Stretching Function Technique Based on The Filled Function Algorithm for Solving Unconstrained Global Optimization Problem: The Univariate Case

Ridwan Pandiya* and Emi Iryanti

Abstract—**This paper intends to develop a new integral-filled function method with four properties. First, the proposed filled function is non-exponential and non-logarithmic. This property is aimed at avoiding the overflow effect during the computational stage. Second, our method does not contain parameters, so the possibility of non-converging iterations can be prevented. Third, the filled function proposed in this paper is continuously differentiable. By this property, any local minimization procedures can be applied. Finally, the suggested filled function is dependent on the objective function, and thus it can provide any information for locating a minimum of the filled function. To show that the proposed method is competitive, a comparison has been made with one of the deterministic approaches, namely the DIRECT method.**

Index Terms—**Global optimization, filled function method, nonlinear programming, global minima, auxiliary function approach.**

I. Introduction

MANY scholars struggle to develop an effective approach to solve the unconstrained global optimization problem. Their efforts led to many new algorithms. For examples, some methods categorized as a stochastic approach offer the advantage of solving the black box optimization, i.e., the problem when the objective function's properties are unclear. However, the stochastic approach has an expensive computational issue, and the solution obtained needs better accuracy. On the other hand, the deterministic approach ensures convergence with accurate results. Nevertheless, this approach can nly be applied to the optimization problem with a clear mathematical structure.

Stochastic methods are typically inspired by nature. Genetic algorithm [1], particle swarm optimization [2], bee colony [3], simulated annealing [4], [5], firefly algorithm [6], Venus flytrap optimization [7], are such examples. On the other hand, the trajectory method [8], [9], covering method [10], [11], branch and bound method [12], tunneling method [13], and filled function method [14], [15], [16], [17], [18], [19] could be categorized as a deterministic approach. Although the use of stochastic methods is wider in the real world, the deterministic approach still needs to be more useful. On the contrary, many problems can still be transformed into a global optimization model where the mathematical structure of the objective function is previously known. In this situation, the deterministic approach is more suitable to implement.

Among the deterministic methods, the filled function is effective due to its ability to move from one local minimum point of the objective function to another local minimum point with a lower value. In addition, the method is fast convergence and has a high accuracy rate. The method was originally intended as a correction to tunneling, covering, and trajectory methods. The filled function method, at first, is created by Renpu Ge (see [14]). The method offered an algorithm that has three main steps included:

1) Localizing the local minimum point of the objective function $h(x)$ employing any suitable iterative formula.
2) Formulating a new function known as the filled function method at the local minimum point achieved by step 1 and finding the local minimum point of the filled function.
3) Utilizing the local minimum point obtained by step 2 to minimize the objective function.

Step 1-3 is done iteratively until a termination criterion is fulfilled. The properties of a new function referred to in step 2 are better known as the filled function definition.

From the aforementioned explanations, the filled function tried to stretch the objective function in the region where $h(x) \geq h(x^*)$, and x^* is a local minimum point of $h(x)$. The second intention is that the local minimum point of the objective function is maintained in the filled function in the region with $h(x) < h(x^*)$. The attempt results in an example of the filled function defined as

$$\omega(x, x^*, \gamma_1, \gamma_2) = \frac{1}{\gamma_1 + h(x)} e^{\left(-\frac{\|x-x^*\|^2}{\gamma_2^2}\right)}. \quad (1)$$

Two parameters in Equation (1) work to keep the filled function from not having a minimum point and a saddle point in the region where $h(x) \geq h(x^*)$ and having a local minimum point in the rest of the feasible region. However, choosing the exact value of the two parameters is a complex matter. In addition, to deal with the filled function algorithm, the parameter value must be corrected iteratively. This situation could be more favorable, especially in the computational implementation. Another drawback of Equation (1) is the use of the exponent function because the rate of change of the value of the filled function given in (1) will be uncontrollable. Thus, the overflow effect becomes predetermined.

Many scholars seize this situation as an opportunity to correct the limitations suffered by Equation (1) by providing some novel filled functions. Efforts to eliminate the exponential function were carried out in [16], [18]. The filled function

Manuscript received May 20, 2023; revised August 29, 2023.

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in [17], [20] used an $arctan$ function. The reason for using such a function is to eliminate the overflow effect since the $arctan$ function is bounded. However, those filled functions still included parameters, which was discussed earlier, and using a parameter is not expected. Meanwhile, the filled functions discussed in [18], [19] aimed to reduce the number of parameters from two to only one parameter. Unluckily, determining the parameter's value includes the unknown number such as the Lipschitz number, global minimum value, etc.

The previous discussion concludes that the filled functions will act effectively if it does not involve parameters. The first non-parameter filled function was given in [21], which has the following formula:

$$\omega\left(x,x^{*}\right)=-sign\left(h\left(x\right)-h\left(x^{*}\right)\right)\left\|x-x^{*}\right\|^{2}, \tag{2}$$

where $sign\left(.\right)$ is a signum function, i.e.,

$$sign\left(w\right)=\begin{cases}1, & w\geq 0\\ -1, & w<0\end{cases}.$$

Although the change of the value of the function given in (2) is not as fast as the exponent function, the rate of change can be reduced by other functions. By this reason, non-parameter filled function (2) was corrected by the authors of [22] by providing a new non-parameter filled function such as following

$$\omega\left(x,x^{*}\right)=-sign\left(h\left(x\right)-h\left(x^{*}\right)\right)\arctan\left(\left\|x-x^{*}\right\|^{2}\right). \tag{3}$$

By Equation (3), the minimization of the filled function is more stable. However, filled functions (2) and (3) are discontinuous at the point such that $h\left(x\right)=h\left(x^{*}\right)$, thus there only a few local minimum procedures could be applied.

Continuously differentiable filled functions were then offered to correct the discontinuous property of Equations (2) and (3). The authors in [23] proposed

$$\omega\left(x,x^{*}\right)=-\left\|x-x^{*}\right\|^{2}\phi\left(h\left(x\right)-h\left(x^{*}\right)\right), \tag{4}$$

where

$$\phi\left(t\right)=\begin{cases}1 & t\geq 0\\ -\exp\left(t^{2}\right)+2 & t<0\end{cases}.$$

From the filled function displayed in (4), it is clear that function (4) is continuously differentiable. However, it contains an exponential function. Thus, the limitation suffered by Ge's filled function (1) could be experienced.

Some improvements were done in the literature [24], [25], [26]. The filled functions proposed by those three literature are displayed in Equations (5), (6), and (7).

$$\omega\left(x,x^{*}\right)=\frac{1}{1+\left\|\ell\right\|^{2}}\phi\left(h\left(x\right)-h\left(x^{*}\right)\right), \tag{5}$$

where $\ell=x-x^{*}$, $\phi\left(b\right)=\frac{1}{2}\pi$ for every $b\geq 0$, and

$$\phi\left(b\right)=-\arctan\left(b^{2}\right)+\frac{1}{2}\pi$$

such that $b<0$.

$$\omega\left(x,x^{*}\right)=\cosh\left(\frac{1}{1+\left\|\ell\right\|^{2}}\right)\phi\left(h\left(x\right)-h\left(x^{*}\right)\right), \tag{6}$$

where $\ell=x-x^{*}$, $\phi\left(b\right)=1$ for $b\geq 0$, and $\phi\left(b\right)=1-b^{2}$ for $b<0$.

$$\omega\left(x,x^{*}\right)=\phi\left(h\left(x\right)-h\left(x^{*}\right)\right)-\arctan\left(\left\|\ell\right\|^{2}\right), \tag{7}$$

where $\ell=x-x^{*}$ and

$$\phi\left(b\right)=\begin{cases}0, & b\geq 0\\ -arctan(b^{2}), & b<0\end{cases}.$$

However, those filled functions are independent of the objective function in the region where $h\left(x\right)\geq h\left(x^{*}\right)$ and dependent on the objective function in the domain with $h\left(x\right)<h\left(x^{*}\right)$. From the global optimization point of view, this property is not advantageous since the constructed filled function may not provide any information for locating a local minimum point of $h\left(x\right)$. Our analytical study found that integration has a "stretching effect" on a function. The line integral filled function was found in [27], which has the following formula:

$$\omega\left(x,x^{*}\right)=\int\limits_{[x^{*},x]}\left[h\left(y\right)-h\left(x^{*}\right)+\gamma\right]ds, \tag{8}$$

where $[x^{*},x]$ is a line segment, $y\in[x^{*},x]$, and γ is a parameter satisfying $0\leq\gamma\leq h\left(x^{*}\right)-h\left(x^{**}\right)$, x^{**} denotes the global minimum point of $h\left(x\right)$.

From Equation (8), it can be investigated that a single parameter γ is used. However, the condition to determine γ involves x^{**}, which is unknown. In addition, to refuse the integration, the algorithm in [27] implemented Newton's method to minimize its filled function. As widely known that Newton's method converges when the initial point is near enough to the solution.

To overcome the imperfection of the filled function displayed in (8), this paper proposes a new integral filled function that has the following properties:

1) The proposed filled function is not exponential or logarithmic function. The objective is to avoid the overflow effect.
2) Our filled function is a non-parameter filled function. The goal is to streamline the algorithm from choosing the best value of the parameter iteratively.
3) Continuously differentiable is the property of the proposed filled function. By this property, many local minimization procedures can be chosen in the minimizing process of the filled function.
4) The proposed filled function is dependent on the objective function. Thus, as previously discussed, the information of the objective function is carried away when obtaining the local minimum point of the filled function. Thus, unlike the algorithm given in [23], [24], [26], [25], which only minimizes the straight line in the region when $h\left(x\right)\geq h\left(x^{*}\right)$, filled function proposed in this paper contain the objective function in the intended region.

This paper is divided into five sections. Basic knowledge is given in Section 2. Then, a new integral filled function and its analytical properties are provided in Section 3. Section 4 contains algorithm and numerical experiments, and finally, a conclusion is drawn in Section 5.

II. Basic Knowledge

This paper is intended to solve problems that can be transformed into unconstrained global optimization in which the objective function is nonlinear and non-convex. Such a model can be mathematically formed as

$$\begin{array}{l} \min h(x) \\ x \in R^n \end{array} . \tag{9}$$

Some assumptions of the objective function $h(x)$ are listed as follows:

1) The objective function $h(x)$ is continuously differentiable;
2) The objective function $h(x)$ has finite local minimum points with different function values;
3) $\lim\limits_{\|x\| \to +\infty} h(x) = +\infty$.

Assumption (3) reveals the existence of a closed and bounded set Θ such that all the global minimum points of $h(x)$ are contained in Θ. Thus, the implication is that Problem (9) is equivalent with

$$\begin{array}{l} \min h(x) \\ x \in \Theta \end{array} . \tag{10}$$

It has been mentioned in the introduction that the filled function definition, which contains three axioms, is given in [14]. However, the definition used a basin of attraction definition. This concept is quite abstract and needs to be determined exactly. Therefore, Yang and Shang in [28] proposed a more simple definition.

Definition 2.1: Let Assumption 1-3 be satisfied, and x^* is a local minimum point of the objective function $h(x)$. A function $\omega(x, x^*)$ is called a filled function if it is satisfied three axioms:

1) The point x^* is a local maximum point of $h(x)$
2) The set $\Theta_1 = \{x \in \Theta : h(x) \geq h(x^*)\} \setminus \{x^*\}$ does not contain stationary points of $\omega(x, x^*)$
3) The set $\Theta_2 = \{x \in \Theta : h(x) < h(x^*)\}$ contains local minimum points of $\omega(x, x^*)$.

Definition 2.1 is widely implemented since it is simpler than the definition proposed in [14]. The proposed integral filled function used Definition 2.1 to formulate the function. Two sets, Θ_1 and Θ_2, will be used throughout this paper.

III. Integral Filled Function and Its Properties

Axiom (2) in the filled function definitions was intended to eliminate all the local minimum points of the objective function because the information of the objective function is not captured in the filled function in Θ_1 . There are some techniques to achieve this goal. Filled function in [20] and [29] implement the flatten function technique. A function $\beta(x, x^*)$ is defined as a flatten function of $h(x)$ at x^* if

$$\beta(x, x^*) = h(x^*) + \frac{1}{2}\{1 - sign(\alpha)\}\alpha,$$

where $\alpha = h(x) - h(x^*)$. If the objective function $h(x)$ transforms into $\beta(x, x^*)$, the constructed filled function is free from any stationary point in Θ_1 as desired by axiom (2) in Definition 2.1.

Function discussed in [27] (integral function) has a "stretching effect". To give a better understanding, an example is given in Example 3.1.

Example 3.1: Assume that $h(x) = \sin(x) + \sin\left(\frac{2}{3}x\right), x \in [-5, 20]$ is an objective function. One of the local minimum points of $h(x)$ is $x_2^* = 5.36225$.
The graph of h is illustrated in Figure 1.

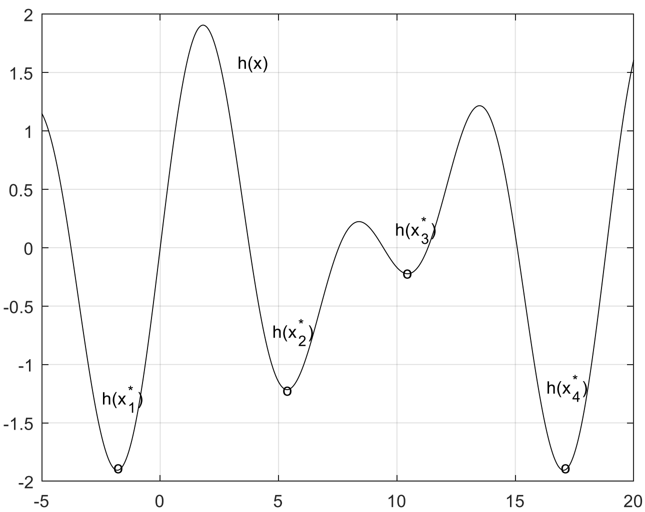

Fig. 1: Objective Function

The aim is to ignore the local minimum point x_2^*. If $r(x) = h(x) - h(x_1^*)$, then the graph of r is shown in Figure 2.

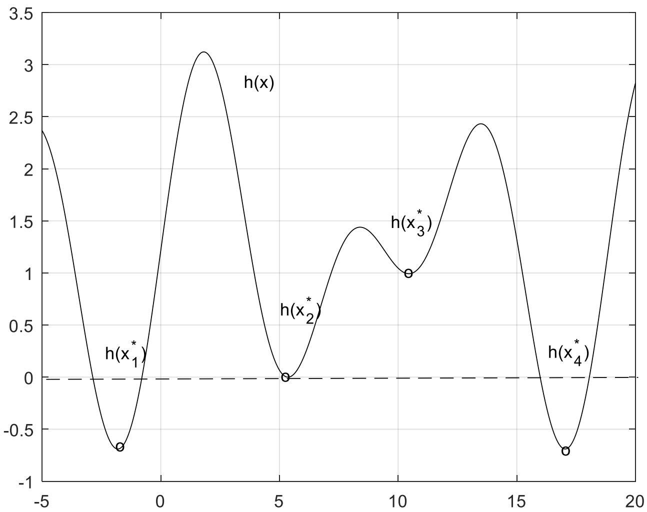

Fig. 2: Graph of $r(x)$

The graph of r in Figure 2 which the value is non-negative, is expected to be a non-stationary descent function. Now define the integral function

$$g(x, x_2^*) = -\int_a^b (h(s) - h(x_2^*))\, ds, \tag{11}$$

$a = x$, $b = x_2^*$, if $x < x_2^*$ and $b = x$, $a = x_2^*$, if $x \geq x_2^*$, for all $x \in \Theta_1$. The graph of $g(x, x_2^*)$ is displayed in Figure 3.

From Figure 3, it can be seen that function given in (11) ($g(x, x_2^*)$) decreases and does not contain stationary points

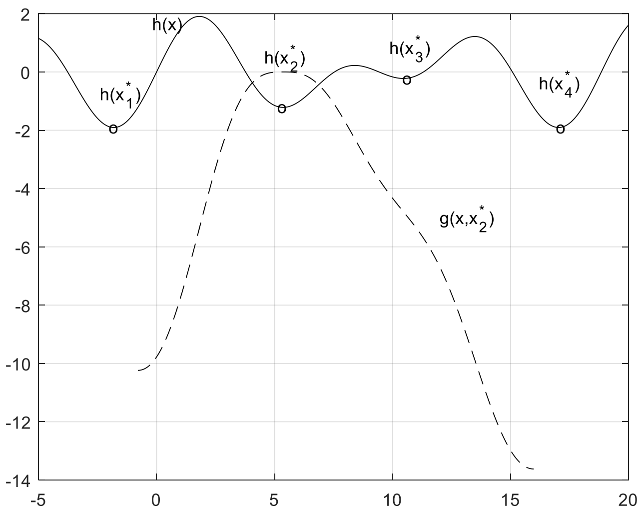

Fig. 3: Graph of $g\left(x, x_2^*\right)$

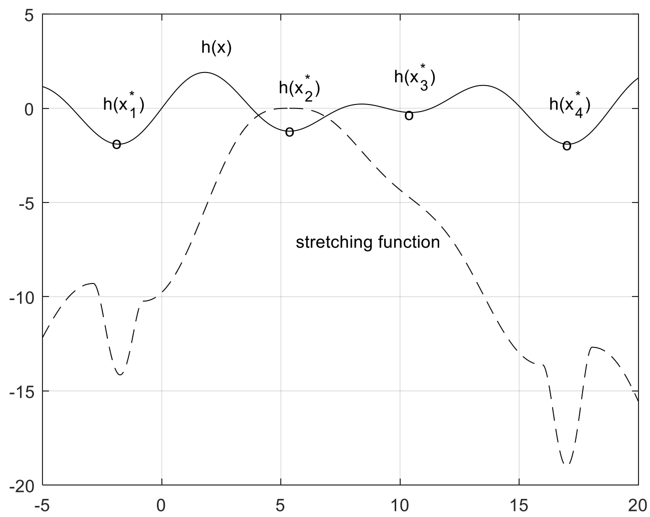

Fig. 4: Integral-filled Function of $h\left(x\right)$ at x_2^*

in Θ_1. All the local minimum and maximum of $h\left(x\right)$ are "stretching" by the integration. The dream is that the local minimum points in Θ_2 are unchanged. It means that the constructed filled function has the same local minimum points as the objective function. However, that condition is hard to achieve. Thus, one possible effort is to multiply $g\left(x, x_2^*\right)$ with another function such that the filled function has a local minimum point in the region Θ_2. By this intuition, we propose a non-parameter integral filled function, stretching function, for short, which is defined as

$$\omega\left(x, x^*\right) = \vartheta_1\left(x\right)\vartheta_2\left(r\right), \tag{12}$$

where $r = h\left(x\right) - h\left(x^*\right)$, x^* is a local minimum point of $h\left(x\right)$, the function ϑ_1 is defined as

$$\vartheta_1\left(x\right) = -\int_x^{x^*} \left(h\left(s\right) - h\left(x^*\right)\right) ds,$$

if $x < x_2^*$, and

$$\vartheta_1\left(x\right) = -\int_{x^*}^{x} \left(h\left(s\right) - h\left(x^*\right)\right) ds,$$

if $x \geq x_2^*$. The function $\vartheta_2\left(r\right)$ is also a piecewise function, which has the form

$$\vartheta_2\left(r\right) = \begin{cases} 1, & r \geq 0 \\ -\arctan\left(r^2\right) - 1, & r < 0 \end{cases}.$$

Therefore, $\omega\left(x, x^*\right)$ is a piecewise function that has four different functions, which are as follows:

1) $\omega_1\left(x, x^*\right) = -\int_x^{x^*} \ell(s) ds$, if $x < x^*$ and $r \geq 0$
2) $\omega_2\left(x, x^*\right) = \left(-\int_x^{x^*} \ell\left(s\right) ds\right)\left(-\arctan\left(r^2\right) - 1\right)$, if $x < x^*$ and $r < 0$
3) $\omega_3\left(x, x^*\right) = -\int_{x^*}^{x} \ell\left(s\right) ds$, if $x \geq x^*$ and $r \geq 0$
4) $\omega_4\left(x, x^*\right) = \left(-\int_{x^*}^{x} \ell\left(s\right) ds\right)\left(-\arctan\left(r^2\right) - 1\right)$, if $x \geq x^*$ and $r < 0$.

where $\ell\left(s\right) = h\left(s\right) - h\left(x^*\right)$.

The geometric interpretation of the proposed integral-filled function can be seen in Figure 4.
Figure 4 is the integral-filled function of $h\left(x\right)$ at x_2^*, where the objective function used is the same as h in Example 3.1. From the geometrical interpretation, the value of $\omega\left(x, x_2^*\right)$ is 0 at x_2^*, and $\omega\left(x, x_2^*\right)$ is decreasing and has no local minimum point and inflection point in the region Θ_1. Finally, $\omega\left(x, x_2^*\right)$ has a local minimum point in Θ_2. The detail of the properties of $\omega\left(x, x^*\right)$ will be discussed through some theorems. For simplicity, when discussing the theorems, H is a set of all local minimum points of $h\left(x\right)$ in Θ.

Theorem 3.2: If $x^* \in$ H, then x^* is a local maximum point of $\omega\left(x, x^*\right)$

Proof: Since $x^* \in$ H, then $h\left(x\right) \geq h\left(x^*\right)$, for all $x \in B\left(x^*, \delta\right)$, where $B\left(x^*, \delta\right)$ is a neigborhood of x^* with a radius $\delta > 0$. Because $h\left(x\right) \geq h\left(x^*\right)$, thus the integral-filled function is

$$\omega_1\left(x, x^*\right) = -\int_x^{x^*} \left(h\left(s\right) - h\left(x^*\right)\right) ds,$$

for $x < x^*$ and

$$\omega_3\left(x, x^*\right) = -\int_{x^*}^{x} \left(h\left(s\right) - h\left(x^*\right)\right) ds,$$

for $x \geq x^*$. We know that $\omega_1\left(x^*, x^*\right) = 0$ and $\omega_3\left(x^*, x^*\right) = 0$, for all $x = x^*$. Since $\ell\left(s\right) = h\left(s\right) - h\left(x^*\right) \geq 0$, hence

$$\omega_1\left(x, x^*\right) = -\int_x^{x^*} \ell\left(s\right) ds \leq 0 = \omega\left(x^*, x^*\right)$$

and

$$\omega_3\left(x, x^*\right) = -\int_{x^*}^{x} \ell\left(s\right) ds \leq 0 = \omega\left(x^*, x^*\right)$$

for all $x \in B\left(x^*, \delta\right)$. Therefore, x^* is a local maximum point of ω in Ω. ■

Theorem 3.2 reveals that the proposed filled function satisfies axiom 1 of Definition 2.1. From the theorem, it also can be drawn that the value of ω is always 0 at x^*.

Now imagine that x^* is a starting position to explore all the feasible domains Θ. Since the problems solved in this paper are the univariate case, thus there are always two possible directions. In order to get a local minimum point of $\omega(x, x^*)$, then $\omega(x, x^*)$ should be decreasing in the region Θ_1, both for $x \geq x^*$ and $x < x^*$. The decreasing property of $\omega(x, x^*)$ is a necessary condition of axiom 2 in Definition 2.1. This will be given in Theorem 3.3.

Theorem 3.3: Assume that $x^* \in$ H. If $x \in \Theta_1$, then the integral-filled function $\omega(x, x^*)$ is strictly decreasing function when x^* is a starting point.

Proof: Since $x \in \Theta_1$, then $h(s) \geq h(x^*)$. Therefore, the integral filled functions are:

$$\omega_1(x, x^*) = -\int_x^{x^*} h(s) - h(x^*)\, ds, \text{for } x < x^* \tag{13}$$

and

$$\omega_3(x, x^*) = -\int_{x^*}^{x} h(s) - h(x^*)\, ds, \text{for } x \geq x^*. \tag{14}$$

Assume that $x_1, x_2 \in \Theta_1$, with $x_1 < x^*$ and $x^* < x_2$. It will be proved that $\omega_1(x_1, x^*) < \omega_1(x^*, x^*)$ and $\omega_3(x^*, x^*) > \omega_3(x_2, x^*)$. From Theorem 3.2, it has been proved that x^* is a maximum point of $\omega(x, x^*)$ in Θ_1, where the local maximum value is $\omega(x^*, x^*) = 0$. Since $h(x) \geq h(x^*)$, then

$$\int_x^{x^*} h(s) - h(x^*)\, ds > 0, \text{for } x < x^*$$

and

$$\int_{x^*}^{x} h(s) - h(x^*)\, ds > 0, \text{for } x \geq x^*.$$

This implies that $\omega_1(x_1, x^*)$ and $\omega_3(x_2, x^*)$ are both negative. Hence, for all $x_1 < x^*$, $\omega_1(x_1, x^*) < \omega_1(x^*, x^*)$ holds, and for all $x^* < x_2$, $\omega_3(x^*, x^*) > \omega_3(x_2, x^*)$ also holds. These prove the theorem. ■

Theorem 3.3 is a necessary condition for the integral-filled function to satisfy point (2) of Definition 2.1. However, there may be some stationary points in Θ_1. When the local minimization procedure necessitates a stationary point as a terminal criterion, then the minimization process of $\omega(x, x^*)$ will stop in the region Θ_1. When it happens, the local minimum point of $\omega(x, x^*)$, which will be used as an initial point to minimize the objective function $h(x)$, is not in the basin of attraction of the local minimum point of $h(x)$. Theorem 3.4, together with Theorem 3.3, ensures that the interval Θ_1 is clean from any stationary point and $\omega(x, x^*)$ is strictly decreasing. Hence, by those two properties, the integral-filled function algorithm will pass through Θ_1 without any obstacle. That is the ultimate desire of the integral-filled function algorithm in Θ_1, to eliminate all the local minimum points of $\omega(x, x^*)$ in Θ_1.

Theorem 3.4: Assume that $x^* \in$ H. If $x \in \Theta_1$, then there are no stationary points of $\omega(x, x^*)$ contained in Θ_1.

Proof: Since $x \in \Theta_1$, then Equations (13) and (14) are held. Since h is continuous in Θ_1, thus the fundamental theorem of Calculus occurs, and we have

$$\omega'_1(x, x^*) = -[h(x) - h(x^*)], \text{ for } x < x^*$$

and

$$\omega'_3(x, x^*) = -[h(x) - h(x^*)], \text{ for } x \geq x^*.$$

Since $x \in \Theta_1$, then from the definition of set Θ_1, $h(x) - h(x^*) \geq 0$. However, x^* does not belong to Θ_1. Therefore, $\omega'_1(x, x^*)$ and $\omega'_3(x, x^*)$ are negative. Hence, stationary points of $\omega(x, x^*)$ will never be found in Θ_1. ■

Theorem 3.3 and 3.4 prove that axiom 2 of Definition 2.1 is satisfied by the proposed filled function (12). The two theorems are necessary for the minimization phase of the integral filled function. For example, if Newton's method or steepest descent method is performed, then Theorem 3.4 ensures that the iteration points of Newton's method or steepest descent method are never be encountered a stationary point, which is the stopping criterion of those methods, in the region Θ_1. However, it is possible that the non-gradient-based local minimum procedures, such as Hooke and Jeeves method, are employed. If so, the minimized function's decreasing property needs to be ensured. In this case, the guarantee is Theorem 3.3. Summarily, by Theorem 3.3 and Theorem 3.4, the minimization of our integral filled function will pass the region Θ_1 successfully without any disturbance.

Up to this point, the proposed integral function has been proved to have two filling properties that must be possessed to be categorized as a filled function. Subsequent attempts are made to show that if the integral filled function has a local minimum point, it must be in the region Θ_2.

Theorem 3.5: Assume that $x^* \in$ H. If $\hat{x}^*$ is a local minimum or inflection point of $\omega(x, x^*)$, then $\hat{x}^*$ is the element of Θ_2.

Proof: Assume that the theorem is not true. Thus, $\hat{x}^* \notin \Theta_2$ and $h(x) \geq h(x^*)$. From Theorem 3.2, x^* is a strict local maximum point, and $\hat{x}^*$ is a local minimum point of $\omega(x, x^*)$ in Θ_1. Therefore, $\hat{x}^* \neq x^*$. From Theorem 3.3, it was proved that $\omega(x, x^*)$ is strictly decreasing in Θ_1. Thus if $\hat{x}^*$ is a local minimum point, it contradicts Theorem 3.3. On the other hand, Theorem 3.4 indicates that there is no stationary or inflection point of $\omega(x, x^*)$ in Θ_1. Hence, if $\hat{x}^*$ is an inflection point of $\omega(x, x^*)$ in Θ_1, it contradicts Theorem 3.4. Consequently, $\hat{x}^*$ is the element of Θ_2. ■

Theorem 3.5 reveals that the local minimum point of the integral filled function is never in Θ_1 but in Θ_2. However, there is no guarantee that $\omega(x, x^*)$ has a local minimum point or inflection point in Θ_2. This will be proved in Theorem 3.6.

Theorem 3.6: Assume that $x^* \in$ H. If $\Theta_2 \neq \emptyset$, then there exists $\widehat{x}^* \in \Theta_2$ such that $\omega'\left(\widehat{x}^*, x^*\right) = 0$, where $\omega'\left(\widehat{x}^*, x^*\right)$ is the derivative of ω at $\widehat{x}^*$.

Proof: Assume that $\bar{\Theta}_2 = \{x \in \Theta : h(x) \leq h(x^*)\}$, where $\bar{\Theta}_2$ is a closure of Θ_2. Since the set $\bar{\Theta}_2$ is a subset of Θ, then $\bar{\Theta}_2$ is bounded. Therefore, $\bar{\Theta}_2 \neq \emptyset$ is a closed and bounded set. From the property of ω, ω is continuously differentiable on R. Hence, ω is continuously differentiable on $\bar{\Theta}_2$. From the Weirstrass extreme value theorem, $\omega(x, x^*)$ has a local minimum point $\widehat{x}^*$ in $\bar{\Theta}_2$. From the fact that $\omega(x, x^*)$ is differentiable at $\widehat{x}^*$; consequently, $\omega'\left(\widehat{x}^*, x^*\right) = 0$. From Theorem 3.4, $\omega(x, x^*)$ does not have a stationary point in the set $\hat{\Theta}_2 = \{x \in \Theta : h(x) = h(x^*)\}$. Since $\Theta_2 \neq \emptyset$, $\widehat{x}^* \in \Theta_2$. This proves the theorem. ∎

From all the theorems discussed in this section, it can be concluded that the proposed integral filled function can be categorized as a filled function.

IV. Algorithm and Numerical Simulations

The proposed method that has been discussed will be tested for its reliability. For this purpose, the global minimum algorithm has been developed. The integral filled function $\omega(x, x^*)$ will be employed in one of the stages in the proposed algorithm. To be more concise, the algorithm will be named as IFFM algorithm. In the numerical implementation stage, IFFM is then written in Matlab language programming code. In the IFFM algorithm, $\aleph(\ell, \iota)$ is defined as the minimization process of a function ℓ using the initial point ι.

IFFM Algorithm

1) Initialization phase
   a) Choose initial point $x^0 \in \Theta$ randomly
   b) Choose $\alpha^0 \in (0, \alpha)$, where $\alpha > 0$ is a small real number
   c) Choose $\lambda > 0$, for instance $\lambda = 0.1$
   d) Set the coordinate direction e_i. Since this paper solves the univariate global optimization problems, thus there are only two directions, i.e., positive and negative axis directions.
   e) Set $k = 1$
2) Looping phase
   a) $\aleph\left(h, x^0\right)$. The first local minimum point x^* is obtained from this phase.
   b) Construct integral filled function at x^*,
   $$\omega(x, x^*) = \vartheta_1(x)\,\vartheta_2(r),$$
   as has been displayed in Equation (12).
   c) **Set** $i = 1$
   d) **while** $(\alpha^0 < \alpha)$ **do**
      i) $x_k = x^* + \alpha^0 e_i$
      ii) **while** $(i \leq 2)$ **do**
         A) $\aleph(\omega, x_k)$. This step will obtain a local minimum point $x^\dagger$ of $\omega(x, x^*)$
         B) **if** $h\left(x^\dagger\right) > h(x^*)$ **then** $i = i + 1$
         C) **else** $\aleph\left(h, x^\dagger\right)$ to find a better local minimum point x^*_* of $h(x)$, $x^* \leftarrow x^*_*$, and set $k = k + 1$
         D) **end if**
         E) $\alpha^0 = \alpha^0 + \lambda$
         F) $i = i + 1$
      iii) **end while**
      iv) x^* is considered as a global minimum point of h.
   e) **end while**

The looping phase of the IFFM algorithm begins with objective function minimization. This is one of the special features of the algorithm with the auxiliary function approach, namely the use of the local minimization procedure in its algorithm. Our algorithm implements the BFGS method to obtain x^*, the local minimum point of $h(x)$. The construction of (12) is done after x^* is found. The looping process using "while" logic is carried out in the next stage, where the looping will stop if $\alpha^0 > \alpha$ is satisfied. The value of α^0 and α are selected at the initialization phase. Inside the looping, some procedures are done; they are as follows:

1) Since x^* is the maximum point of $\omega(x, x^*)$, the initial point to minimize $\omega(x, x^*)$ needs to be formed in the neighborhood of x^*. The constructed initial point is defined as
   $$x_k = x^* + \alpha^0 e_i,$$
   where e_i is coordinate directions. Since this paper is limited to obtaining the global minimum point of the univariate functions, e_i is the positive and negative x-axis.
2) "while" logic is performed provided that $i \leq 2$. The integral-filled function (12) in this path is minimized using the initial point $x_k = x^* + \alpha^0 e_i$. The local minimum point of $\omega(x, x^*)$ is notated by $x^\dagger$.
3) After $x^\dagger$ is found, we perform "if" logic because there are two possibilities regarding the value of the objective function at $x^\dagger$.
4) Condition $h\left(x^\dagger\right) > h(x^*)$ has implications for repeating the minimization process of $\omega(x, x^*)$ using the other direction.
5) Conversely, if $h\left(x^\dagger\right) < h(x^*)$, the objective function will be minimized by using $x^\dagger$ as a new initial point. In this phase, x^*_*, a better local minimum point of $h(x)$ will be obtained, and the process will be back to the looping phase until $\alpha^0 > \alpha$ is satisfied. The value of α^0 increases by adding $\lambda > 0$.

To confirm that the filled function (12) can be implemented as one of the alternatives in solving univariate global optimization problems, the IFFM algorithm has been implemented by involving some benchmark functions taken from [30]. The intended test functions are displayed in Table I. The numerical results obtained for each problem are displayed in Table II - XVI. The symbols used in Table II - XVI can be described as follows:

1) For $j = 1$, x_0^j denotes the initial point to minimize the objective function $h(x)$, with the value at that point defined as $h\left(x_0^j\right)$.
2) For $i > 1$, x_0^j denotes the local minimum point of the filled function, and $h\left(x_0^j\right)$ is its value.
3) x_j^* and $h\left(x_j^*\right)$ denote the local minimum points and the minimum value of $h(x)$, respectively.

The numerical results in Table II - XVI show that the proposed method can solve the unconstrained global optimization problems for univariate cases. Table XVII indicates the computational performance of the integral-filled function (12). IFFM algorithm is programmed in MATLAB working on Windows 10 with Intel(R) Core(TM) i3-7020U CPU and

TABLE I: Test Functions

P	Function	Θ
1	$h(x) = \frac{1}{6}x^6 - \frac{52}{25}x^5 + \frac{39}{80}x^4 + \frac{71}{10}x^3 - \frac{79}{20}x^2 - x + \frac{1}{10}$	$[-1.5, 11]$
2	$h(x) = \sin\frac{10x}{3} + \sin x$	$[1, 10]$
3	$h(x) = -\sum_{j=1}^{5} j\sin((j+1)x + j)$	$[-10, 10]$
4	$h(x) = 0.1\cos(5\pi x) + x^2$	$[-1, 1]$
5	$h(x) = 3x\sin 18x - 1.4\sin 18x$	$[-4.2, 0]$
6	$h(x) = 3 - 0.84x + \ln x + \sin x + \sin\frac{10}{3}x$	$[0.8, 10]$
7	$h(x) = -\sum_{j=1}^{5} j\cos((j+1)x + j)$	$[-10, 10]$
8	$h(x) = \sin x + \sin\frac{2}{3}x$	$[-3.1, 20]$
9	$h(x) = -x\sin x$	$[0, 30]$
10	$h(x) = x^2 - \cos(18x)$	$[-2, 2]$
11	$h(x) = \frac{1}{2}\sum_{k=1}^{n}\left(x_k^4 - 16x_k^2 + 5x_k\right), n = 1$	$[-5, 5]$
12	$h(x) = -e^{-x}\sin 2\pi x$	$[0, 4]$
13	$h(x) = \cos\left(\frac{3}{5}x\right)\cos(2x) + \sin(x)$	$[0.5, 12]$
14	$h(x) = \sin(2x)\sin(x) + \sin\left(\frac{2}{3}x\right)$	$[0, 20]$
15	$h(x) = -x - 1 + \sin 3x$	$[0, 12]$

TABLE II: Computational Result of Problem 1

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	-1.5000	-11.0886	-1.4142	-11.4180
2	2.0633	-12.5028	10	-2.9763e+04

TABLE III: Computational Result of Problem 2

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	1.0000	0.6509	1.3983	-0.0135
2	2.9031	-0.0135	3.3873	-1.1999
3	5.0656	-1.8618	5.1457	-1.8996

TABLE IV: Computational Result of Problem 3

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	-3.7000	-2.5707	-3.7392	-2.6596
2	-1.9217	-5.1495	-1.7255	-9.4947
3	-6.7710	-12.0292	-6.7746	-12.0312

TABLE V: Computational Result of Problem 4

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	-0.5000	0.2500	-0.5505	0.2318
2	-0.3741	0.2318	0.1849	-0.0630

TABLE VI: Computational Result of Problem 5

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	0.2000	0.3540	0.0794	-1.1502
2	-0.9319	-3.6747	-0.9621	-4.2830
3	-1.9987	-7.3102	-2.0084	-7.4233
4	-4.0842	-12.9947	-4.1022	-13.7056

TABLE VII: Computational Result of Problem 6

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	0.8000	3.3115	1.4071	2.2027
2	2.8028	2.2027	3.4353	0.2983
3	4.5787	0.2983	5.1964	-1.3934
4	9.0033	-2.5822	9.1079	-2.6361

TABLE VIII: Computational Result of Problem 7

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	2.1000	-0.9453	7.6032	-2.9276
2	-3.0030	-3.6137	-3.0032	-3.6137
3	-2.1223	-4.8164	-2.0072	-6.1698
4	-1.0050	-7.8078	-0.8003	-14.5080

TABLE IX: Computational Result of Problem 8

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	3.1000	0.9211	5.3622	-1.2160
2	-1.7458	-1.9031	-1.8104	-1.9060

TABLE X: Computational Result of Problem 9

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	6.7000	-2.7125	7.9787	-7.9167
2	13.2987	-8.8915	14.2074	-14.1724
3	19.6845	-14.5911	20.4692	-20.4448
4	26.0496	-20.6754	26.7409	-26.7222

TABLE XI: Computational Result of Problem 10

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	-1.7000	2.2048	-1.7346	2.0274
2	-1.4752	2.0274	-1.3877	0.9376
3	-1.1170	0.9376	-1.0408	0.0899
4	-0.7573	0.0899	-0.6938	-0.5156
5	-0.3954	-0.5156	-0.3469	-0.8789
6	-0.0275	-0.8789	7.9797e-17	-1

TABLE XII: Computational Result of Problem 11

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	4.0000	10.0000	2.7468	-25.0294
2	-2.0838	-30.5205	-2.9035	-39.1662

TABLE XIII: Computational Result of Problem 12

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	3.3000	-0.0351	3.2249	-0.0393
2	2.4267	-0.0393	2.2249	-0.1067
3	1.4267	-0.1067	1.2249	-0.2901
4	0.2955	-0.7139	0.2249	-0.7887

4.00 GB RAM. The notations used in Table XVII are as follows:

TABLE XIV: Computational Result of Problem 13

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	0.5000	0.9956	1.3410	0.3523
2	2.7349	0.3474	3.6915	-0.7951
3	5.5698	-0.7951	5.9460	-1.0421
4	10.3374	-1.0421	10.9598	-1.9544

TABLE XV: Computational Result of Problem 14

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	1.7000	0.6524	2.1684	0.2231
2	3.6576	0.2231	4.2310	-0.4121
3	5.8759	-0.4121	6.3598	-0.8787
4	7.8616	-0.8787	8.3621	-1.3934
5	16.6402	-1.7638	16.6483	-1.7640

TABLE XVI: Computational Result of Problem 15

j	x_0^j	$h\left(x_0^j\right)$	x_j^*	$h\left(x_j^*\right)$
1	0.9000	-1.4726	1.6841	-3.6269
2	3.0489	-3.7744	3.7785	-5.7213
3	5.1115	-5.7468	5.8729	-7.8157
4	7.2013	-7.8238	7.9673	-9.9101
5	9.2945	-9.9136	10.0617	-12.0045

I_T: the number of iterations, which means how many local minimum points are obtained by the proposed algorithm, where the last local minimum point is a global one.

α^0: The real number used as a stopping criterion in the IFFM algorithm

F_E: Function evaluation of the objective and integral-filled functions before termination.

T: CPU time in seconds for the IFFM algorithm to stop.

Table XVII summarizes the computational performance of the integral-filled function method for all the problems. From the function evaluation point of view, the proposed method is effective. F_E1 and F_E2 in Table XVIII denote the number of function evaluations obtained by the integral-filled function proposed in the paper and obtained by the DIRECT method [31], respectively. To indicate the competitiveness of the proposed method, in Table XVIII, we compare the results obtained by our method and the most famous method to solve the unconstrained global optimization problems, which is the DIRECT method. The DIRECT method becomes very useful since it is superior when implemented to solve black-box problems (i.e., the problems where the mathematical structures are not available). Of 15 problems solved, 86.67% have smaller function evaluations obtained by the proposed method than the function evaluation achieved by the DIRECT method. For problem 15, the iteration points of the DIRECT method are out of the feasible domain. Thus, we conclude that the DIRECT method fails to locate the global minimum point. Nevertheless, from the numerical experiments and the comparison, it can be concluded that the integral-filled function is reliable.

Figures 5 and 6 are the geometric illustration of the convergence of the DIRECT method for Problems 2 and 5. The objective function of Problems 2 and 5 has the global minimum points at $x_j^* = 5.1457$ and $x_j^* = -4.1022$ with the global minimum values $h\left(x_j^*\right) = -1.8996$ and $h\left(x_j^*\right) = -13.7056$, respectively. The strength of the DIRECT method lies in its ability to partition the search area so that it does not need a starting point like the filled function method. If the iteration points of minimization with the DIRECT method are close to the global minimum point, the search process usually slows down because the partition is narrower. This is the drawback of the DIRECT method. For Problems 2 and 5, the DIRECT method has converged since the 81st and 93rd function evaluations. Although this method is efficient, based on the comparison results shown in Table XVIII, the method

TABLE XVII: Computational performance of the proposed integral-filled function (12)

Problem	I_T	α^0	F_E	T
1	2	1.00	27	0.439381
2	3	0.10	40	0.430665
3	3	0.01	40	0.228277
4	2	0.10	27	0.198790
5	4	1.00	70	0.263361
6	4	0.10	56	1.174794
7	4	0.01	63	0.449774
8	2	0.10	27	0.246004
9	4	0.10	56	1.142269
10	6	0.10	81	0.209939
11	2	0.10	24	0.197185
12	4	0.10	51	0.216852
13	4	0.10	56	0.982040
14	5	0.10	81	1.693732
15	5	0.10	81	0.432175

TABLE XVIII: Comparison between integral-filled function and DIRECT method

Problem	x_j^*	$h\left(x_j^*\right)$	F_E1	F_E2
1	10.0000	-2.9763e+04	27	37
2	5.1457	-1.8996	40	81
3	-6.7746	-12.0312	40	43
4	0.1849	-0.0630	27	39
5	-4.1022	-13.7056	70	93
6	9.1079	-2.6361	56	61
7	-0.8003	-14.5080	63	105
8	-1.8104	-1.9060	27	45
9	26.7409	-26.7222	56	51
10	7.9797e-17	-1.0000	81	3
11	-2.9035	-39.1662	24	39
12	0.2249	-0.7887	51	63
13	10.9598	-1.9544	56	67
14	16.6483	-1.7640	81	87
15	10.0617	-12.0045	81	failed

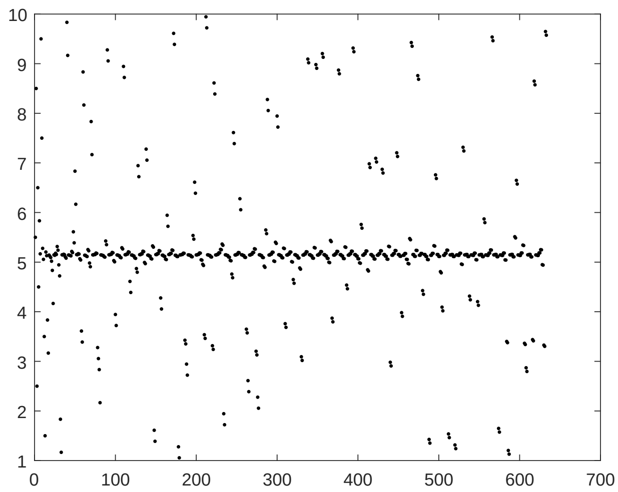

Fig. 5: Computational performance of the DIRECT method for Problem 2

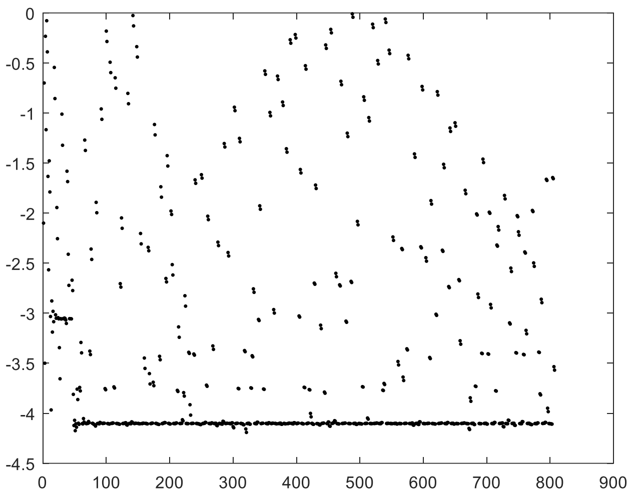

Fig. 6: Computational performance of the DIRECT method for Problem 5

proposed in this study is more efficient.

V. Conclusion

Inspired by the filled function method, we introduced the integral-filled function method. The main feature of this function is dependent on the objective function. This property differs from the continuously differentiable parameter-free filled functions, which are all independent of the objective function in the region where the value is greater than the value at the incumbent local minimum point of the objective function. This valuable property provides information for obtaining a minimum point of the proposed integral-filled function, which is very helpful in the computational stage. Numerical experiments have been carried out to support the claim for superior properties of the proposed method. In the final stage, a numerical comparison with the DIRECT method is also done to show that the proposed method is competitive and reliable. The integral-filled function approach proposed in this paper is an integral function. How to generalize the function is still an open problem.

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