Perturbed Initial Value Problem for Chaplygin System with Combustion

Yujin Liu and Wenhua Sun

***Abstract*—In the present paper, the authors consider the perturbed initial value problem of the Chapman-Jouguet model for the Chaplygin gas. We obtain the unique solution by analyzing the elementary waves under the global entropy conditions. We observe that the combustion wave solution may be extinguished after perturbation which tells the instability of the unburnt gas. And we also capture the transitions between the deflagration wave and the detonation wave.**

***Index Terms*—Wave interaction, Riemann problem, Detonation wave, Deflagration wave, Chaplygin gas.**

I. INTRODUCTION

IN this study we investigate the ideal combustible Chaplygin gas equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho E)_t + (\rho u E + pu)_x = 0, \\ Q(x,t) = \begin{cases} 0, & \text{if } \sup\limits_{0\le z\le t} T(x,z) > T_i; \\ Q(x,0), & \text{if } \sup\limits_{0\le z\le t} T(x,z) \le T_i, \end{cases} \end{cases} \tag{1}$$

where ρ is the density, u is the velocity, $p < 0$ is the pressure. T and T_i are respectively the temperature and the ignition temperature. $E = \frac{u^2}{2} + e + Q$ and $e = -\frac{p}{2\rho}$ is the internal energy, Q is the chemical binding energy. The state equation is $p = -\frac{1}{\rho}$. We suppose that the process of combustion is exothermic [1]. The discussions about the Chaplygin gas are shown in [2], [3], [4], [5], [6].

In [7], they discussed the delta shock and the vacuum state problem in detail by letting $p \to 0$.

In [8], they studied the Riemann solutions, and investigated the asymptotic behavior. The Riemann problem was studied in [9], [10], [11].

We usually apply the two simplified models [1], [12] to study the combustion phenomena. In [13], they studied the idealized CJ model in Lagrangian coordinates

$$\begin{cases} u_t + p_x = 0, \\ \tau_t - u_x = 0, \\ E_t + (up)_x = 0, \\ Q(x,t) = \begin{cases} 0, & \text{if } \sup\limits_{0\le z\le t} T(x,z) > T_i; \\ Q(x,0), & \text{if } \sup\limits_{0\le z\le t} T(x,z) \le T_i, \end{cases} \end{cases} \tag{2}$$

here $\tau = \frac{1}{\rho}$, $p > 0$.

Manuscript received April 5, 2023; revised September 8, 2023. This work is supported by the Foundation for Young Scholars of Shandong University of Technology (No. 115024).

Yujin Liu is an Associate Professor in School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, 255000, P. R. China. (Corresponding author, e-mail: yjliu98@126.com)

Wenhua Sun is a Professor in School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, 255000, P. R. China. (e-mail: sunwenhua@sdut.edu.cn)

In [14] they discussed the SZND model

$$\begin{cases} u_t + p_x = 0, \\ \tau_t - u_x = 0, \\ E_t + (up)_x = 0, \\ Q(x,t) = -\frac{k}{t}\phi(T)q, \end{cases} \tag{3}$$

here ϕ is the Heaviside function and they got the solution uniquely. In [15] the authors studied the limit behaviors of the solutions.

In [16], we investigated the Riemann problem for (1) and

$$(\tau, u, p, Q) = \begin{cases} (\tau_l, u_l, p_l, Q_l), & \text{when } x < 0, \\ (\tau_r, u_r, p_r, Q_r), & \text{when } x > 0. \end{cases} \tag{4}$$

In [17], we studied the wave interactions for (1) when the solutions contain no S_δ.

In the present study, we analyze (1) and

$$(\tau, u, p)(x, 0) = \begin{cases} (\tau_l, u_l, p_l), & -\infty < x < -\epsilon, \\ (\tau_m, u_m, p_m), & -\epsilon \le x \le \epsilon, \\ (\tau_r, u_r, p_r), & \epsilon < x < \infty, \end{cases} \tag{5}$$

where the parameter $\epsilon > 0$ is arbitrary and small enough. We observe that for the most part (1) and (4) can preserve the structure of the original solution after the perturbation, while for some situations, the perturbation can bring about significant changes. It is shown that the combustion wave may be extinguished, and we capture the transitions between the deflagration wave and the detonation wave.

This article is organized as follows. we give the preliminaries in II. In Section III, we construct uniquely the solution for (1) and (5) according the different cases. Section IV gives the main conclusions.

II. PRELIMINARIES

For the later study, we give some preliminaries [16], [18], [19], [20]. The characteristic roots of (1) are described by

$$\lambda_1 = u - \sqrt{\frac{-p}{\rho}}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{\frac{-p}{\rho}}. \tag{6}$$

The right characteristic vector of λ_1, λ_2, λ_3 is given respectively by

$$\vec{\nu}_1 = (1, -\frac{1}{\rho}\sqrt{-\frac{p}{\rho}}, -\frac{p}{\rho})^\top, \qquad \vec{\nu}_2 = (1, 0, 0)^\top,$$

$$\vec{\nu}_3 = (1, \frac{1}{\rho}\sqrt{-\frac{p}{\rho}}, -\frac{p}{\rho})^\top.$$

It follows from $\nabla\lambda_i \cdot \vec{\nu}_i \equiv 0, i = 1, 2, 3.$

$\overrightarrow{R}(-)$(or $\overleftarrow{R}(-)$) is given by

$$\begin{cases} p\rho = p_-\rho_-, \\ u = u_- \pm \frac{p - p_-}{\sqrt{-p_-\rho_-}}, \quad (p > p_-, \text{or } p < p_-), \end{cases} \tag{7}$$

and $\overrightarrow{S}(-)$(or $\overleftarrow{S}(-)$) is given by

$$\begin{cases} p\rho = p_-\rho_-, \\ \frac{u-u_-}{p-p_-} = \pm\sqrt{-\frac{1}{p_-\rho_-}}, \quad (p_- > p, \text{or } p_- < p). \end{cases} \tag{8}$$

J is described by

$$\begin{cases} [u] = [p] = 0, \\ \rho_- \neq \rho_+. \end{cases} \tag{9}$$

Suppose

$$\rho = \rho_0 + \sigma(t)\delta(x - x(t)), \quad \rho_0 = \begin{cases} \rho_-, & x < x(t), \\ \rho_+, & x > x(t), \end{cases} \tag{10}$$

$$u(x,t) = \begin{cases} u_-, & x < x(t), \\ u_\delta, & x = x(t), \\ u_+, & x > x(t), \end{cases} \tag{11}$$

$$p(x,t) = \begin{cases} p_-, & x < x(t), \\ 0, & x = x(t), \\ p_+, & x > x(t). \end{cases} \tag{12}$$

When $\rho_+ \neq \rho_-$,

$$\sigma(t) = \sqrt{\rho_-\rho_+(u_+ - u_-)^2 - (\rho_+ - \rho_-)(p_+ - p_-)}\ t$$

$$u_\delta = \frac{\rho_+u_+ - \rho_-u_- + \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}}{\rho_+ - \rho_-},$$

when $\rho_+ = \rho_-$,

$$\sigma(t) = (\rho_-u_- - \rho_+u_+)t$$

$$u_\delta = \frac{1}{2}(u_+ + u_-).$$

Further, the entropy condition of S_δ is given by

$$u_+ + \sqrt{-\frac{p_+}{\rho_+}} < \frac{\mathrm{d}x(t)}{\mathrm{d}t} < u_- - \sqrt{-\frac{p_-}{\rho_-}}. \tag{13}$$

The noncombustion wave curves are given in (u,p) (Fig. 1.(i) and Fig. 1.(ii))

On the other hand, the R-H relations

$$\begin{cases} \zeta[u] = [p], \\ \zeta[\tau] = -[u], \\ \zeta[E] = [up], \end{cases}$$

reveal that

$$-\tau_r p + p_r\tau = 2q_0 > 0.$$

$\overrightarrow{D}(+)$ in (u,p) (Fig. 2.) is described by

$$\overrightarrow{D}(+): \qquad \frac{u-u_+}{p-p_+} = \sqrt{-\frac{2q_0 + \tau_+(p-p_+)}{p_+(p-p_+)}}, \tag{14}$$

where $p_+ < p < 0$ or $p < p_+ - \frac{2q_0}{\tau_r}$. The detonation wave curve corresponds to $p_+ < p < 0$ and the deflagration wave curve corresponds to $p < p_+ - 2q_0\rho_+$ [1].

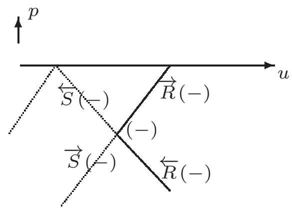

Fig. 1.(i) The backward wave curves.

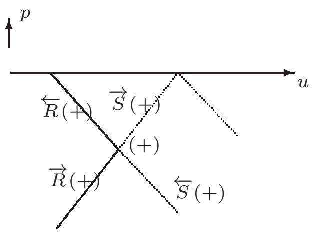

Fig. 1.(ii) The forward wave curves.

When $Q_r > 0$, $\overrightarrow{W}(r)$ is given by as follows

$$\overrightarrow{W}(r) \doteq \overrightarrow{W}_S(r) \cup S_\delta(r) \cup \overrightarrow{DF}(r) \cup \overrightarrow{DT}(r),$$

and the forward noncombustion wave curve $\overrightarrow{R}(r) \cup \overrightarrow{S}(r)$ is described by $\overrightarrow{W}_S(r)$.

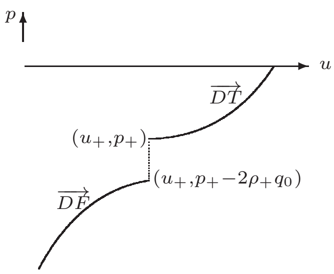

Fig. 2. Combustion wave in (u,p).

For the situation that $Q_r = 0$, $Q_l = 0$, it is discussed in the paper [19]. Therefore, we investigate the following cases.

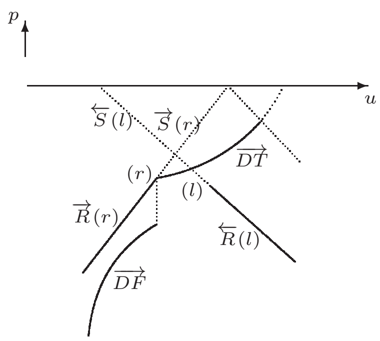

Fig. 3. When $Q_l = 0,\ Q_r = q_0 > 0$.

Case 2.1. When $Q_l = 0,\ Q_r = q_0 > 0$ (Fig. 3.).

Notice that $\overleftarrow{W}(l) = \overleftarrow{W}_S(l) \cup S_\delta(l)$, and $\overrightarrow{W}(r) = \overrightarrow{W}_S(r) \cup S_\delta(r) \cup \overrightarrow{DF}(r) \cup \overrightarrow{DT}(r)$.

Subcase 2.1.1 If $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, and the intersection point of $\overleftarrow{W}(l)$ and $\overrightarrow{W}(r)$ is not unique, we should get uniquely the solution from *global entropy conditions (GEC):*

(i) η is as small as possible, where η is defined by the oscillation frequency of $T(\eta)$ between $\{\eta \in R^1 : T(\eta) \leq T_i\}$ and $\{\eta \in R^1 : T(\eta) > T_i\}$;

(ii) the combustion wave is as many as possible.

$\star_1$ and $\star_2$ are respectively the intersection points of $\overleftarrow{W}_S(l)$ with $\overrightarrow{W}_S(r)$, $\overrightarrow{DF}(r)$ or $\overrightarrow{DT}(r)$. The temperature at the point $\star_1$, $\star_2$ is respectively denoted by T_1, T_2 (Fig. 5. and Fig. 6.).

(1) If $T_l > T_i$, $T_2 > T_i$, then $\eta(\star_1) = 1$, $\eta(\star_2) = 1$, due to *(ii)*, we select $\star_2$ and gain $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DF} or \overrightarrow{DT}$ (Fig. 4.);

(2) if $T_l > T_i$, $T_2 \leq T_i (\Rightarrow T_1 \leq T_i)$, then $\eta(\star_1) = 1$, $\eta(\star_2) = 3$, due to *(i)*, we select $\star_1$ and gain $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$ (Fig. 5.);

(3) if $T_l \leq T_i$, $T_1 \leq T_i$, then $\eta(\star_1) = 0$, $\eta(\star_2) = 2$, due to *(i)*, we select $\star_1$ and gain $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$ (Fig. 5.);

(4) if $T_l \leq T_i$, $T_1 > T_i (\Rightarrow T_2 > T_i)$, then $\eta(\star_1) = 2$, $\eta(\star_2) = 2$, due to *(ii)*, we select $\star_2$ and gain $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DF} or \overrightarrow{DT}$(Fig. 4.).

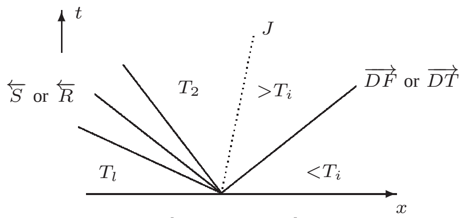

Fig. 4. Combustion wave solution.

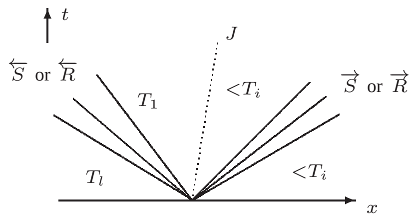

Fig. 5. Non-combustion wave solution.

Subcase 2.1.2 As $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we obtain the delta shock wave solution S_δ (see [17], [18], [19]).

Case 2.2. When $Q_l > 0,\ Q_r > 0$ (Fig. 6.).

We know that $\overleftarrow{W}(l) = \overleftarrow{W}_S(l) \cup S_\delta(l) \cup \overleftarrow{DT}(l) \cup \overleftarrow{DF}(l)$, and $\overrightarrow{W}(r) = \overrightarrow{W}_S(r) \cup S_\delta(r) \cup \overrightarrow{DT}(r) \cup \overrightarrow{DF}(r)$.

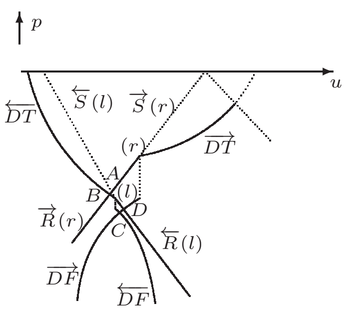

Fig. 6. When $Q_l > 0,\ Q_r > 0$.

Subcase 2.2.1 When $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$ (Fig. 7.(i)(ii)(iii)(iv)).

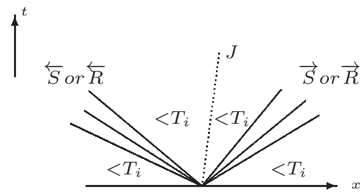

Fig. 7.(i) Solution of the point A.

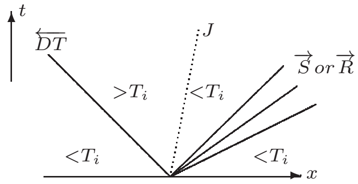

Fig. 7.(ii) Solution of the point B.

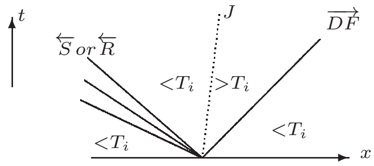

Fig. 7.(iii) Solution of the point C.

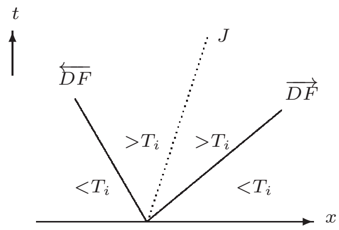

Fig. 7.(iv) Solution of the point D.

Since we have $\eta = 0$ for Fig. 7.(i), $\eta = 2$ for the other cases, we select A and gain $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

Subcase 2.2.2 When $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we gain the S_δ solution.

Theorem 2.1 There is unique solution of (1) and (4).

III. Initial probem of (1) and(5)

When S_δ appears, we just investigate the cases containing the interesting combustion phenomena for simplicity. In the following we discuss the six kinds of wave interactions: S_δ and $\overleftarrow{DT}$, S_δ and $\overleftarrow{DF}$, S_δ and $\overrightarrow{DT}$, S_δ and $\overrightarrow{DF}$, J and $\overleftarrow{DF}$, J and $\overleftarrow{DT}$.

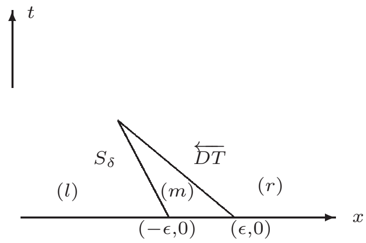

Fig. 8. The interaction of S_δ and $\overleftarrow{DT}$.

Case 3.1. S_δ and $\overleftarrow{DT}$ (Fig. 8.) We divide the discussions into four subcases as follows.

Case 3.1.1 (l) is burnt, (m) is unburnt and (r) is unburnt.

Since $u_m - \sqrt{-\frac{p_m}{\rho_m}} < \sigma_\delta < u_l - \sqrt{-\frac{p_l}{\rho_l}}$ for S_δ, and $u_r - \sqrt{-\frac{p_r}{\rho_r}} < \sigma < u_m - \sqrt{-\frac{p_m}{\rho_m}}$ for $\overleftarrow{DT}$, S_δ will overtake $\overleftarrow{DT}$ in the finite time. Notice that $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we proceed as follows. If $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get the S_δ solution; otherwise (Fig. 9.), we find the perturbed solution is $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DT}$.

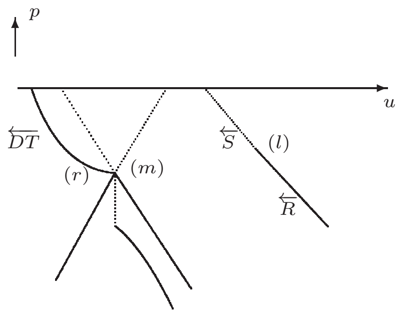

Fig. 9. The wave curves in (u, p) for Subcase 3.1.1.

Case 3.1.2 (l) is unburnt, (m) is unburnt and (r) is unburnt.

If $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we know that the perturbed solution is the delta shock wave solution; otherwise, we get the perturbed solution under *GEC* since for this case there is at most three intersection points (Fig. 10.).

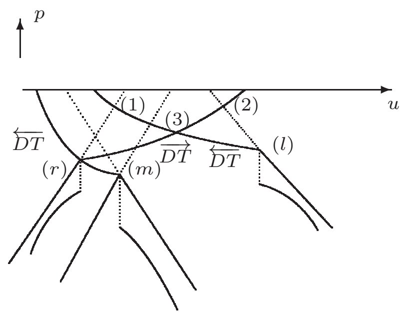

Fig. 10. The wave curves in (u, p) for Subcase 3.1.2.

Since $\eta = 2$ for intersection point 1, 2, 3 (Fig. 11.(i)-(iii)), from *GEC*, we get $\overleftarrow{DT} + J + \overrightarrow{DT}$ (Fig. 11.(iii)).

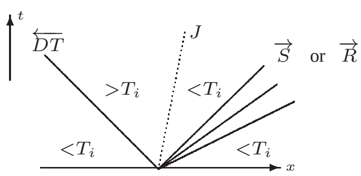

Fig. 11.(i) Solution of the point 1.

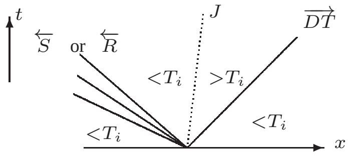

Fig. 11.(ii) Solution of the point 2.

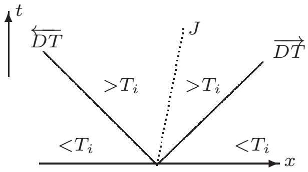

Fig. 11.(iii) Solution of the point 3.

Case 3.1.3 (l) is unburnt, (m) is unburnt and (r) is burnt.

If $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we obtain S_δ, otherwise, $\overleftarrow{W}(l)$ intersects with $\overrightarrow{W}(r)$ uniquely, it yields that $\overleftarrow{DT} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$ (Fig. 12.).

Case 3.1.4 (l) is burnt, (m) is unburnt and (r) is burnt.

From $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get the S_δ solution (Fig. 13.).

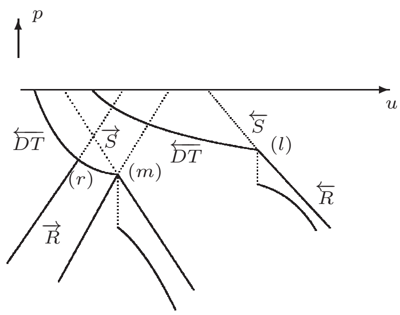

Fig. 12. The wave curves in (u, p).

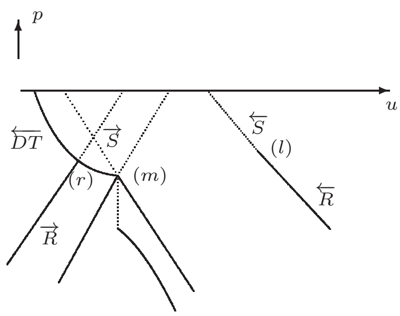

Fig. 13. The wave curves in (u, p).

Theorem 3.1 When S_δ intersects with $\overleftarrow{DT}$, the combustion wave may be extinguished. And for this case, it may produce the combustion wave in the opposite direction.

Case 3.2. S_δ and $\overrightarrow{DT}$ (Fig. 14.)

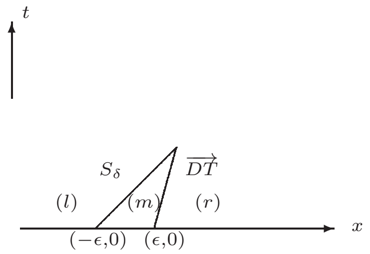

Fig. 14. The interaction of S_δ and $\overrightarrow{DT}$.

Case 3.2.1 (l) is burnt, (m) is burnt and (r) is unburnt. Since S_δ will overtake the combustion wave, we consider the fact that $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, and get that $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$. It yields that the perturbed solution is described by the delta shock wave (Fig. 15.)

Case 3.2.2 (l) is unburnt, (m) is burnt and (r) is unburnt. Due to $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we konw $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, and get the S_δ soltuin(Fig. 16.). Similar with Case 3.1.2, we get $\overleftarrow{DT} + J + \overrightarrow{DT}$.

Theorem 3.2 When S_δ intersects with $\overrightarrow{DT}$, the detonation wave may be extinguished. And the detonation wave may persist and the combustion wave in the opposite direction may produce.

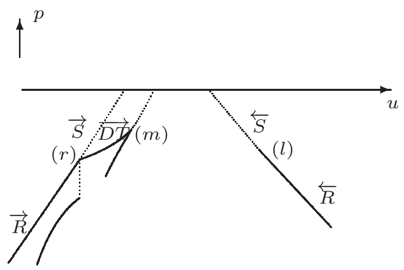

Fig. 15. The wave curves in (u, p).

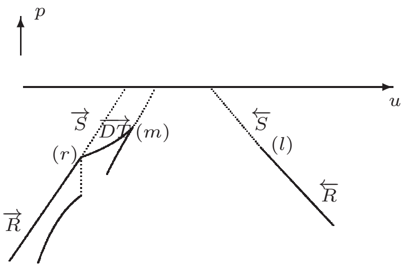

Fig. 16. The wave curves in (u, p).

Case 3.3. S_δ and $\overleftarrow{DF}$ (Fig. 17.)

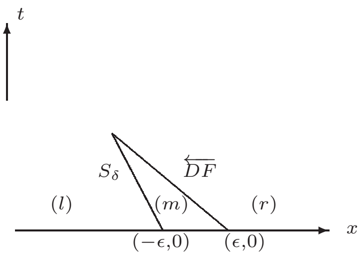

Fig. 17. The interaction of S_δ and $\overleftarrow{DF}$.

Case 3.3.1 (l) is burnt, (m) is unburnt and (r) is unburnt. Since $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we discuss as follows. When $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get the S_δ solution; when $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, $\overleftarrow{W}(l)$ intersects with $\overrightarrow{W}(r)$ uniquely, the perturbed solution is $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DT}$, othersize, there are at most two intersection points (Fig. 18.), we should select the solution according to *GEC*.

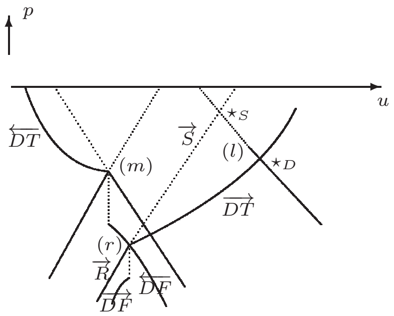

Fig. 18. The wave curves in (u, p).

For simplicity, we denote respectively $\star_S$, $\star_D$ the intersection point of $\overleftarrow{W}_S(l)$ and $\overrightarrow{W}_S(r)$, $\overrightarrow{DT}(r)$. The temperature is T_S, T_D at the point $\star_S$, $\star_D$ respectively.

(1) When $T_l > T_i$, $T_D > T_i$, then $\eta(\star_S) = 1$, $\eta(\star_D) = 1$, we pick out $\star_D$ and get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DT}$.

(2) When $T_l > T_i, T_D \leq T_i (\Rightarrow T_S \leq T_i)$, then $\eta(\star_S) = 1$, $\eta(\star_D) = 3$, we pick out $\star_S$ and get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

(3) When $T_l \leq T_i$, $T_S \leq T_i$, then $\eta(\star_S) = 0$, $\eta(\star_D) = 2$, we pick out $\star_S$ and get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

(4) When $T_l \leq T_i, T_S > T_i (\Rightarrow T_D \geq T_i)$, then $\eta(\star_S) = 2$, $\eta(\star_D) = 2$, we pick out $\star_D$ and get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DT}$.

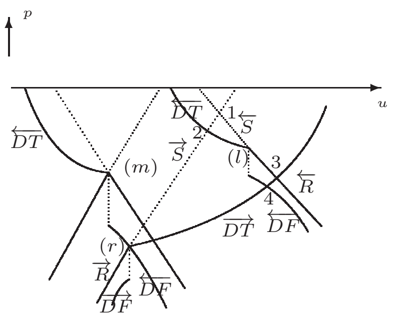

Fig. 19. The wave curves in (u, p).

Case 3.3.2 (l) is unburnt, (m) is unburnt and (r) is unburnt.

Since $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we discuss as follows. When $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get the S_δ solution; when $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, and $\overleftarrow{W}(l)$ intersects with $\overrightarrow{W}(r)$ uniquely, we get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{DT}$, otherwise (Fig. 19.), Since we have $\eta = 0$ corresponding to the intersection point 1, and $\eta = 2$ for the other intersection points 2, 3, 4, from *GEC*, we pick out the point 1 and get $\overleftarrow{S} + J + \overrightarrow{S}$.

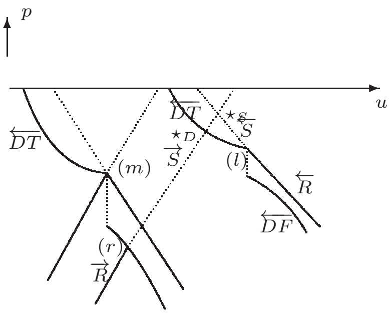

Fig. 20. The wave curves in (u, p).

Case 3.3.3 (l) is unburnt, (m) is unburnt and (r) is burnt. Due to $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we go on discussing. When $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get the S_δ solution; when $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, if $\overleftarrow{W}(l)$ intersects with $\overrightarrow{W}(r)$ uniquely, it follows that $\overleftarrow{DT} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$, otherwise, there are at most two intersection points (Fig. 20.), we should select the solution according to *GEC*.

(1) When $T_r > T_i$, $T_D > T_i$, then $\eta(\star_S) = 1$, $\eta(\star_D) = 1$, we pick out $\star_D$ and get $\overleftarrow{DT} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

(2) When $T_r > T_i, T_D \leq T_i (\Rightarrow T_S \leq T_i)$, then $\eta(\star_S) = 1$, $\eta(\star_D) = 3$, we pick out $\star_S$ and get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

(3) When $T_r \leq T_i$, $T_S \leq T_i$, then $\eta(\star_S) = 0$, $\eta(\star_D) = 2$, we pick out $\star_S$ and get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

(4) When $T_r \leq T_i$, $T_S > T_i (\Rightarrow T_D > T_i)$, then $\eta(\star_S) = 2$, $\eta(\star_D) = 2$, we pick out $\star_D$ and get $\overleftarrow{DT} + J + \overrightarrow{S}\, or\, \overrightarrow{R}$.

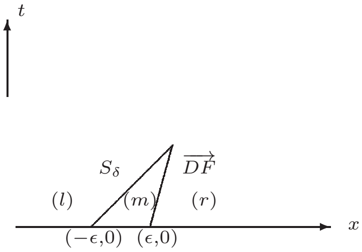

Fig. 21. The interaction of S_δ and $\overrightarrow{DT}$.

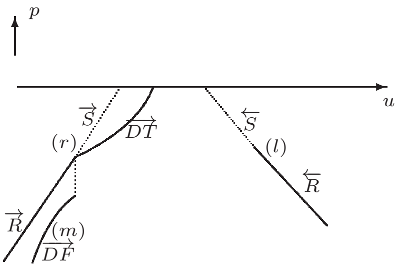

Fig. 22. The wave curves in (u, p).

Case 3.3.4 (l) is burnt, (m) is unburnt and (r) is burnt. Due to $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we proceed to discuss. When $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get the S_δ solution; when $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get $\overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}$.

Theorem 3.3 In this case, as S_δ intersects with $\overleftarrow{DF}$, the combustion wave is extinguished. Furthermore, we observe DF may be transformed into DT.

Case 3.4. S_δ and $\overrightarrow{DF}$ (Fig. 21.)

Case 3.4.1 (l) is burnt, (m) is burnt and (r) is unburnt.

Due to $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we know that $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$ (Fig. 22.), and the perturbed solution is the delta shock wave.

Case 3.4.2 (l) is unburnt, (m) is burnt and (r) is unburnt.

Due to the fact $u_l - u_m \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_m}{\rho_m}}$, we know that if $u_l - u_r \geq \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$, we get S_δ; otherwise, similar discussions with Case 3.1.2, we get $\overleftarrow{DT} + J + \overrightarrow{DT}$ (Fig. 23.).

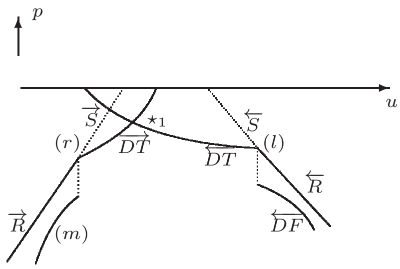

Fig. 23. The wave curves in (u, p).

Theorem 3.4 In this case, when S_δ intersects with $\overrightarrow{DF}$, the deflagration wave is extinguished for some case. Furthermore, we find that DF may be transformed into DT.

Case 3.5. J and $\overleftarrow{DF}$ (Fig. 24.)

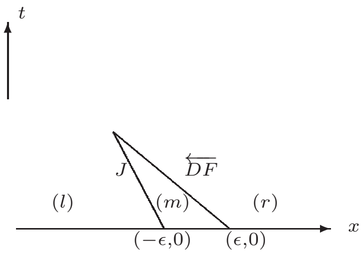

Fig. 24. The interaction of J and $\overleftarrow{DF}$.

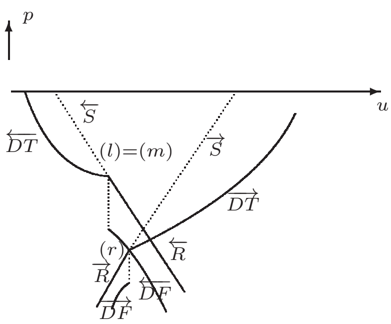

Fig. 25. The wave curves in (u, p).

Case 3.5.1 (l) is burnt, (m) is burnt and (r) is unburnt. There are at most two intersection points (Fig. 25.), we should select the solution according to *GEC*.

(1) When $T_l > T_i$, $T_2 > T_i$, then $\eta(\star_1) = 1$, $\eta(\star_2) = 1$, we pick out $\star_2$ and get $\overleftarrow{R} + J + \overrightarrow{DT}$.

(2) When $T_l > T_i$, $T_2 \leq T_i (\Rightarrow T_1 \leq T_i)$, then $\eta(\star_1) = 1$, $\eta(\star_2) = 3$, we pick out $\star_1$ and get $\overleftarrow{R} + J + \overrightarrow{S}$.

(3) When $T_l \leq T_i$, $T_1 \leq T_i$, then $\eta(\star_1) = 0$, $\eta(\star_2) = 2$, we pick out $\star_1$ and get $\overleftarrow{R} + J + \overrightarrow{S}$.

(4) When $T_l \leq T_i$, $T_1 > T_i (\Rightarrow T_2 > T_i)$, then $\eta(\star_1) = 2$, $\eta(\star_2) = 2$, we pick out $\star_2$ and get $\overleftarrow{R} + J + \overrightarrow{DT}$.

Case 3.5.2 (l) is unburnt, (m) is unburnt and (r) is unburnt (Fig. 25.).

Since we have $\eta = 0$, from *GEC*, we get $\overleftarrow{R} + J + \overrightarrow{S}$.

Case 3.5.3 (l) is unburnt, (m) is unburnt and (r) is burnt. There are at most two intersection points (Fig. 25.), we should select the solution according to *GEC*. For simplicity, we denote respectively $\star_3$, $\star_4$ the intersection point of $\overleftarrow{W}_S(l)$, $\overleftarrow{DF}(l)$ and $\overrightarrow{W}_S(r)$. The temperature is T_3, T_4 at the point $\star_3$, $\star_4$ respectively.

(1) When $T_r > T_i$, $T_4 > T_i$, then $\eta(\star_3) = 1$, $\eta(\star_4) = 1$, we pick out $\star_4$ and get $\overleftarrow{DF} + J + \overrightarrow{S}$.

(2) When $T_r > T_i$, $T_4 \leq T_i(\Rightarrow T_3 \leq T_i)$, then $\eta(\star_3) = 1$, $\eta(\star_4) = 3$, we pick out $\star_3$ and get $\overleftarrow{R} + J + \overrightarrow{S}$.

(3) When $T_r \leq T_i$, $T_3 \leq T_i$, then $\eta(\star_3) = 0$, $\eta(\star_4) = 2$, we pick out $\star_3$ and get $\overleftarrow{R} + J + \overrightarrow{S}$.

(4) When $T_r \leq T_i$, $T_3 > T_i(\Rightarrow T_4 > T_i)$, then $\eta(\star_3) = 2$, $\eta(\star_4) = 2$, we pick out $\star_4$ and get $\overleftarrow{DF} + J + \overrightarrow{S}$.

Case 3.5.4 (l) is burnt, (m) is burnt and (r) is burnt. In this case we obtain $\overleftarrow{R} + J + \overrightarrow{S}$.

Theorem 3.5 In this case, when J intersects with $\overleftarrow{DF}$, the combustion wave is extinguished for some case. Furthermore, we observe that DF may be transformed into DT.

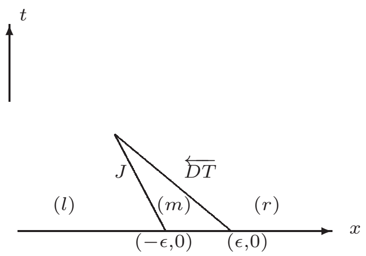

Fig. 26. The interaction of J and $\overleftarrow{DT}$.

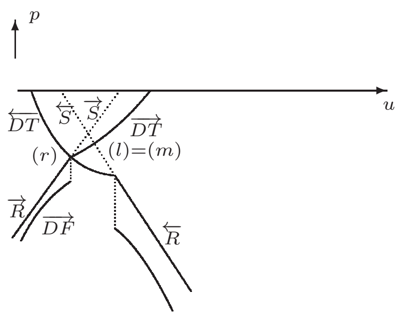

Fig. 27. The wave curves in (u, p).

Case 3.6. J and $\overleftarrow{DT}$ (Fig. 26.)

Case 3.6.1 (l) is burnt, (m) is burnt and (r) is unburnt. There are at most two intersection points (Fig. 27.), we should select the solution from *GEC*.

For simplicity, we denote respectively $\star_5$, $\star_6$ the intersection point of $\overleftarrow{W}_S(l)$ and $\overrightarrow{W}_S(r)$, $\overrightarrow{DT}(r)$. The temperature is T_5, T_6 at the point $\star_5$, $\star_6$ respectively.

(1) When $T_l > T_i$, $T_6 > T_i$, then $\eta(\star_5) = 1$, $\eta(\star_6) = 1$, we pick out $\star_6$ and get $\overleftarrow{S} + J + \overrightarrow{DT}$.

(2) When $T_l > T_i$, $T_6 \leq T_i(\Rightarrow T_5 \leq T_i)$, then $\eta(\star_5) = 1$, $\eta(\star_6) = 3$, we pick out $\star_5$ and get $\overleftarrow{S} + J + \overrightarrow{S}$.

(3) When $T_l \leq T_i$, $T_5 \leq T_i$, then $\eta(\star_5) = 0$, $\eta(\star_6) = 2$, we pick out $\star_5$ and get $\overleftarrow{S} + J + \overrightarrow{S}$.

(4) When $T_l \leq T_i$, $T_5 > T_i(\Rightarrow T_6 > T_i)$, then $\eta(\star_5) = 2$, $\eta(\star_6) = 2$, we pick out $\star_6$ and get $\overleftarrow{S} + J + \overrightarrow{DT}$.

Case 3.6.2 (l) is unburnt, (m) is unburnt and (r) is unburnt.

Since $\eta = 0$ for the intersection point of $\overleftarrow{W}_S(l)$ and $\overrightarrow{W}_S(r)$, and $\eta > 0$ for the other intersection points, we get $\overleftarrow{S} + J + \overrightarrow{S}$.

Case 3.6.3 (l) is unburnt, (m) is unburnt and (r) is burnt. There are at most two intersection points (Fig. 27.), we should select the solution from *GEC*. For simplicity, we denote respectively $\star_7$, $\star_8$ the intersection point of $\overleftarrow{W}_S(l)$, $\overleftarrow{DF}(l)$ and $\overrightarrow{W}_S(r)$. The temperature is T_7, T_8 at the point $\star_7$, $\star_8$ respectively.

(1) When $T_r > T_i$, $T_8 > T_i$, then $\eta(\star_7) = 1$, $\eta(\star_8) = 1$, we pick out $\star_8$ and get $\overleftarrow{DT} + J + \overrightarrow{S}$.

(2) When $T_r > T_i$, $T_8 \leq T_i(\Rightarrow T_7 \leq T_i)$, then $\eta(\star_7) = 1$, $\eta(\star_8) = 3$, we pick out $\star_7$ and get $\overleftarrow{S} + J + \overrightarrow{S}$.

(3) When $T_r \leq T_i$, $T_7 \leq T_i$, then $\eta(\star_7) = 0$, $\eta(\star_8) = 2$, we pick out $\star_7$ and get $\overleftarrow{S} + J + \overrightarrow{S}$.

(4) When $T_r \leq T_i$, $T_7 > T_i(\Rightarrow T_8 > T_i)$, then $\eta(\star_7) = 2$, $\eta(\star_8) = 2$, we pick out $\star_8$ and get $\overleftarrow{DT} + J + \overrightarrow{S}$.

Case 3.6.4 (l) is burnt, (m) is burnt and (r) is burnt. Since there is unique intersection point of $\overleftarrow{W}(l)$ and $\overrightarrow{W}(r)$ (Fig. 27.), the solution is $\overleftarrow{S} + J + \overrightarrow{S}$.

Theorem 3.6 In this case, as J intersects with $\overleftarrow{DT}$, the combustion wave may be extinguished.

IV. Conclusion

Now we conclude our main results as follows.

There exists uniquely the solution of (1) and (5). By investigating the detailed elementary wave interactions, we capture that for some cases the combustion process is extinguished which shows the instability of the unburnt gas. Moreover, we also see the transition from the deflagration wave to the detonation wave.

We assume that the reaction rate of (1) is infinite for simplicity. It is the important model to investigate the combustion problem in many applications, while it has some limitations due to the idealized assumptions. In the further works, we will discuss the construction of solutions for the self-similar ZND model which has the finite reaction rate.

References

[1] R. Courant and K.O. Friedrichs, "Supersonic Flow and Shock Waves", *Interscience*, New York, 1948.

[2] Y. Brenier, "Solutions with concentration to the Riemann problem for one-dimensional Chaplygin gas equations", *Journal of Mathematical Fluid Mechanics*, vol. 7, 326-331, 2005.

[3] S. Chaplygin, "On gas jets", *Moscow University Mathematic Physics,*, vol. 21, pp. 1-121, 1904.

[4] L.H. Guo, W.C. Sheng and T. Zhang, "The two-dimensional Riemann Problem for isentropic Chaplygin gas dynamics system", *Communications on Pure and Applied Analysis*, vol. 9, no. 2, pp. 431-458, 2010.

[5] G. Lai, W.C. Sheng and Y. X. Zheng, "Simple waves and pressure delta waves for a Chaplygin gas in two-dimensions", *Discrete Continuous Dynamical Systems*, vol. 31, no. 2, pp. 489-523, 2011.

[6] D. Serre, "Multidimensional shock interaction for a Chaplygin gas", *Archive for Rational Mechanics and Analysis*, vol. 191, pp. 539-577, 2009.

[7] Q.X. Ding and L.H. Guo, The Vanishing Pressure Limit of Riemann Solutions to the Non-Isentropic Euler Equations for Generalized Chaplygin Gas", *Advances in Mathematical Physics*, vol. 2019, pp. 1-12, 2019.

[8] Y.F. Zhang and M.N. Sun, Concentration Phenomenon of Riemann Solutions for the Relativistic Euler Equations with the Extended Chaplygin Gas", *Acta Applicandae Mathematicae*, vol. 170, pp. 539-568, 2020.

[9] Y.J. Liu and W.H. Sun, "The Riemann problem for the simplified combustion model in magnetogasdynamics," *IAENG International Journal of Applied Mathematics*, vol. 49, No. 4, pp. 513-520, 2019.

[10] Y.J. Liu and W.H. Sun, "Analysis of the Stability of the Riemann Solutions for the Suliciu Relaxation System," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 4, pp. 1014-1019, 2022.

[11] Wenhua Sun and Yujin Liu, "The Generalized Riemann Problem for the Simplified Model in Magnetogasdynamics with Combustion," *Engineering Letters*, vol. 29, no. 2, pp. 391-399, 2021.

[12] F.A. Williams, Combustion Theory", *Benjamin Commings*, Menlo Park, CA, 1985.

[13] T. Zhang and Y.X. Zheng, Riemann problem for gasdynamic combustion", *Journal of Differential Equations*, vol. 77, pp. 203-230, 1989.

[14] D.C. Tan and T. Zhang, Riemann problem for the selfsimilar ZND model in gas dynamical combustion", *Journal of Differential Equations*, vol. 95, pp. 331-369, 1992.

[15] C.H. Hsu and S.S. Lin, Some qualitative properties of the Riemann problem in gas dynamical combustion", *Journal of Differential Equations*, vol. 140, pp. 10-43, 1997.

[16] Y.J. Liu, Riemann problem for Chaplygin gas with combustion", *Applicable Analysis*, vol. 94, no. 8, 1877-1887, 2015.

[17] Y.J. Liu, Wave Interactions for Chaplygin Gas with Combustion", *Journal of Applied Mathematics and Physics*, vol. 9, 683-693, 2021.

[18] Y.J. Liu, The generalized Riemann problem for Chaplygin gas with combustion", *Journal of Applied Mathematics and Physics*, vol. 7, 2738-2750, 2019.

[19] L. Zhu, Chaplygin Gas Dynamic System in Adiabatic Flow", Master Thesis, Shanghai University, 2010.

[20] W.C. Sheng and T. Zhang, The Riemann problem for the transportation equations in gas dynamics", *Memoirs of the American Mathematical Society*, vol. 137, no. 654, 1999.