Black-Litterman Portfolio Optimization Using Gaussian Process Regression

Zhen Li, Changfei Li*, Liangyu Min, *Member*, *IAENG*, Dijia Lin

***Abstract*—The Black-Litterman portfolios based on the predictions provided by Gaussian Process are constructed in this study. Besides the expert views generated by the Gaussian Process, an customized algorithm quantifying the confidence level of the given investor opinions is also designed, which can be inputted into the Black-Litterman framework to revise the posterior parameters estimations. Low-risk anomaly is observed from the numerical experiments through the grouping method base on stock β, demonstrating the potential irrationality for even giant companies and brands on the advanced market. Empirical analysis shows that Gaussian Process is able to model the low β stock effectively, while not feasible for stocks with high volatility. Thus, the proposed BLGPlo portfolio outperform the benchmarks in terms of cumulative excess return and Sharpe ratio. Moreover, the BLGPlo performance can be further improved by allocating higher confidence level for the Gaussian Process-derived investor opinions.**

***Index Terms*—Portfolio selection, Machine learning, Gaussian Process, Black-Litterman**

I. Introduction

THE Black-Litterman (BL) portfolio model[1], [2], [3], [4], [5] is a successful extension of the classical Markowitz model[6], where the subjective investor opinions can be integrated into the classical mean-variance (MV) portfolio model by the Bayesian formula. Upon the expert and visionary investor opinions, some notorious drawbacks such as parameter-sensitivity and over-concentrated portfolio weight of the MV model can be overcome by the BL portfolio to some extent.

However, it is expensive to obtain high quality expert views for individual investor, and the issue of potential conflicts among these expert opinions should be tackled carefully. Besides that, how to quantify the confidence level of these subjective opinions has not reached consensus within the academic community. Existing studies[7], [8], [9], [10], [11], [12] have pointed that machine learning is an available technique to generate predictions about market tendency which can be used to substitute the subjective views to some extent. Whereas the issue of gauging the confidence level for the derived views seldom be involved and discussed, which motivate us to investigate some approaches to quantify the views confidence level.

This paper designs the BL portfolio based on the predictive investor opinions provided by the Gaussian Process, which essentially belongs to the supervised machine learning algorithm. Different from some other artificial intelligence techniques such as Support Vector Machine, Random Forest, and Neural Networks, the associated uncertainty level for the given predictions can be obtained easily, paving the pay for developing strategy to evaluate the confidence level for the generated investor view. Further, we also analyse the BL portfolio performance with different confidence levels.

The phenomenon of low-risk anomaly is also demonstrated using the grouping method, where the stocks are divided into two groups according to their respective β value. Numerical experiments select 7 giant brands listed on the S&P 500 index, and the portfolio using low β stocks outperform the same portfolio using high β stocks whether in terms of risk or return. The low-risk anomaly contradicts the classical financial theory such as CAPM[13], [14], [15], where high return should compensate for the high risk undertaken by one stock. Therefore, limited efficiency and irrationality can be observed even in the mature US market, laying the practical foundation for building machine learning-based portfolio models[16], [7], [8], by which the dynamic and non-linear properties of financial data can be captured and modelled. In addition, grouping method provides preselection stage[17], [10] for portfolio formation, which is feasible for multiple investment strategies.

The rest of this paper is structured as follows: Section II introduces the Black-Litterman-Bayes model, constructing the theoretical framework for the proposed portfolio strategy. Section III details the Gaussian Process and presents the relationship between the Gaussian Process and the proposed portfolio model (BLGP). Section IV illustrates the proposed BLGP models, as well as the grouping method. The associated numerical experiments are implemented in Section V, where the portfolio performances of the BLGP models and the benchmarks are presented, compared, and analyzed. Finally, Section VI concludes the paper.

In this paper, the lower case bold letters such as $\boldsymbol{x}$ refer to vectors, while the matrices are indicated by upper case bold letters such as $\boldsymbol{X}$. The plain white letters such as X and x refers to scalars.

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II. Black-Litterman-Bayes framework

Assume that the available universe of assets include N securities, and the return vector $\boldsymbol{r}$ follows the multivariate Gaussian distribution, that is, $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. According to the assumption of Black-Litterman-Bayes (BLB) framework, the expected return vector $\boldsymbol{\mu}$ can be split into two parts, the first part is the implied market equilibrium return $\boldsymbol{\Pi}$, and the second part is the residual vector also following the Gaussian distribution, $\boldsymbol{\varepsilon}_\mu \sim \mathcal{N}(\boldsymbol{0}, \tau\boldsymbol{\Sigma})$. Therefore, the expected return vector $\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma})$.

The BLB framework takes the investor opinions as the prior distribution, where K investor opinions for the $M(M \leq N)$ securities are expressed using the pick matrix $\boldsymbol{P} \in \mathbb{R}^{K\times M}$ and the quantitative opinion vector $\boldsymbol{Q} \in \mathbb{R}^K$. To be consistent with Bayesian theory, the investor options are viewed as the likelihood function, while the moments estimating from the historical samples are used to calibrate the prior distribution. It can be observed the relationship between $\boldsymbol{P}$ and $\boldsymbol{Q}$ is $\boldsymbol{P\mu} = \boldsymbol{Q} + \boldsymbol{\varepsilon}_\nu, \boldsymbol{\varepsilon}_\nu \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Omega}) \rightarrow \boldsymbol{P\mu} \sim \mathcal{N}(\boldsymbol{Q}, \boldsymbol{\Omega})$. Therefore, the posterior distribution for the expected return vector $\boldsymbol{\mu}$ can be derived using the Bayesian formula as follows:

$$\mathcal{N}([(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{Q}], \\ [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}\boldsymbol{P}]^{-1})$$

On the basis of the posterior distribution, BLB builds the mean-variance portfolio, that is, $\hat{\boldsymbol{\mu}} = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{Q}]$ and $\hat{\boldsymbol{\Sigma}} = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}\boldsymbol{P}]^{-1}$. Empirical studies[7], [2], [18], [19], [20], [5] demonstrate the effectiveness of the Black-Litterman portfolio.

III. Gaussian Process

Gaussian Process (GP) is a powerful supervised machine learning technique that has been accepted by academics and practitioners due to its ability to model complex and non-linear relationship in data[21], [22], [23]. Essentially, GP can be defined as an infinite version of multivariate Gaussian distributions regarding of real-valued variables[11]. Note that the parameter τ in the BLB framework denoting the uncertainty level of the investor opinion, which is difficult to gauge accurately and reasonably based on the classical research results[2], [4], [3]. Fortunately, GP provides a feasible solution to quantify the uncertainty of parameter selection by introducing the kernel method into the variance-covariance matrix[11], [24].

Define $\boldsymbol{x}_i \in \mathbb{R}^n$ as the input vector, $y_i \in \mathbb{R}$ as the output variable, $f(\cdot)$ as the mapping function. In GP, the following multivariate Gaussian relationship holds:

$$\begin{bmatrix} f(\boldsymbol{x}_1) \\ \vdots \\ f(\boldsymbol{x}_m) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m(\boldsymbol{x}_1) \\ \vdots \\ m(\boldsymbol{x}_m) \end{bmatrix}, \begin{bmatrix} k(\boldsymbol{x}_1,\boldsymbol{x}_1) & \dots & k(\boldsymbol{x}_1,\boldsymbol{x}_m) \\ \vdots & \ddots & \vdots \\ k(\boldsymbol{x}_m,\boldsymbol{x}_1) & \dots & k(\boldsymbol{x}_m,\boldsymbol{x}_m) \end{bmatrix} \right)$$

For the in-sample observations $(\boldsymbol{Y}, \boldsymbol{X})$, GP has the following regression:

$$y_i = f(\boldsymbol{x}_i) + \varepsilon_i, \varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

where $y_i \in \boldsymbol{Y}, \boldsymbol{x}_i \in \boldsymbol{X}$. And, for the out-of-sample observations $(\boldsymbol{Y}^*, \boldsymbol{X}^*)$, the conditional distribution of $\boldsymbol{Y}^*$ is $\boldsymbol{Y}^*|\boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{X}^* \sim \mathcal{N}(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}^*)$, where $\boldsymbol{\mu}^* = \boldsymbol{K}(\boldsymbol{X}^*, \boldsymbol{X})(\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X}))^{-1}\boldsymbol{Y}$, $\boldsymbol{\Sigma}^* = \boldsymbol{K}(\boldsymbol{X}^*, \boldsymbol{X}^*) + \sigma^2\boldsymbol{I} - \boldsymbol{K}(\boldsymbol{X}^*, \boldsymbol{X})(\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X}) + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X}^*)$, and $\boldsymbol{K}(\boldsymbol{X}, \boldsymbol{X})$ denotes the matrix with the element at the position (i, j) is $k(\boldsymbol{x}_i, \boldsymbol{x}_j)$.

In the proposed BLB portfolio, the confidence of a investor view generated by GP is closely associated with the volatility derived by the GP prediction, on which we give the corresponding heuristic method to quantify the confidence level of the investor opinion.

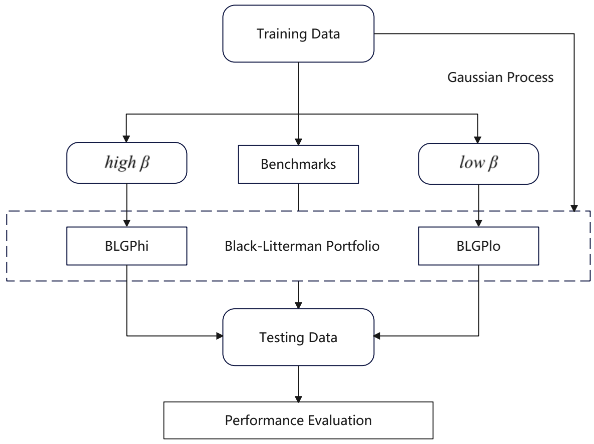

Fig. 1. Flowchart of the proposed portfolio model.

IV. Proposed portfolio model

According to the classical financial theory, β is a crucial indicator for evaluating a stock. $\beta > 1$ means the stock has higher volatility than the market, while $\beta < 1$ indicates the stock has lower volatility than the benchmark. Therefore, this study splits the candidate securities into two groups, the first group consisting of the stocks with high β, and the second group includes the stocks with low β.

Then, the BL portfolio on the high β group (BLGPhi) and the BL portfolio on the low β group (BLGPlo) are constructed respectively. During the process of portfolio modelling, GP plays an important role in providing the investor opinion and the associated volatility. Fig. 1 presents the flowchart of modelling. In this study, the portfolio out-of-sample performance is detailed evaluated and analyzed. Algorithm 1 illustrates the basic logic of the proposed BLGP portfolio model.

V. Numerical experiments

In the empirical analysis, 7 giant brands listed on the S&P 500 are selected: Agilent Technologies (A), Apple Inc. (AAPL), Boeing (BA), Alphabet Inc. (GOOG), Goldman Sachs (GS), JPMorgan Chase (JPM), Microsoft (MSFT). Total 156 months records ranging from 2010.1 to 2022.12 are involved in the experiment, where the first 106 months are set as the training set, and the rest months are set as the testing set.

Algorithm 1 BLGP portfolio framework.

Input: Stock data with price & volume information D
Output: BLGPlo and BLGPhi portfolios; Investor views Q and the associated standard deviation V

1: Calculate the β for each stock based on the formula $\beta_i = \frac{cov(r_i, r_M)}{\sigma_M^2}$
2: Split the stocks into two groups (high β group and low β group) according to the calculated β
3: Obtain the prediction $q_i \in Q$ and the associated standard deviation $v_i \in V$ for each stock using Gaussian Process Regression
4: Build the Black-Litterman portfolio on the low β group, BLGPlo
5: Build the Black-Litterman portfolio on the high β group, BLGPhi
6: **return** BLGPlo, BLGPhi

TABLE I
PREDICTIONS OF THE GAUSSIAN PROCESS REGRESSION.

Stocks	Predictions (q_i)	Stdev. (s_i)	View Confidence (v_i)
A	0.0155	0.0735	63.40%
AAPL	0.0216	0.0723	63.61%
BA	0.0208	0.0652	65.12%
GOOG	0.0157	0.0672	64.65%
GS	0.0064	0.0769	62.80%
JPM	0.0143	0.0648	64.40%
MSFT	0.0170	0.0615	66.00%

A. Gaussian process predictions

Table I presents the predictions and the associated standard deviation of Gaussian Process regression, where the RBF kernel with scale 1.0 are used as the kernel function.

It can be observed that MSFT has the lowest predicted standard deviation, which should be assigned the highest confidence level, but GS has the highest predicted standard deviation, which should be assigned the lowest confidence level. To quantify such a relationship, this section develops the following empirical formula: $v_i = 50\% + \frac{1/s_i}{\sum s_i}$, where 50% represents the naive investor view using the average return for evaluating the stocks, and $\frac{1/s_i}{\sum 1/s_i}$ indicates the expert opinion generating by the Gaussian Process Regression algorithm.

B. Grouping

CAPM[25] laid the theoretical foundation for grouping the risky assets based on the individual β. Table II presents the β of each candidate stock as well as the group information.

High β portfolio BLGPhi includes Agilent Technologies, Boeing, Goldman Sachs, JPMorgan Chase, whereas low β BLGPlo portfolio includes Apple Inc., Alphabet Inc., and Microsoft. As a result, BLGPhi is designed for investors with risk-loving preferences, trying to pursue high profit while assuming more risk than risk-averse investors. Meanwhile, BLGPlo caters to the preferences of conservative investors.

The proposed framework aims to assist diverse investors in configuring individual portfolio models that effectively meet their requirements, where the forecasting information is integrated into the constructed portfolio model in the form of investor opinions via GP algorithm. If these investor opinions are reliable, the investor's utility would be satisfied splendidly, and the associated BLB portfolios would outperform the benchmarks.

TABLE II
GROUPS OF THE STOCKS.

Stock	β	Group	Portfolio
A	1.2667	high β	BLGPhi
AAPL	1.1508	low β	BLGPlo
BA	1.3440	high β	BLGPhi
GOOG	1.0849	low β	BLGPlo
GS	1.4551	high β	BLGPhi
JPM	1.2734	high β	BLGPhi
MSFT	1.0007	low β	BLGPlo

Notes: The market capital (in billion) of the involved stocks are as follows: 46.095 for A; 2311 for AAPL; 126.193 for BA; 1261 for GOOG; 118.46 for GS; 411.741 for JPM; 1847 for MSFT.

C. Portfolio performance

This section reports the experimental results on the testing set, where $1/N$ strategy (EW)[26], global minimum-variance portfolio are chosen as the baseline models for comparison. The portfolio programs are coded on the Python 3.8 platform, where the PyPortfolioOpt[27] package is imported for efficiently building the basic Black-Litterman framework. For the baseline strategies, they are also implemented using python and solved by Gurobi.

Table III presents the portfolio weights of the group-based BLGP portfolios and the benchmarks. For each portfolio model, the only budget constraint $\mathbf{1}'\boldsymbol{x} = 1$ is considered, and each risk stock can be shorted freely to maximize investor utility.

Besides the fundamental return and volatility, this study uses the following financial indicators to compare the proposed BLGP portfolios and the benchmarks comprehensively:

- Sharpe ratio (SR), the indicator gauges the risk-adjusted return, which can be calculated using the formula $\frac{r_p - r_f}{\sigma_p}$, where r_p represents the portfolio annual return, $r_f = 2\%$ in this paper, σ_p indicates the annual standard deviation of the evaluated portfolio.
- Maximum drawdown (MDD), the indicator measures the maximum observed loss from a peak to a trough of a portfolio, before a new peak reached. The formula used for MDD is $\frac{V_p - V_t}{V_p}$, where V_p is the peak value of the portfolio and V_t is the trough value of the portfolio.
- CVaR(5%), refers to conditional value at risk, also known as expected shortfall. It is a risk management measure that quantifies the potential loss that an investment portfolio or a trading strategy could suffer under adverse market conditions, beyond a certain confidence level (we set the confidence level 5% in the numerical experiment).

Table IV summarizes the portfolio performances on the testing period. BLGPlo achieves the highest cumulative excess return of 132.81%, followed by GMV, 111.1%, but BLGPhi has poor performance on the evaluation

TABLE III
PORTFOLIO WEIGHTS.

Item	A	AAPL	BA	GOOG	GS	JPM	MSFT
benchmarks							
1/N	14.28%	14.28%	14.28%	14.28%	14.28%	14.28%	14.28%
GMV	21.26%	0%	0%	4.52%	0%	15.02%	59.19%
BLGP-based portfolios							
BLGPlo	/	17.47%	/	12.99%	/	/	69.54%
BLGPhi	60.17%	/	8.28%	/	−60.96%	92.52%	/
BLGPall	24.94%	10.69%	3.15%	16.20%	−73.90%	49.51%	69.40%

Notes: 1/N is the equal-weighted portfolio; GMV indicates the global minimum variance portfolio minimizing the objective function $\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}$; BLGPlo and BLGPhi are the BLGP-based portfolios using the low β and high β stocks group respectively; BGLPall is the BLGP-based portfolio considering all of the risky assets. The annual risk-free rate is 2%.

TABLE IV
PORTFOLIO PERFORMANCE.

Item	Cumulative excess return	Annual return	Volatility	SR	MDD	CVaR(5%)
benchmarks						
1/N	83.28%	18.00%	24.66%	0.65	28.45%	0.14
GMV	111.1%	20.60%	20.76%	0.90	29.12%	0.09
BLGP-based portfolios						
BLGPlo	132.81%	23.44%	22.72%	0.94	29.20%	0.10
BLGPhi	35.01%	10.67%	25.68%	0.34	35.93%	0.14
BLGPall	80.93%	16.61%	19.85%	0.74	35.36%	0.11

Notes: The portfolio cumulative excess return is calculated as follows: $R_p = -1 + \prod_{i=1}^{n}(1 + r_i)$, where r_i is the portfolio mean monthly return. The portfolio annual return $r_p = r_i \times 12$, and the portfolio annual volatility $\sigma_p = \sigma_i \times \sqrt{12}$, where σ_i is the portfolio monthly volatility.

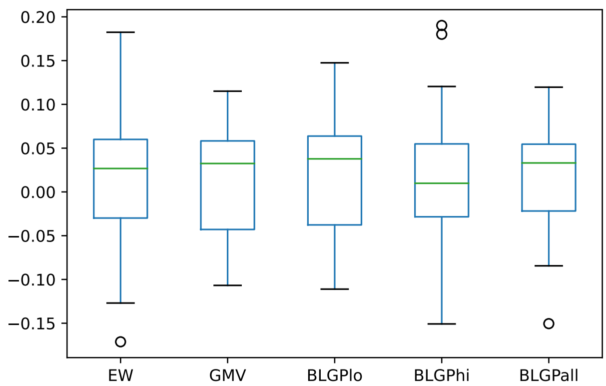

Fig. 2. Monthly returns the tested portfolio model.

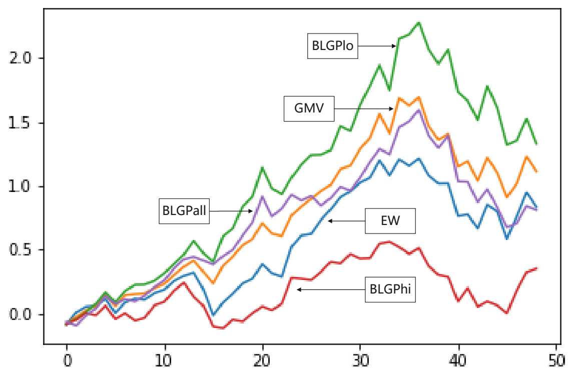

Fig. 3. Cumulative returns the tested portfolio model.

indicators, only 35.01% cumulative excess return and 10.67% annual return. In terms of volatility, BLGPall shows the most stable performance with lowest volatility of 19.85%, GMV ranks the second place with 20.76% volatility. As far as the risk-adjusted indicators, BLGPlo reaches the highest SR of 0.94, then GMV, with 0.90 SR. $1/N$ has the lowest MDD of 28.45%, and GMV has the lowest CVaR(5%) of 0.09. Besides appealing return indicators, BLGPlo also presents competitive risk-related performance, with 29.20% MDD and 0.10 CVaR(5%).

Fig. 2 visualizes the monthly returns of the tested portfolios, and Fig. 3 shows the cumulative returns of the portfolios. From the box-plots presented in Fig. 3, the long lower tails can be observed on the $1/N$ strategy and the BLGPhi portfolio, corresponding to the high values of the CVaR(5%) and illustrating potential tail risk. Accordingly, the cumulative returns of $1/N$ strategy and BLGPhi portfolio demonstrate the poor performance for models with long lower tails. Conversely, GMV, BLGPlo, and BLGPall have short lower tails with high cumulative portfolio returns. Notably, BLGPall portfolio have shorter tails than BLGPlo and BLGPhi except for individual outliers, implicating the stable return characteristic of the BLGP portfolio considering multiple risky assets. The effectiveness of the grouping based on stock β is also demonstrated, where the portfolio composing of low β stocks show better out-of-sample performance than the portfolio using high β stocks.

Since the performance of the BLB-based portfolio is highly dependent on the quality of investor opinions, but

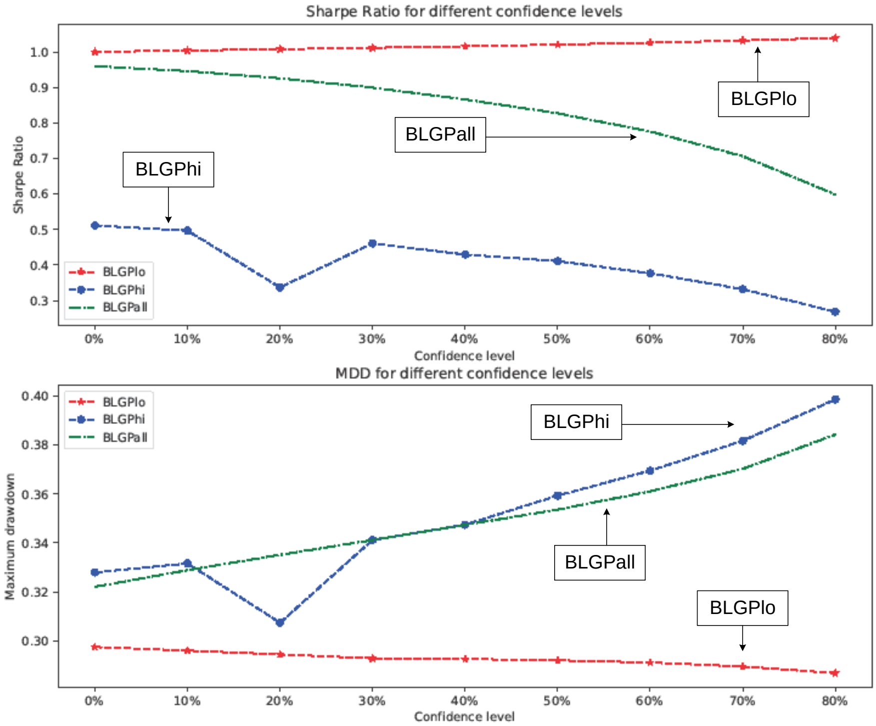

Fig. 4. BLGP portfolios with different confidence levels for investor opinion.

the credibility of these subjective views are difficult to quantify. In the proposed BLGP framework, 50% are set as the baseline confidence level which essentially is a heuristic value. To strictly analyze the influences bring by different confidence levels of investor opinion on the BLGP-based portfolio from particular group, the comparative experiment is carried out in the sequel.

D. Further analysis on the investor opinions

This section presents the BLGP portfolios performances considering different baseline confidence levels of investor opinions, where the portfolio Sharpe ratio and the portfolio MDD are reported in Fig. 4.

It can be observed that the Sharpe ratio of BLGPlo rises with the increasing of baseline confidence level, and the MDD of BLGPlo decreases as the baseline confidence level increases. However, BLGPhi gives the diametrically opposite performance comparing with BLGPlo, high confidence level of investor opinion seems to have negative impact on the Sharpe ratio and Maximum drawdown of the BLGPhi portfolio. The performance of BLGPall reconciles the properties of BLGPlo and BLGPhi, but it is more inclined to conform the characteristic of BLGPhi.

According to the result shown in Fig. 4, the influences of the baseline confidence level are asymmetric. For the low β group, increasing baseline confidence level is beneficial to portfolio performance, whereas the investor opinions generated from Gaussian Process does not provide reliable forecasting about the high β group stocks. In general, stocks with high volatility are hard to be predicted accurately, without any exception to the method of Gaussian Process, but the goal of this study is to construct appealing portfolios based on the forecasting provided by Gaussian Process, grouping according to stock β can effectively exploit the modelling ability of Gaussian Process on the low β stocks, which alleviates the issue of forecasting accuracy. However, a direct method to handle the forecasting accuracy problem is to develop sophisticated algorithms with fine-tuned parameters such as deep neural networks, but it beyond the scope of this paper.

VI. Conclusions & discussions

We mainly discuss the BLB-based portfolios in this paper, where the Gaussian Process is applied to generate investor opinions, and grouping method based on stock β is used to provide further analysis. Through the numerical experiments on the companies listed on S&P 500 index, some key findings are as follows. First of all, the Black-Litterman-Bayes framework provides an effective approach to bridge the sample estimations and the subjective investor opinions. Based on that, the portfolio can be constructed using the posterior parameters. Besides the estimations, Gaussian Process also provide the associated uncertainty level for the generated predictions, which is beneficial to quantify the confidence level for the investor opinion. Numerical experiments illustrate that

BLGP portfolio is suitable for the low β stocks, but not very applicable for the high β stocks due to the limited predictive power of Gaussian Process.

The two benchmarks, $1/N$ and GMV portfolios only inferior to the BLGPlo strategy, but outperform the BLGPhi and BLGPall on the out-of-sample dataset. The result is intuitive and consistent with the existing conclusions, where the low volatility portfolio usually has better out-of-sample performance than the high volatility portfolio. Accordingly, the numerical results support the rationality of using $1/N$ and GMV as the baseline models in evaluating portfolio out-of-sample performance strongly.

Future researches will revolve around developing volatility-based forecasting algorithm to improve the ability of describing the characteristics of high β stocks. Since the proposed framework manages to quantify the confidence level of investor opinions resort to the designed algorithm based on the associated uncertainty given by the GP predictions. Therefore, a customized mechanism should also be specified for the volatility-based approach to gauge the confidence level.

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