph(s), \aleph(s), \aleph(s))ds.$$

Now

$$\begin{aligned}
& \zeta\left(G(T(\aleph, \wp, \varpi)(t), T(\aleph, \wp, \varpi)(t), T(\alpha, \beta, \gamma)(t))\right) \\
= \ & \frac{2}{5} G(T(\aleph, \wp, \varpi)(t), T(\aleph, \wp, \varpi)(t), T(\alpha, \beta, \gamma)(t)) \\
= \ & \frac{4}{5} |R(\aleph, \wp, \varpi)(t) - R(\alpha, \beta, \gamma)(t)| \\
= \ & \frac{2}{40} \max\left\{ \begin{array}{c} |f(\aleph)(t) - f(\alpha)(t)|, \\ |f(\wp)(t) - f(\beta)(t)|, \\ |f(\varpi)(t) - f(\gamma)(t)| \end{array} \right\} \\
= \ & \frac{1}{40} \max\left\{ \begin{array}{c} G(f\aleph, f\aleph, f\alpha), \\ G(f\wp, f\wp, f\beta), \\ G(f\varpi, f\varpi, f\gamma) \end{array} \right\} \\
= \ & \frac{1}{20}\left(\frac{1}{2} \max\left\{ \begin{array}{c} G(f\aleph, f\aleph, f\alpha), \\ G(f\wp, f\wp, f\beta), \\ G(f\varpi, f\varpi, f\gamma) \end{array} \right\}\right) \\
\leq \ & \frac{7}{30}\left(\frac{1}{2} \max\left\{ \begin{array}{c} G(f\aleph, f\aleph, f\alpha), \\ G(f\wp, f\wp, f\beta), \\ G(f\varpi, f\varpi, f\gamma) \end{array} \right\}\right) \\
\leq \ & \frac{2}{5}\left(\frac{1}{2} \max\left\{ \begin{array}{c} G(f\aleph, f\aleph, f\alpha), \\ G(f\wp, f\wp, f\beta), \\ G(f\varpi, f\varpi, f\gamma) \end{array} \right\}\right) \\
& -\frac{1}{6}\left(\frac{1}{2} \max\left\{ \begin{array}{c} G(f\aleph, f\aleph, f\alpha), \\ G(f\wp, f\wp, f\beta), \\ G(f\varpi, f\varpi, f\gamma) \end{array} \right\}\right) \\
\leq \ & \frac{2}{5}\left(\frac{1}{2} \max\left\{ \begin{array}{c} \max\{\ G(f\aleph, f\aleph, f\alpha), 0\ \}, \\ \max\{\ G(f\wp, f\wp, f\beta), 0\ \}, \\ \max\{\ G(f\varpi, f\varpi, f\gamma), 0\ \} \end{array} \right\}\right) \\
& -\frac{1}{6}\left(\frac{1}{2} \max\left\{ \begin{array}{c} \max\{\ G(f\aleph, f\aleph, f\alpha), 0\ \}, \\ \max\{\ G(f\wp, f\wp, f\beta), 0\ \}, \\ \max\{\ G(f\varpi, f\varpi, f\gamma), 0\ \} \end{array} \right\}\right) \\
\leq \ & \zeta\left(\lambda M(\aleph, \wp, \varpi, \aleph, \wp, \varpi, \alpha, \beta, \gamma)\right) \\
& -\chi\left(\lambda M(\aleph, \wp, \varpi, \aleph, \wp, \varpi, \alpha, \beta, \gamma)\right)
\end{aligned}$$

The equation (11) has a unique solution in $C(I, \mathcal{R})$, as deduced by Corollary III.2.

B. APPLICATION TO HOMOTOPY

Theorem III.6 Let $(\Im, G)$ be the complete G-metric space, U and $\overline{U}$ be an open and closed subset of $\Im$ such that $U \subseteq \overline{U}$. Consider the operator $\mathcal{H_G} : \overline{U}^3 \times [0,1] \to \Im$ as meeting the following conditions:

(χ_0) $\aleph'_\star \neq \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \tau)$, $\wp'_\star \neq \mathcal{H_G}(\wp'_\star, \varpi'_\star, \aleph'_\star, \xi)$, $\varpi'_\star \neq \mathcal{H_G}(\varpi'_\star, \aleph'_\star, \wp'_\star, \xi)$, for each $\aleph'_\star, \wp'_\star, \varpi'_\star \in \partial U$ and $\xi \in [0,1]$ (Here ∂U is boundary of U in $\Im$)

(χ_1)

$$\begin{aligned}
& G\left(\begin{array}{c} \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \\ \mathcal{H_G}(\rho'_\star, \varrho'_\star, \varsigma'_\star, \xi) \end{array} \right) \\
\leq \ & \lambda \max\left\{ \begin{array}{c} G(\aleph'_\star, \aleph'_\star, \rho'_\star), G(\wp'_\star, \wp'_\star, \varrho'_\star), \\ G(\varpi'_\star, \varpi'_\star, \varsigma'_\star) \end{array} \right\}
\end{aligned}$$

for all $\aleph'_\star, \wp'_\star, \varpi'_\star, \rho'_\star, \varrho'_\star, \varsigma'_\star \in \overline{U}$ and $\xi \in [0,1]$,

(χ_2) $\exists\ \ M \geq 0 \ni, G\left(\begin{array}{c} \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \\ \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \\ \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \zeta) \end{array} \right) \leq M|\xi - \zeta|$
for every $\aleph'_\star, \wp'_\star, \varpi'_\star \in \overline{U}$ and $\xi, \zeta \in [0,1]$.

Then $\mathcal{H_G}(.,0)$ has a tripled fixed point $\iff$ $\mathcal{H_G}(.,1)$ has a tripled fixed point.

Proof: Let the set,

$$\Im = \left\{ \begin{array}{c} \xi \in [0,1] : \mathcal{H_G}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi) = \aleph'_\star, \\ \mathcal{H_G}(\wp'_\star, \varpi'_\star, \aleph'_\star, \xi) = \wp'_\star, \\ \mathcal{H_G}(\varpi'_\star, \aleph'_\star, \wp'_\star, \xi) = \varpi'_\star \\ \text{for some } \aleph'_\star, \wp'_\star, \varpi'_\star \in U \end{array} \right\}.$$

We have that $(0,0,0)$ in $\Im$ because $\mathcal{H_G}(.,0)$ has a tripled fixed point in U^3. Therefore, the set $\Im$ is not empty. Now we prove that $\Im$ is both closed and open in $[0,1]$ and consequently by the connectedness $\Im = [0,1]$. As a result, the fixed point of $\mathcal{H_G}(.,1)$ is tripled and is located in U^3. We begin by demonstrating that $\Im$ closed in $[0,1]$. To see this, Let $\{\xi_n\}_{n=1}^{\infty} \subseteq \Im$ with $\xi_n \to \xi \in [0,1]$ as $n \to \infty$. We must show that $\xi \in \Im$. Since $\xi_n \in \Im$ for $n = 0, 1, 2, 3, \cdots$, there exists sequences $\{\aleph'_{\star n}\}, \{\wp'_{\star n}\}, \{\varpi'_{\star n}\}$ with $\aleph'_{\star n+1} = \mathcal{H_G}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n)$,

$\wp'_{\star n+1} = \mathcal{H}_{\mathcal{G}}(\wp'_{\star n}, \varpi'_{\star n}, \aleph'_{\star n}, \xi_n)$ and
$\varpi'_{\star n+1} = \mathcal{H}_{\mathcal{G}}(\varpi'_{\star n}, \aleph'_{\star n}, \wp'_{\star n}, \xi_n)$
Consider

$$\begin{aligned} & G(\aleph'_{\star n+1}, \aleph'_{\star n+1}, \aleph'_{\star n+2}) \\ = \ & G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}, \xi_{n+1}) \end{pmatrix} \\ \leq \ & G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}\xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}\xi_n) \end{pmatrix} \\ & + G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}\xi_{n+1}) \end{pmatrix} \\ \leq \ & G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}, \xi_n) \end{pmatrix} \\ & + M|\xi_n - \xi_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} & \lim_{n\to\infty} G(\aleph'_{\star n+1}, \aleph'_{\star n+1}, \aleph'_{\star n+2}) \\ \leq \ & \lim_{n\to\infty} G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n}, \wp'_{\star n}, \varpi'_{\star n}, \xi_n), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}, \xi_n) \end{pmatrix} \\ \leq \ & \lim_{n\to\infty} \lambda \max\left\{ \begin{array}{c} G(\aleph'_{\star n}, \aleph'_{\star n}, \aleph'_{\star n+1}), \\ G(\wp'_{\star n}, \wp'_{\star n}, \wp'_{\star n+1}), \\ G(\varpi'_{\star n}, \varpi'_{\star n}, \varpi'_{\star n+1}) \end{array} \right\}. \end{aligned}$$

Similar lines follows, we have

$$\begin{aligned} & \lim_{n\to\infty} \max\left\{ \begin{array}{c} G(\aleph'_{\star n+1}, \aleph'_{\star n+1}, \aleph'_{\star n+2}), \\ G(\wp'_{\star n+1}, \wp'_{\star n+1}, \wp'_{\star n+2}) \\ G(\varpi'_{\star n+1}, \varpi'_{\star n+1}, \varpi'_{\star n+2}) \end{array} \right\} \\ \leq \ & \lim_{n\to\infty} \lambda \max\left\{ \begin{array}{c} G(\aleph'_{\star n}, \aleph'_{\star n}, \aleph'_{\star n+1}), \\ G(\wp'_{\star n}, \wp'_{\star n}, \wp'_{\star n+1}) \\ G(\varpi'_{\star n}, \varpi'_{\star n}, \varpi'_{\star n+1}) \end{array} \right\} \\ & \vdots \\ \leq \ & \lim_{n\to\infty} \lambda^n \max\left\{ \begin{array}{c} G(\aleph'_{\star 0}, \aleph'_{\star 0}, \aleph'_{\star 1}), \\ G(\wp'_{\star 0}, \wp'_{\star 0}, \wp'_{\star 1}), \\ G(\varpi'_{\star 0}, \varpi'_{\star 0}, \varpi'_{\star 1}) \end{array} \right\} = 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} G(\aleph'_{\star n+1}, \aleph'_{\star n+1}, \aleph'_{\star n+2}) = 0$, $\lim_{n\to\infty} G(\wp'_{\star n+1}, \wp'_{\star n+1}, \wp'_{\star n+2}) = 0$ and $\lim_{n\to\infty} G(\varpi'_{\star n+1}, \varpi'_{\star n+1}, \varpi'_{\star n+2}) = 0$. By use of the rectangle inequality, for $n > m$, we get

$$\begin{aligned} & \lim_{m\to\infty} G\left(\aleph'_{\star n}, \aleph'_{\star n}, \aleph'_{\star m}\right) \\ \leq \ & \lim_{m\to\infty} G\left(\aleph'_{\star m}, \aleph'_{\star m+1}, \aleph'_{\star m+1}\right) \\ & + \lim_{m\to\infty} G\left(\aleph'_{\star m+1}, \aleph'_{\star n}, \aleph'_{\star n}\right), \\ \leq \ & \lim_{m\to\infty} G\left(\aleph'_{\star m}, \aleph'_{\star m+1}, \aleph'_{\star m+1}\right) \\ & + \lim_{m\to\infty} G\left(\aleph'_{\star m+2}, \aleph'_{\star m+2}, \aleph'_{\star m+1}\right) \\ & + \cdots + \lim_{m\to\infty} G\left(\aleph'_{\star n-1}, \aleph'_{\star n}, \aleph'_{\star n}\right) = 0. \end{aligned}$$

By similar arguments, $G\left(\wp'_{\star n}, \wp'_{\star n}, \wp'_{\star m}\right) \to 0$, $G\left(\varpi'_{\star n}, \varpi'_{\star n}, \varpi'_{\star m}\right) \to 0$ as $m \to \infty$. This shows that $\{\aleph'_{\star n}\}, \{\wp'_{\star n}\}, \{\varpi'_{\star n}\}$ are Cauchy sequences in the G-metric space $(\Im, G)$ and by completeness of $(\Im, G)$, there exist $\imath'_\star, \jmath'_\star, \ell'_\star \in \Im$ with

$$\lim_{n\to\infty} \aleph'_{\star n+1} = \imath'_\star \quad \lim_{n\to\infty} \wp'_{\star n+1} = \jmath'_\star$$
$$\lim_{n\to\infty} \varpi'_{\star n+1} = \ell'_\star$$

By using (χ_1). we have

$$\begin{aligned} & G\left(\mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi), \imath'_\star\right) \\ = \ & \lim_{n\to\infty} G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi), \\ \mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star n+1}, \wp'_{\star n+1}, \varpi'_{\star n+1}, \xi) \end{pmatrix} \\ \leq \ & \lim_{n\to\infty} \lambda \max\left\{ \begin{array}{c} G(\imath'_\star, \imath'_\star, \aleph'_{\star n+1}), \\ G(\jmath'_\star, \jmath'_\star, \wp'_{\star n+1}), \\ G(\ell'_\star, \ell'_\star, \varpi'_{\star n+1}) \end{array} \right\} \end{aligned}$$

Similar lines follows, we have

$$\begin{aligned} & \max\left\{ \begin{array}{c} G\left(\mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi), \imath'_\star\right), \\ G\left(\mathcal{H}_{\mathcal{G}}(\jmath'_\star, \ell'_\star, \imath'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\jmath'_\star, \ell'_\star, \imath'_\star, \xi), \jmath'_\star\right) \\ G\left(\mathcal{H}_{\mathcal{G}}(\ell'_\star, \imath'_\star, \jmath'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\ell'_\star, \imath'_\star, \jmath'_\star, \xi), \ell'_\star\right) \end{array} \right\} \\ \leq \ & \lim_{n\to\infty} \lambda \max\left\{ \begin{array}{c} G(\imath'_\star, \imath'_\star, \aleph'_{\star n+1}), \\ G(\jmath'_\star, \jmath'_\star, \wp'_{\star n+1}), \\ G(\ell'_\star, \ell'_\star, \varpi'_{\star n+1}) \end{array} \right\} = 0 \end{aligned}$$

It follows that $\mathcal{H}_{\mathcal{G}}(\imath'_\star, \jmath'_\star, \ell'_\star, \xi) = \imath'_\star, \mathcal{H}_{\mathcal{G}}(\jmath'_\star, \ell'_\star, \imath'_\star, \xi) = \jmath'_\star$, $\mathcal{H}_{\mathcal{G}}(\ell'_\star, \imath'_\star, \jmath'_\star, \xi) = \ell'_\star$ Thus $\xi \in \Im$. Hence $\Im$ is closed in $[0, 1]$. Let $\xi_0 \in \Im$, then there exist $\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0} \in U$ with
$\aleph'_{\star 0} = \mathcal{H}_{\mathcal{G}}(\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0}, \xi_0)$,
$\wp'_{\star 0} = \mathcal{H}_{\mathcal{G}}(\wp'_{\star 0}, \varpi'_{\star 0}, \aleph'_{\star 0}, \xi_0)$,
$\varpi'_{\star 0} = \mathcal{H}_{\mathcal{G}}(\varpi'_{\star 0}, \aleph'_{\star 0}, \wp'_{\star 0}, \xi_0)$. Since U is open, then there exist $r > 0$ such that $B_G(\aleph'_{\star 0}, r) \subseteq U$.
Choose $\xi \in (\xi_0 - \epsilon, \xi_0 + \epsilon)$ such that
$|\xi - \xi_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$, then for

$$\begin{aligned} & \aleph'_\star \in \overline{B_G(\aleph'_{\star 0}, r)} \\ = \ & \left\{\aleph'_\star \in \Im / G(\aleph'_\star, \aleph'_\star, \aleph'_{\star 0}) \leq r + G(\aleph'_{\star 0}, \aleph'_{\star 0}, \aleph'_{\star 0})\right\}, \end{aligned}$$

Also

$$\begin{aligned} & G\left(\mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \aleph'_{\star 0}\right) \\ = \ & G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0}, \xi_0) \end{pmatrix} \\ \leq \ & G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0) \end{pmatrix} \\ & + G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0}, \xi_0) \end{pmatrix} \\ \leq \ & M|\xi - \xi_0| + G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0}, \xi_0) \end{pmatrix} \\ \leq \ & \frac{1}{M^{n-1}} + G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0}, \xi_0) \end{pmatrix}. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} & G\left(\mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi), \aleph'_{\star 0}\right) \\ \leq \ & G\begin{pmatrix} \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_\star, \wp'_\star, \varpi'_\star, \xi_0), \\ \mathcal{H}_{\mathcal{G}}(\aleph'_{\star 0}, \wp'_{\star 0}, \varpi'_{\star 0}, \xi_0) \end{pmatrix} \\ \leq \ & \lambda \max\left\{ \begin{array}{c} G(\aleph'_\star, \aleph'_\star, \aleph'_{\star 0}), \\ G(\wp'_\star, \wp'_\star, \wp'_{\star 0}), \\ G(\varpi'_\star, \varpi'_\star, \varpi'_{\star 0}) \end{array} \right\}. \end{aligned}$$

Similar lines follows, Thus we have,

$$\max \left\{ \begin{array}{c} G\left(\mathcal{H}_{\mathcal{G}}(\aleph'_{\star}, \wp'_{\star}, \varpi'_{\star}, \xi), \mathcal{H}_{\mathcal{G}}(\aleph'_{\star}, \wp'_{\star}, \varpi'_{\star}, \xi), \aleph'_{\star 0}\right) \\ G\left(\mathcal{H}_{\mathcal{G}}(\wp'_{\star}, \varpi'_{\star}, \aleph'_{\star}, \xi), \mathcal{H}_{\mathcal{G}}(\wp'_{\star}, \varpi'_{\star}, \aleph'_{\star}, \xi), \wp'_{\star 0}\right) \\ G\left(\mathcal{H}_{\mathcal{G}}(\varpi'_{\star}, \aleph'_{\star}, \wp'_{\star}, \xi), \mathcal{H}_{\mathcal{G}}(\varpi'_{\star}, \aleph'_{\star}, \wp'_{\star}, \xi), \varpi'_{\star 0}\right) \end{array} \right\}$$

$$\leq \lambda \max \left\{ \begin{array}{c} r + G(\aleph'_{\star 0}, \aleph'_{\star 0}, \aleph'_{\star 0}), \\ r + G(\wp'_{\star 0}, \wp'_{\star 0}, \wp'_{\star 0}) \\ r + G(\varpi'_{\star 0}, \varpi'_{\star 0}, \varpi'_{\star 0}) \end{array} \right\}.$$

Thus for each fixed $\xi \in (\xi_0 - \epsilon, \xi_0 + \epsilon)$,
$\mathcal{H}_{\mathcal{G}}(., \xi) : \overline{B_G(\aleph'_{\star 0}, r)} \to \overline{B_G(\aleph'_{\star 0}, r)}$,
$\mathcal{H}_{\mathcal{G}}(., \xi) : \overline{B_G(\wp'_{\star 0}, r)} \to \overline{B_G(\wp'_{\star 0}, r)}$ and
$\mathcal{H}_{\mathcal{G}}(., \xi) : \overline{B_G(\varpi'_{\star 0}, r)} \to \overline{B_G(\varpi'_{\star 0}, r)}$.
Afterward, Theorem (III.6)'s are all met. Our conclusion is that $\mathcal{H}_{\mathcal{G}}(., \xi)$ has a tripled fixed point in $\overline{U}^3$. But it has to be in U^3. As a result, (χ_0) is true. Accordingly, $\xi \in \Im$ for any $\xi \in (\xi_0 - \epsilon, \xi_0 + \epsilon)$. Consequently, $(\xi_0 - \epsilon, \xi_0 + \epsilon) \subseteq \Im$. It is obvious that [0, 1] is open for $\Im$.
We follow the same approach for the opposite inference.

IV. Conclusion

By using a generalised contractive condition in G-metric space that involves altering distance function, we were able to guarantee the presence and individuality of a common tripled fixed point for two mappings. There are two applications with illustrations.

Significance Statement

In order to establish tripled fixed point results, this paper provided a methodology that involved altering distance function in G-metric spaces under generalised contractive conditions. Researchers will be able to generalise various contractions in G-metric spaces with applications to integral equations and homotopy theory with the aid of this study. As a result, a fresh framework for G-metric spaces might be developed.

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