Spectral Properties of Partial Chain and Partial Threshold Graphs

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Abstract—**Chain graphs and threshold graphs play an important role in Spectral Graph Theory. Nesting in the neighborhood of vertices in these graphs has gained the attention of various researchers. Motivated by this structure, recently two new classes of graphs, namely partial chain graphs and partial threshold graphs have been defined. In this article, we give a few bounds on the spectral radius and energy of partial chain graphs and partial threshold graphs in terms of the total number of vertices. We obtained a class of partial chain graphs and partial threshold graphs with exactly two main eigenvalues. The energy of some classes of partial threshold graphs and partial chain graphs are obtained.**

Index Terms—**Threshold graphs, Chain graphs, Divisor matrix, Equitable partition, Energy.**

I. Introduction

A Collection $S = \{S_1, S_2, \ldots, S_n\}$ of sets is said to form a chain with respect to set inclusion, if for every $S_i, S_j \in S$ either $S_i \subseteq S_j$ or $S_j \subseteq S_i$. We write $u \sim v$ if the vertices u and v are adjacent in G, $u \nsim v$ if they are not.

The neighborhood of the vertex $u \in V(G)$ is the set $N(u)$ consisting of all the vertices v such that $v \sim u$ in G. For a graph G, we write $det(G)$ and $per(G)$ for determinant and permanent of adjacency matrix $A(G)$ of G. Its characteristic polynomial is denoted by $\chi(G)$. The spectral radius $\lambda_1(G)$ of a graph is the largest eigenvalue of its adjacency matrix.

Readers are referred to [16] for all the elementary notations and definitions not described but used in this paper.

Definition 1.1: A chain graph is a bipartite graph in which the neighborhoods of the vertices in each partite set form a chain with respect to set inclusion.
In other words, for every two vertices u and v in the same partite set and their neighborhoods $N(u)$ and $N(v)$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. We note that, every partite set in a chain graph has at least one dominating vertex, that is, a vertex adjacent to all the vertices of the other partite set. The color classes of a chain graph $G(V_1 \cup V_2, E)$ can be partitioned into h non-empty cells given by $V_1 = V_{11} \cup V_{12} \cup \ldots \cup V_{1h}$ *and* $V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2h}$ such that $N(u) = V_{21} \cup V_{22} \cup \ldots \cup V_{2\ h-i+1}$, for any vertex $u \in V_{1i},\ 1 \leq i \leq h$. If $m_i = |V_{1i}|$ and $n_i = |V_{2i}|$, then we write $G = DNG(m_1, \ldots, m_h; n_1, \ldots, n_h)$. Due to this nesting property, the chain graphs are also called double nested graphs (DNGs).

Manuscript received 15 July 2023; revised 27 September 2023;
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A split graph is a graph which admits a partition of its vertex set into two parts W_1 and W_2 such that W_1 induces a complement of a clique (co-clique) and W_2 induces a clique. Every other edge, called a cross edge, joins a vertex of W_1 with a vertex of W_2. A threshold graph is a split graph in which the adjacencies defined by the cross edges satisfy the following nesting property. Both W_1 and W_2 can be partitioned into h non-empty cells, say, $W_1 = W_{11} \cup W_{12} \cup \ldots \cup W_{1h}$ *and* $W_2 = W_{21} \cup W_{22} \cup \ldots \cup W_{2h}$ such that $N(u) = W_{21} \cup W_{22} \ldots \cup W_{2\ h-i+1}$, for any vertex $u \in W_{1i},\ 1 \leq i \leq h$. It is also called a nested split graph (NSG). If $m_i = |W_{1i}|$ and $n_i = |W_{2i}|$, then we write $G = NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Readers are referred to [1]–[5], [8] for more results on chain and threshold graphs.

Motivated by the nesting property of the extremal graphs (chain and threshold graphs), the authors of the article [12] defined a new class of graphs, whose vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$ and has the nesting property. Formal definition is given below.

Definition 1.2: [12] A graph G on n vertices is said to be a partial threshold graph if its vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that the following conditions are satisfied.

1 $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$.
2 The set $\{V_i \cap N(v)\} \neq \phi$ form a chain with respect to set inclusion for every $v \in V_j, j \neq i, 1 \leq i, j \leq 2$.

We denote $N_1(u) = N(u) \cap V_1, u \in V_2$ and $N_2(v) = N(v) \cap V_2, v \in V_1$. The subsets V_1 and V_2 can be further partitioned into h non-empty cells $V_1 = V_{11} \cup \ldots \cup V_{1h}$ and $V_2 = V_{21} \cup \ldots \cup V_{2h}$ which satisfies the following nesting property:
For every vertex $u \in V_{1i}, 1 \leqslant i \leqslant h, N_2(u) = V_{21} \cup \ldots \cup V_{2\ h-i+1}$ and for $v \in V_{2j}, 1 \leqslant j \leqslant h$, $N_1(v) = V_{11} \cup \ldots \cup V_{1\ h-j+1}$. If $|V_{1i}| = m_i$ and $|V_{2i}| = n_i$, then we write $G = PTG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Unlike the chain graphs or threshold graphs, $G = PTG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ is not representing a single graph, instead it represents a graph family G_f with nesting property as explained earlier. It does not Specify the structure of $\langle V_1 \rangle$ or $\langle V_2 \rangle$. Thus, we write $G_f = PTG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. We use the notion $G \in G_f = PTG(m_1, \ldots, m_h; n_2, \ldots, n_h)$ of graphs which have the bipartition $V(G) = V_1 \cup V_2$ such that $\langle V_1 \rangle \cong \langle \overline{V_2} \rangle$.

Note that all the graphs $G \in G_f = PTG(m_1, m_2, \ldots, m_h; n_2, n_2, \ldots, n_h)$ have same number of edges and same number of vertices. Also, when $G \in G_f = PTG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ with $\langle V_1 \rangle \cong K_{\frac{n}{2}}$, we get a threshold graph on n vertices.
The graphs (Figure 1) $G_1, G_2 \in G_f = PTG(1, 2, 1; 1, 1, 2)$.

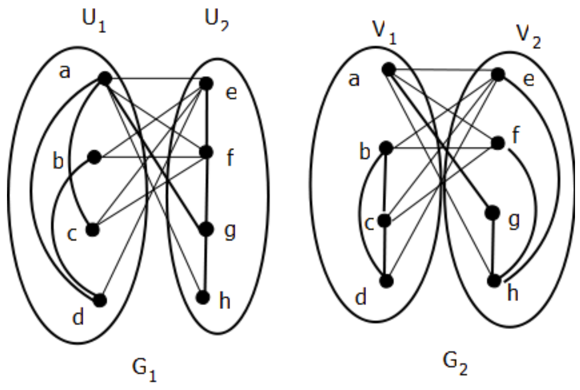

Fig. 1. $G_1, G_2 \in G_f = PTG(1, 2, 1; 1, 1, 2)$

Definition 1.3: [12] Consider a partial threshold graph $G(V_1 \cup V_2, E)$ with $V_1 = \{u_1, u_2, \ldots, u_p\}$ and $V_2 = \{u'_1, u'_2, \ldots, u'_p\}$ and $N_2(u_i) \subseteq N_2(u_{i-1}), 2 \leqslant i \leqslant p$. Then G is said to be a strong partial threshold graph if there exists a bijective mapping $\Phi : V_1 \to V_2$ satisfying the following conditions:
(i) $u_i \sim u_j$ implies $\Phi(u_i) \nsim \Phi(u_j)$, for all $1 \leq i \neq j \leq p$.
(ii) $N_1(\Phi(u_i)) \subseteq N_1(\Phi(u_{i-1})), 2 \leqslant i \leqslant p$.
We denote $\Phi(u_i)$ by u'_i.

The nested split graphs with $|V_1| = |V_2|$ and $PTG(p; p)$ are strong partial threshold graphs. The graph G_1 of Figure 1 is a strong partial threshold graph with $\Phi(a) = e, \Phi(b) = f, \Phi(c) = g$ and $\Phi(d) = h$. For G_2 of Figure 1, $\Phi(a) = h, \Phi(b) = e, \Phi(c) = f$ and $\Phi(d) = g$. But as $N_1(e) \nsubseteq N_1(h)$, G_2 is not a strong partial threshold graph.

In [12], the authors developed an algorithm which returns (if exists) a strong partial threshold graph with Wiener index k for a given input value k.

Definition 1.4: [9] A graph G is said to be a partial chain graph if its vertex set can be partitioned into two subsets V_1 and V_2 such that the following conditions are satisfied.

1) At least one of the partite sets is independent.
2) If a partite set V_i $(i = 1, 2)$ is independent, then neighborhoods of vertices of V_i form a chain with respect to the operation of set inclusion. If not, $\{V_j \cap N(v)\} \neq \phi$ $(j \neq i)$ for every vertex $v \in V_i$.

Clearly, if V_i is not independent, then the neighborhoods of its vertices do not form a chain. Further, when both the partite sets are independent, we get a chain graph. When V_1 is independent and $\langle V_2 \rangle \cong K_n$ for some $n \geq 1$, we get a threshold graph. Due to the nesting property of neighborhoods, it is possible to further partition each of V_i $(i = 1, 2)$ into h cells $V_1 = V_{11} \cup V_{12} \cup \ldots \cup V_{1h}$ and $V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2h}$ such that $N(u) = V_{21} \cup V_{22} \cup \ldots \cup V_{2\ h-i+1}$ for all $u \in V_{1i}, 1 \leq i \leq h$. Suppose $m_i = |V_{1i}|$ and $n_i = |V_{2i}|$, then we write $G = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$, where $|V_1| = \sum_{i=1}^{h} m_i$ and $|V_2| = \sum_{i=1}^{h} n_i$. The structure induced by the partite set V_2 (which need not be independent) is not taken into account in the above-said approach and the notation. Similar to the partial threshold graph, $G = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ does not represent a single graph, but a family of graphs G_f with nesting as said above. Thus, we write $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ (instead of just G). A threshold graph with $|V_1| = |V_2|$ is also a partial chain graph as well as a partial threshold graph. Unlike a chain graph and threshold graph, a partial threshold graph and a partial chain graph can contain any graph as its induced subgraph.

The graphs G_1 and G_2 (Figure 2) are the partial chain graphs in the family $G_f = PCG(2, 1, 1; 1, 1, 3)$.

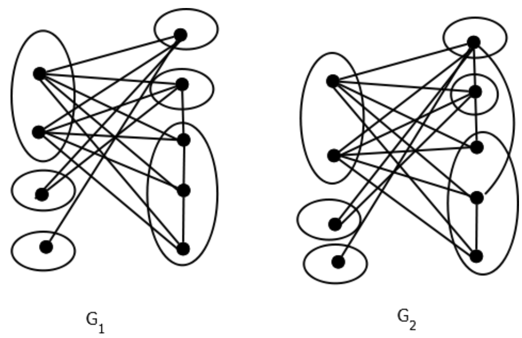

Fig. 2. The graph $G_1, G_2 \in G_f = PCG(2, 1, 1; 1, 1, 3)$

The authors of the article [9] gave the expressions for rank, determinant and permanent of partial chain graphs.

We obtain energy of some classes of partial chain graphs and partial threshold graphs with respect to their adjacency matrix using the concept of equitable partition of the graph. In the theory of graph spectra, equitable partitions play an important role.

For a given graph G, a partition $D : W_1 \cup W_2 \cup \ldots \cup W_k$ of $V(G)$ is called an equitable partition if every vertex in W_i has the same number of neighbours in W_j, say d_{ij}, for all $i, j \in 1, 2, \ldots, k$. Then the $k \times k$ matrix with entries $[d_{ij}]$ is called the divisor matrix of D.

II. Preliminary Results

Some of the important results which are useful in the next sections are listed below.

Theorem 2.1: [7] Let M be a real symmetric matrix with a divisor matrix D. Then the characteristic polynomial of D divides the characteristic polynomial of M.

Theorem 2.2: [6] Let D be an equitable partition of the connected graph G. Then $A(G)$ and the divisor matrix A_D of D have the same spectral radius $\lambda_1(G)$.
An eigenvalue of a graph is said to be a main eigenvalue if it has an eigenvector not orthogonal to the main vector $J = (1, 1, \ldots, 1)^T$.

Theorem 2.3: [14] Let D be an equitable partition of the connected graph G. Then an eigenvalue λ of G is main eigenvalue if and only if it is a main eigenvalue of the divisor matrix A_D of D.

Theorem 2.4: [10] Let G be a graph of order n and size m. Then

$$2\sqrt{m} \leq E(G) \leq 2m.$$

Theorem 2.5: [10] Let G be a graph of order n and size m. Then,

$$E(G) \leq \sqrt{2mn}.$$

Theorem 2.6: [10] Let G be a graph of order n and size m. Then,

$$E(G) \leq \lambda_1(G) + \sqrt{(n-1)(2m - \lambda_1(G)^2)},$$

where $\lambda_1(G)$ is the largest eigenvalue of the graph G.

Theorem 2.7: [15] For a connected graph G,

$$\lambda_1(G) \leq \sqrt{2m-n+1}$$

with equality if and only if G is isomorphic to the star graph or the complete graph.

Theorem 2.8: [13] For any graph G of order n and size m,

$$\lambda_1(G) \leq \frac{1}{2}\left(-1+\sqrt{1+8m}\right)$$

with equality holds when $m = \binom{n}{2}$.

Theorem 2.9: [12] Let G be any partial threshold graph of order n and size m. Then,

$$\binom{n/2}{2} + n - 1 \leq m \leq \binom{n/2}{2} + \frac{n^2}{4}.$$

Theorem 2.10: [8] Let $G = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a partial chain graph of order n and size m. Then

$$\sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-j+1} n_i\right) \leq m \leq \sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-j+1} n_i\right) + \frac{k(k-1)}{2}$$

where $k = \sum_{i=1}^{h} n_i$.

The following lemma gives the multiplicity of the eigenvalues 0 and -1 of $A(G)$ in the graph G based on its structure.

Lemma 2.11: Given a graph G, let S be a subset of $V(G)$ of the size p.
(i) If S is a clique (i.e., induces a complete subgraph of G) and $N(u) \setminus S = N(v) \setminus S$ for all $u, v \in S$, then -1 is an eigenvalue of $A(G)$ with multiplicity at least $p-1$.
(ii) If S is a co-clique (i.e., induces an empty subgraph of G) and $N(u) = N(v)$ for all $u, v \in S$, then 0 is an eigenvalue of $A(G)$ with multiplicity at least $p-1$.

III. Partial Threshold Graphs

In this section, we obtain the bounds for the energy and spectral radius of partial threshold graphs. The class of partial threshold graphs with exactly two main eigenvalues is obtained along with the spectrum of some strong partial threshold graphs.

Theorem 3.1: Let G be a partial threshold graph of order n. Then,

$$E(G) \leq \frac{n}{2}\sqrt{3n-2}$$

with equality if and only if $n = 2$.

Proof: Proof follows from Theorems 2.5 and 2.9. ■

Using Theorems 2.4 and 2.9, the next theorem follows.

Theorem 3.2: Let G be a partial threshold graph of order n. Then,

$$\sqrt{\frac{1}{2}(n^2+6n-8)} \leq E(G) \leq \frac{n}{4}(3n-2)$$

with equality if and only if $n = 2$.

Theorem 3.3: Let G be a partial threshold graph of order n with largest eigenvalue $\lambda_1(G)$, of the adjacency matrix of the graph G. Then,

$$E(G) \leq \lambda_1(G) + \frac{1}{2}\sqrt{(n-1)(3n^2-2n-4\lambda_1(G)^2)}.$$

Proof: From Theorems 2.6 and 2.9, the result follows. ■

The following theorem gives a class of strong partial threshold graphs with zero determinant and permanent.

Theorem 3.4: Let $G = PTG(m_1, m_2, \ldots, m_h; 1, n_2, \ldots, n_h = m_h)$ be a strong partial threshold graph on n vertices and $deg(v) = 1$, $v \in V_{1h}$ where $m_h > 1$. Then, $per(G) = det(G) = 0$ and $0, -1$ are the eigenvalues of $A(G)$ with multiplicity at least $m_h - 1$.

Proof: From Lemma 2.11, we observe that 0 and -1 are the eigenvalues of $A(G)$ with multiplicity at least $m_h - 1$. Hence $det(G) = 0$. As G has more than one pendant vertices, there is no elementary subgraph which spans all the vertices of G. Hence $per(G) = 0$. ■

Theorem 3.5: Let G be a partial threshold graph of order n. Then,

$$\lambda_1(G) \leq \frac{1}{2}\sqrt{3n^2-6n+4}.$$

The equality in the above inequality for $n > 2$, will never hold.

Proof: From Theorems 2.7 and 2.9, we have

$$\begin{aligned}\lambda_1(G) &\leq \sqrt{2m-n+1} \\ &\leq \sqrt{2\left(\binom{\frac{n}{2}}{2} + \frac{n^2}{4}\right) - n + 1} \\ &= \frac{1}{2}\sqrt{3n^2-6n+4}.\end{aligned}$$

As the equality in the above expression holds if and only if G is complete (or a star graph), which is not possible for $n > 2$. ■

Theorem 3.6: Let G be a partial threshold graph of order n, then

$$\lambda_1(G) \leq \frac{1}{2}\left(-1+\sqrt{3n^2-2n+1}\right).$$

The equality in the above inequality for $n > 2$, will never hold.

Proof: From Theorems 2.8 and 2.9, we have

$$\begin{aligned}\lambda_1(G) &\leq \frac{1}{2}\left[-1+\sqrt{1+8m}\right] \\ &\leq \frac{1}{2}\left[-1+\sqrt{1+8\left(\binom{\frac{n}{2}}{2} + \frac{n^2}{4}\right)}\right] \\ &= \frac{1}{2}\left(-1+\sqrt{3n^2-2n+1}\right)\end{aligned}$$

As the equality in the above expression holds if and only if G is complete, which is not possible for $n > 2$. ■

The spectral radius of a strong partial threshold graph $G = PTG(p;p)$, with the graph induced by one of the sets is a regular graph, is given below.

Theorem 3.7: Let $G = PTG(p;p)$ be a strong partial threshold graph on $n = 2p$ vertices and the graph induced by the set V_1 is a regular graph with regularity r. Then,

$$\lambda_1(G) = \frac{(p-1)+\sqrt{5p^2-2p+4r(r+1-p)+1}}{2}.$$

Proof: Checking the structure of graph G, we can obtain an equitable partition $D : V_1 \cup V_2$ of G. The divisor matrix

A_D of D is given by

$$A_D = \begin{bmatrix} r & p \\ p & p-r-1 \end{bmatrix}.$$

Thus, $\chi(A_D) = \lambda^2 + \lambda(1-p) + rp - r^2 - r - p^2$. By Theorem 2.2, the result follows. ■

Recently, Alazemi et al. [1] obtained the chain graphs with exactly 2 main eigenvalues. The next theorem discusses a class of partial threshold graphs with exactly 2 main eigenvalues.

Theorem 3.8: Let $G = PTG(p;p)$ be a non-regular partial threshold graph with $\langle V_1 \rangle$ is a regular with regularity r. Then, G has exactly two main eigenvalues.

Proof: We have $A_D = \begin{bmatrix} r & p \\ p & p-r-1 \end{bmatrix}$.
From Theorem 2.3, all main eigenvalues of G are also the main eigenvalues of any divisor matrix. If $r \neq \frac{p-1}{2}$, we show that the 2 eigenvalues of A_D are main, using the fact that "a graph G has two main eigenvalues if and only if $\{J, A(G)J, A(G)^2J\}$ are linearly dependent and $\{J, A(G)J\}$ are linearly independent". It is easy to observe that $\{J, A(G)J\}$ is linearly independent, if $r \neq \frac{p-1}{2}$. Consider $A_D^2 J = \alpha J + \beta A_D J$.

$$\Longrightarrow \begin{bmatrix} r^2+p^2 & p^2-p \\ p^2-p & 2p^2+r^2-2rp-2p+2r+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} + \beta \begin{bmatrix} r & p \\ p & p-r-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} r^2+2p^2-p \\ 3p^2+r^2-2rp-3p+2r+1 \end{bmatrix} = \begin{bmatrix} \alpha+\beta(r+p) \\ \alpha+\beta(2p-r-1) \end{bmatrix}$$

On solving we get the values of α and β which are not zero always, provided $r \neq \frac{p-1}{2}$. ■

The energy of few strong partial threshold graphs are discussed below.

Theorem 3.9: Let $G = PTG(m_1, m_2; m_1, m_2)$ be a strong partial threshold graph on $n = 2p$, $p \geq 2$ vertices with $\langle V_1 \rangle \cong DNG(m_1; m_2)$. Then,
$E(G) = \sum_{i=1}^{4} |\lambda_i| + p - 2$, where $\lambda_1, \ldots, \lambda_4$ are the roots of the polynomial,
$\lambda^4 + (2-m_1-m_2)\lambda^3 + (1-m_1-m_1^2-m_2-2m_1m_2)\lambda^2 + (-m_1^2-4m_1m_2+m_1^2m_2+2m_1m_2^2)\lambda + (-m_1m_2-m_1^2m_2+m_1m_2^2+2m_1^2m_2^2)$.

Proof: From Lemma 2.11, it is observed that λ and $\lambda+1$ are the factors of $\chi(G)$ with multiplicity at least $p-2$. The partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ induces an equitable partition of G.

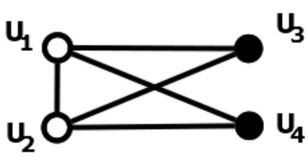

Fig. 3. Partial Threshold Graph : U_1 and U_2 are co-cliques and U_3, U_4 are cliques, each thick line indicates the edge set of a complete bipartite subgraph between U_i, U_j.

The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_2 & m_1 & m_2 \\ m_1 & 0 & m_1 & 0 \\ m_1 & m_2 & m_1-1 & 0 \\ m_1 & 0 & 0 & m_2-1 \end{bmatrix}.$$

Now, $|A_D - \lambda I| = \begin{vmatrix} -\lambda & m_2 & m_1 & m_2 \\ m_1 & -\lambda & m_1 & 0 \\ m_1 & m_2 & m_1-1-\lambda & 0 \\ m_1 & 0 & 0 & m_2-1-\lambda \end{vmatrix}$.

By performing, $R_i \longrightarrow R_i + \left(\frac{m_1}{\lambda}\right) R_1 \ for \ i = 2, 3 \ and \ 4$, we get
$|A_D - \lambda I| =$
$$\begin{vmatrix} -\lambda & m_2 & m_1 & m_2 \\ 0 & \frac{-\lambda^2+m_1m_2}{\lambda} & \frac{m_1\lambda+m_1^2}{\lambda} & \frac{m_1m_2}{\lambda} \\ 0 & \frac{m_2\lambda+m_1m_2}{\lambda} & \frac{(m_1-1-\lambda)\lambda+m_1^2}{\lambda} & \frac{m_1m_2}{\lambda} \\ 0 & \frac{m_1m_2}{\lambda} & \frac{m_1^2}{\lambda} & \frac{(m_2-1-\lambda)\lambda+m_1m_2}{\lambda} \end{vmatrix}.$$
$\Longrightarrow |A_D - \lambda I| = \frac{-1}{\lambda^2}$
$$\begin{vmatrix} -\lambda^2+m_1m_2 & m_1\lambda+m_1^2 & m_1m_2 \\ m_2\lambda+m_1m_2 & (m_1-1-\lambda)\lambda+m_1^2 & m_1m_2 \\ m_1m_2 & m_1^2 & (m_2-1-\lambda)\lambda+m_1m_2 \end{vmatrix}.$$
Now, by performing $R_2 \longrightarrow R_2 - \left(\frac{m_2\lambda+m_1m_2}{-\lambda^2+m_1m_2}\right) R_1$, $R_3 \longrightarrow R_3 - \left(\frac{m_1m_2}{-\lambda^2+m_1m_2}\right) R_1$ and by further reduction we get, $|A_D - \lambda I| = \lambda^4 + (2-m_1-m_2)\lambda^3 + (1-m_1-m_1^2-m_2-2m_1m_2)\lambda^2 + (-m_1^2-4m_1m_2+m_1^2m_2+2m_1m_2^2)\lambda + (-m_1m_2-m_1^2m_2+m_1m_2^2+2m_1^2m_2^2)$. Therefore,
$Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ p-2 & p-2 & 1 & 1 & 1 & 1 \end{pmatrix}$
where $\lambda_1, \ldots, \lambda_4$ are the roots of $|A_D - \lambda I|$. ■

Corollary 3.10: Let $G = PTG(1, p-1; 1, p-1)$ be a strong partial threshold graph on $n = 2p$, $p \geq 2$ vertices, with $\langle V_1 \rangle \cong K_{1,p-1}$ with dominating vertex being the central vertex. Then, $E(G) = \sum_{i=1}^{4} |\lambda_i| + p - 2$, where $\lambda_1, \ldots, \lambda_4$ are the roots of the polynomial,
$[\lambda^4 + (2-p)\lambda^3 + (2-3p)\lambda^2 + (4-7p+2p^2)\lambda + 5 - 8p + 3p^2]$, and $per(G) = 0$.

Proof: Proof follows by substituting $m_1 = 1$ and $m_2 = p-1$ in Theorem 3.9 and $per(G) = 0$ by Theorem 3.4. ■

Theorem 3.11: Let $G = PTG(m_1, m_2; n_1, n_2)$ with $m_1 < n_1$ be a strong partial threshold graph on $n = 2p$, $p \geq 3$ vertices and $\langle V_1 \rangle \cong DNG(m_1; m_2)$. Then, $E(G) = \sum_{i=1}^{5} |\lambda_i| + p - 3$, where $\lambda_1, \lambda_2, \ldots, \lambda_5$ are the roots of the polynomial,
$[-\lambda^5 + (p-3)\lambda^4 + (2p+m_1^2+m_1m_2+m_2n_1-3)\lambda^3 + (p+m_1^2+2m_1^3+3m_1m_2+2m_1^2m_2+m_1n_1-2m_1^2n_1+2m_2n_1-m_1m_2n_1+m_1n_2-2m_1^2n_2-m_1m_2n_2-m_2n_1n_2-1)\lambda^2 + (2m_1^3+3m_1m_2+2m_1^2m_2+3m_1^3m_2+m_1n_1-2m_1^2n_1+m_2n_1-3m_1^2m_2n_1+m_1n_2-2m_1^2n_2-2m_1m_2n_2-2m_1^2m_2n_2-m_2n_1n_2)\lambda + m_1m_2(1+3m_1^2+n_1-3m_1n_1-n_2-m_1n_2-2m_1^2n_2-n_1n_2+2m_1n_1n_2)]$.

Proof: From Lemma 2.11, we note that λ and $\lambda+1$ are the factors of $\chi(G)$ with multiplicity at least $p-2$ and $p-3$ respectively.

The partition $D : U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5$ of $V(G)$ as shown in Figure 4 induces an equitable partition of G with $U_1 = V_{11}, U_2 = V_{12}, U_5 = V_{22}$ and $U_3, U_4 \in V_{21}$. Here $|U_3| = m_1$, $|U_4| = n_1 - m_1$.

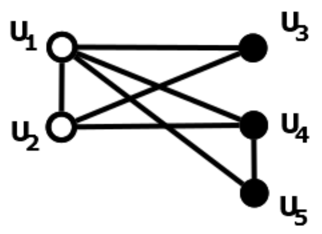

Fig. 4. Partial Threshold Graph : U_1 and U_2 are co-cliques and U_3, U_4, U_5 are cliques, each thick line indicates the edge set of a complete bipartite subgraph between U_i, U_j.

The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_2 & m_1 & n_1-m_1 & n_2 \\ m_1 & 0 & m_1 & n_1-m_1 & 0 \\ m_1 & m_2 & m_1-1 & 0 & 0 \\ m_1 & m_2 & 0 & n_1-m_1-1 & n_2 \\ m_1 & 0 & 0 & n_1-m_1 & n_2-1 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ p-3 & p-2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \dots, \lambda_5$ are the roots of $|A_D - \lambda I|$. Hence the proof. ■

Theorem 3.12: Let $G = PTG(m_1, m_2; n_1, n_2)$ with $m_1 > n_1$ be a strong partial threshold graph on $n = 2p, p \geq 3$ vertices, with $\langle V_1 \rangle \cong DNG(m_1; m_2)$. Then, $E(G) = \sum_{i=1}^{5} |\lambda_i| + p - 3$, where $\lambda_1, \dots, \lambda_5$ are the roots of the polynomial,
$[-\lambda^5 + (p-3)\lambda^4 + (2p + m_1^2 + m_1m_2 + m_2n_1 - 3)\lambda^3 + (p + 2m_1^2 + 4m_1m_2 - 3m_1^2m_2 - m_1m_2^2 + 2m_2n_1 + m_1m_2n_1 - m_2^2n_1 + m_2n_1^2 - 1)\lambda^2 + (m_1^2 + 4m_1m_2 - 4m_1^2m_2 - 2m_1m_2^2 + m_1^2m_2^2 + m_2n_1 + 3m_1m_2n_1 - m_1^2m_2n_1 - m_2^2n_1 - 2m_1m_2^2n_1 + m_2n_1^2 + m_1m_2n_1^2 - m_2^2n_1^2)\lambda + m_1m_2(1 - p + m_1m_2 + 2n_1 - m_1n_1 - 3m_2n_1 + 2m_1m_2n_1 + n_1^2 - 2m_2n_1^2)]$.

Proof: From Lemma 2.11, it is observed that λ and $\lambda+1$ are the factors of $\chi(G)$ with multiplicity at least $p-2$ and $p-3$ respectively.
There exists a partition $D : \cup_{i=1}^{5} U_i$ of $V(G)$ which induces an equitable partition of G with $U_1 = V_{11}, U_2 = V_{12}, U_3 = V_{21}$ and $U_4, U_5 \in V_{22}$. Here $|U_4| = m_1 - n_1$ and $|U_5| = m_2$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_2 & n_1 & m_1-n_1 & m_2 \\ m_1 & 0 & n_1 & 0 & 0 \\ m_1 & m_2 & n_1-1 & m_1-n_1 & 0 \\ m_1 & 0 & n_1 & m_1-n_1-1 & 0 \\ m_1 & 0 & 0 & 0 & m_2-1 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ p-3 & p-2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \dots, \lambda_5$ are the roots of $|A_D - \lambda I|$ and the proof follows. ■

Theorem 3.13: Let $G = PTG(2, r+s; 2, r+s)$ be a strong partial threshold graph on $n = 2p = 2(r+s+2), (r, s \geq 1)$ vertices, with $\langle V_1 \rangle \cong DNG(1, r; 1, s)$ and the dominating vertices of the partial threshold graph be the dominating vertices of $DNG(1, r; 1, s)$. Then, $E(G) = \sum_{i=1}^{8} |\lambda_i| + p - 4$, where $\lambda_1, \dots, \lambda_8$ are the roots of the polynomial,
$[\lambda^8 + (2 - r - s)\lambda^7 - (4 + 7r + 7s)\lambda^6 + (8rs - 14 - 14r + 3r^2 - 14s + 3s^2)\lambda^5 + (12r^2 - 9s + 34rs + 12s^2 - 13 - 9r)\lambda^4 + (r + 15r^2 + s + 46rs - 11r^2s + 15s^2 - 11rs^2 - 4)\lambda^3 + (2r + 6r^2 + 2s + 24rs - 31r^2s + 6s^2 - 31rs^2)\lambda^2 + (4rs - 21r^2s - 21rs^2 + 10r^2s^2)\lambda + 17r^2s^2 - 2r^2s - 2rs^2]$.

Proof: From Lemma 2.11, it is observed that λ and $\lambda+1$ are the factors of $\chi(G)$ with multiplicity at least $p-4$.

There exists a partition $D : \cup_{i=1}^{8} U_i$ of $V(G)$ which induces an equitable partition of G with $U_1, U_2 \in V_{11}, U_3, U_4 \in V_{12}$, $U_5, U_6 \in V_{21}$ and $U_7, U_8 \in V_{22}$. Here $|U_1| = |U_2| = |U_5| = |U_6| = 1, |U_3| = |U_7| = r, |U_4| = |U_8| = s$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & 1 & 0 & s & 1 & 1 & r & s \\ 1 & 0 & r & 0 & 1 & 1 & r & s \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & r & s & 0 & 0 & r & 0 \\ 1 & 1 & r & s & 0 & 0 & 0 & s \\ 1 & 1 & 0 & 0 & 1 & 0 & r-1 & s \\ 1 & 1 & 0 & 0 & 0 & 1 & r & s-1 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_8 \\ p-4 & p-4 & 1 & 1 & \cdots & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \dots, \lambda_8$ are the roots of $|A_D - \lambda I|$. ■

Theorem 3.14: Let $G = PTG(m_1, m_2; m_1, m_2)$ be a strong partial threshold graph on $n = 2p,\ p \geq 3$ vertices and $\langle V_1 \rangle \cong NSG(m_1; m_2)$ with partition $V_1 = V_{11} \cup V_{12}$, where V_{11} induces a clique and V_{12} induces a co-clique. Then, $E(G) = \sum_{i=1}^{4} |\lambda_i| + p - 2$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of the polynomial,
$[\lambda^4 + (2 - p)\lambda^3 + (1 - p - m_1^2 - 2m_1m_2)\lambda^2 + (2m_1m_2^2 - m_1^2 - 3m_1m_2)\lambda + m_1m_2(m_2 + 2m_1m_2 - 1 - m_1)]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $p-2$. The partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ of $V(G)$ induces an equitable partition of G. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} m_1-1 & m_2 & m_1 & m_2 \\ m_1 & 0 & m_1 & 0 \\ m_1 & m_2 & 0 & 0 \\ m_1 & 0 & 0 & m_2-1 \end{bmatrix}.$$

Therefore, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ p-2 & p-2 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of $|A_D - \lambda I|$. ■

Theorem 3.15: Let $G = PTG(m_1, m_2; n_1, n_2)$ be a strong partial threshold graph with $m_1 < n_1$, $n = 2p,\ p \geq 3$ vertices and $\langle V_1 \rangle \cong NSG(m_1; m_2)$ with partition $V_1 = V_{11} \cup V_{12}$, where V_{11} induces a clique and V_{12} induces a co-clique. Then, $E(G) = \sum_{i=1}^{5} |\lambda_i| + p - 3$, where $\lambda_1, \lambda_2, \dots, \lambda_5$ are roots of the polynomial,
$[-\lambda^5 + (p-3)\lambda^4 + (p - 3 + m_1^2 + m_1m_2 + m_2n_1)\lambda^3 + (p - 1 + 2m_1^2 + m_1^3 + 3m_1m_2 + 2m_1^2m_2 - m_1^2n_1 + 2m_2n_1 - m_1m_2n_1 - m_1^2n_2 - m_1m_2n_2 - m_2n_1n_2)\lambda^2 + (m_1^2 + m_1^3 + 3m_1m_2 + 4m_1^2m_2 + m_1^3m_2 - m_1^2n_1 + m_2n_1 - 2m_1m_2n_1 - m_1^2m_2n_1 - m_1^2n_2 - 2m_1m_2n_2 - 2m_1^2m_2n_2 - m_2n_1n_2)\lambda + m_1m_2(1 - p + 2m_1 + m_1^2 - m_1n_1 - 2m_1n_2 - m_1^2n_2 + m_1n_1n_2)]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $p-2$ and $p-3$ respectively. The partition $D : \cup_{i=1}^{5} U_i$ induces an equitable partition of G with $U_1 = V_{11}, U_2 = V_{12}, U_3, U_4 \in V_{21}$ and $U_5 = V_{22}$. Here $|U_3| = m_1, |U_4| = n_1 - m_1$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} m_1-1 & m_2 & m_1 & n_1-m_1 & n_2 \\ m_1 & 0 & m_1 & n_1-m_1 & 0 \\ m_1 & m_2 & 0 & 0 & 0 \\ m_1 & m_2 & 0 & n_1-m_1-1 & n_2 \\ m_1 & 0 & 0 & n_1-m_1 & n_2-1 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ p-2 & p-2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are the roots of $|A_D - \lambda I|$. ■

Theorem 3.16: Let $G = PTG(m_1, m_2; n_1, n_2)$ be a strong partial threshold graph with $m_1 > n_1$ on $n = 2p,\ p \geq 3$ vertices and $\langle V_1 \rangle \cong NSG(m_1; m_2)$ with partition $V_1 = V_{11} \cup V_{12}$, where V_{11} induces a clique and V_{12} induces a co-clique. Then, $E(G) = \sum_{i=1}^{5} |\lambda_i| + p - 2$, where $\lambda_1, \lambda_2, \ldots, \lambda_5$ are roots of the polynomial,
$[-\lambda^5 + (p-2)\lambda^4 + (p-1+m_1n_1+m_2n_1+m_1n_2)\lambda^3 + (m_1n_1+2m_2n_1-m_2^2n_1+m_1n_2-m_1m_2n_2)\lambda^2 + m_2n_1(1+m_1-m_2-m_1m_2-m_1n_2)\lambda + n_1m_1m_2(m_2-m_2^2-n_2+m_2n_2)]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $p-3$ and $p-2$ respectively. The partition $D : U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5$ induces an equitable partition of G with $U_1 = V_{11}, U_2 = V_{12}, U_3 = V_{21}$ and $U_4, U_5 \in V_{22}$. Here $|U_4| = n_2 - m_2, |U_5| = m_2$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} m_1-1 & m_2 & n_1 & n_2-m_2 & m_2 \\ m_1 & 0 & n_1 & 0 & 0 \\ m_1 & m_2 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 & m_2-1 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ p-2 & p-2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \ldots, \lambda_5$ are the roots of $|A_D - \lambda I|$. ■

Theorem 3.17: Let $G = PTG(m_1, m_2; m_1, m_2)$, with $m_1, m_2 \geq 2$ be a strong partial threshold graph on $n = 2p,\ p \geq 4$ vertices, with $\langle V_1 \rangle \cong NSG(1, m_1-1; 1, m_2-1)$ with the partition $V_1 = V_{11} \cup V_{12}$, where V_{11} induces a clique and V_{12} induces a co-clique. Then, $E(G) = \sum_{i=1}^{8} |\lambda_i| + p - 4$, where $\lambda_1, \lambda_2, \ldots, \lambda_8$ are the roots of the polynomial,
$[\lambda^8 + (4-p)\lambda^7 + (6-3p-m_1^2-2m_1m_2)\lambda^6 + (6-4p-4m_1^2-2m_2-5m_1m_2+m_1^2m_2+m_2^2+m_1m_2^2)\lambda^5 + (7-m_1-9m_1^2-9m_2-9m_1m_2+7m_1^2m_2+2m_2^2+5m_1m_2^2)\lambda^4 + (6+4m_1-11m_1^2-8m_2-13m_1m_2+14m_1^2m_2+2m_2^2+8m_1m_2^2-2m_1^2m_2^2)\lambda^3 + (1+7m_1-8m_1^2-m_2-17m_1m_2+16m_1^2m_2+10m_1m_2^2-7m_1^2m_2^2)\lambda^2 + (2m_1-2m_1^2-6m_1m_2+6m_1^2m_2+4m_1m_2^2-4m_1^2m_2^2)\lambda + 1 - 2p + m_1^2 + 4m_1m_2 - 2m_1^2m_2 + m_2^2 - 2m_1m_2^2 + m_1^2m_2^2]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $p-4$. There exists a partition $D : \cup_{i=1}^{8} U_i$ of $V(G)$ which induces an equitable partition of G with $U_1, U_2 \in V_{11}, U_3, U_4 \in V_{12}, U_5, U_6 \in V_{21}$ and $U_7, U_8 \in V_{22}$. Here, $|U_1| = |U_3| = |U_5| = |U_7| = 1$, $|U_2| = |U_6| = m_1 - 1$ and $|U_4| = |U_8| = m_2 - 1$. Then the divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_1-1 & 1 & m_2-1 & 1 & m_1-1 & 1 & m_2-1 \\ 1 & m_1-2 & 1 & 0 & 1 & m_1-1 & 1 & m_2-1 \\ 1 & m_1-1 & 0 & 0 & 1 & m_1-1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & m_1-1 & 0 & 0 \\ 1 & m_1-1 & 1 & m_2-1 & 0 & 0 & 0 & 0 \\ 1 & m_1-1 & 1 & m_2-1 & 0 & 0 & 0 & m_2-1 \\ 1 & m_1-1 & 0 & 0 & 0 & 0 & 0 & m_2-1 \\ 1 & m_1-1 & 0 & 0 & 0 & m_1-1 & 1 & m_2-2 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_8 \\ p-4 & p-4 & 1 & 1 & \cdots & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \ldots, \lambda_8$ are the roots of $|A_D - \lambda I|$. ■

Theorem 3.18: Let $G = PTG(m_1, m_2, \ldots, m_h; m_1, m_2, \ldots, m_h)$ be a strong partial threshold graph on $n = 2p,\ p \geq h$ vertices and $\langle V_1 \rangle \cong K_{m_1, m_2, \ldots, m_h}$. Then, $E(G) = \sum_{i=1}^{2h} |\lambda_i| + p - h$, where $\lambda_1, \lambda_2, \ldots, \lambda_{2h}$ are the eigenvalues of A_D, where A_D is the divisor matrix of the equitable partition D of G of order $2h$ which is given by,

$$\begin{bmatrix} 0 & m_2 & \cdots & m_h & m_1 & m_2 & \cdots & m_h \\ m_1 & 0 & \cdots & m_h & m_1 & m_2 & \cdots & 0 \\ m_1 & m_2 & \cdots & m_h & m_1 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ m_1 & m_2 & \cdots & 0 & m_1 & 0 & \cdots & 0 \\ m_1 & m_2 & \cdots & m_h & m_1-1 & 0 & \cdots & 0 \\ m_1 & m_2 & \cdots & 0 & 0 & m_2-1 & \cdots & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ m_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & m_h-1 \end{bmatrix}.$$

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $p-h$.

There exists a partition $D : U_1 \cup U_2 \cup \ldots \cup U_{2h}$ of $V(G)$ which induces an equitable partition of G with $U_i = V_{1i}, 1 \leq i \leq h$ and $U_{h+j} = V_{2j}, 1 \leqslant j \leqslant h$. Then the divisor matrix A_D of D can be obtained. Therefore, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2h} \\ p-h & p-h & 1 & 1 & \cdots & 1 \end{pmatrix}$ where $\lambda_1, \lambda_2, \ldots, \lambda_{2h}$ are the eigenvalues of $|A_D - \lambda I|$. ■

IV. Partial Chain graphs

In this section, the characteristic polynomials of some partial chain graphs are obtained. First we give a bound for the energy and spectral radius of a partial chain graph.

Theorem 4.1: Let $G = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a partial chain graph of order n and size m, then
$2\sqrt{\sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-i+1} n_i\right)} \leq E(G) \leq 2\sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-i+1} n_i\right) + k(k-1)$ where $k = \sum_{i=1}^{h} n_i$.

Theorem 4.2: Let $G(V_1 \cup V_2, E)$ be a partial chain graph of order n. Then,

$$\lambda_1(G) \leq \sqrt{2\sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-j+1} n_i\right) + k(k-1) - n + 1},$$

where $k = \sum_{i=1}^{n} n_i$. Equality in the upper bound holds if and only if either $|V_1| = 1$, $|V_2| = n-1$ and the graph induced by V_2 is a complete graph on $n-1$ vertices, or $|V_1| = n-1$ and $|V_2| = 1$.

Proof: The proof of the upper bound follows from Theorems 2.7 and 2.10. In order for equality to hold in the inequality 2.7, G must be a complete graph or a star graph of order n, from which the equality conditions hold. ∎

Theorem 4.3: Let $G(V_1 \cup V_2, E)$ be a partial chain graph of order n. Then,

$$\lambda_1(G) \leq \frac{1}{2}\left[-1+\sqrt{8\sum_{j=1}^{n}\left(\sum_{i=1}^{h-j+1} n_i\right)+4k(k-1)-8n+9}\right],$$

where $k = \sum_{i=1}^{n} n_i$. Equality in the upper bound holds if and only if either $|V_1| = 1$, $|V_2| = n-1$ and the graph induced by V_2 is a complete graph on $n-1$ vertices.

Proof: The proof of the upper bound follows from Theorems 2.8 and 2.10. In order for equality to hold in the inequality 2.8, G must be a complete graph of order n, from which the equality holds. ∎

The spectral radius of $G = PCG(p;p)$ with the graph induced by one of the set is regular is given below.

Theorem 4.4: Let $G = PCG(p;p)$ be a partial chain graph on $n = 2p$ vertices and the graph induced by the set V_1 is regular graph with regularity r. Then,

$$\lambda_1(G) = \frac{r+\sqrt{r^2+4p^2}}{2}.$$

Proof: Checking the structure of graph G, we can obtain an equitable partition $D : V_1 \cup V_2$ of G. Then, the divisor matrix A_D of D is given by

$$A_D = \begin{bmatrix} r & p \\ p & 0 \end{bmatrix}.$$

Thus, $\chi(A_D) = \lambda^2 - r\lambda - p^2$. By Theorem 2.2, the result follows. ∎

We show that a non-regular $PCG(p;q)$ with $\langle V_1 \rangle$ a regular graph has exactly 2 main eigenvalues in the following theorem.

Theorem 4.5: Let $G = PCG(p;q)$ be a non-regular graph with $\langle V_1 \rangle$ is a regular with regularity r. Then G has exactly two main eigenvalues.

Proof: We have $A_D = \begin{bmatrix} r & q \\ p & 0 \end{bmatrix}$. As all main eigenvalues of G are also the main eigenvalues of any divisor matrix, we show that the 2 eigenvalues of A_D are main, using the fact that "a graph G has two main eigenvalues if and only if $J, A(G)J, A(G)^2J$ are linearly dependent". Now suppose that

$$A_D^2 J = \alpha J + \beta A_D J$$

$$\implies \begin{bmatrix} r^2+pq & rq \\ pr & pq \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} + \beta\begin{bmatrix} r & q \\ p & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} r^2+pq+rq \\ pr+pq \end{bmatrix} = \begin{bmatrix} \alpha+\beta(r+q) \\ \alpha+\beta p \end{bmatrix}$$

On solving we get, $\alpha = pq$ and $\beta = r$ provided $p - q \neq r$. ∎

Theorem 4.6: Let $G = PCG(m_1, m_2; n_1, n_2)$ be a partial chain graph on $n \geq 4$ vertices, with $\langle V_1 \rangle \cong DNG(m_1; m_2)$. Then, $E(G) = \sum_{i=1}^{4} |\lambda_i|$ where $\lambda_1, \ldots, \lambda_4$ are the roots of the polynomial,
$[\lambda^4 - (m_1m_2 + m_1n_1 + m_2n_1 + m_1n_2)\lambda^2 - 2m_1m_2n_1\lambda + m_1m_2n_1n_2]$.

Proof: From Lemma 2.11, we note that 0 is the eigenvalue of G with multiplicity at least $n-4$. The partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ induces an equitable partition of G. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_2 & n_1 & n_2 \\ m_1 & 0 & n_1 & 0 \\ m_1 & m_2 & 0 & 0 \\ m_1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$ where $\lambda_1, \ldots, \lambda_4$ are the roots of $|A_D - \lambda I|$. ∎

Theorem 4.7: Let $G = PCG(2, r+s; n_1, n_2)$ be a partial chain graph on $n \geq 6$ vertices, with $\langle V_1 \rangle \cong DNG(1, r; 1, s)$ with partition $V_1 = V_{11} \cup V_{12}$ and the dominating vertices of the partial chain graph be the dominating vertices of $DNG(1, r; 1, s)$. Then, $E(G) = \sum_{i=1}^{6} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_6$ are the roots of the polynomial,
$[\lambda^6 - (1 + 2(n_1 + n_2) + (r+s)(n_1+1))\lambda^4 - 2(n_1(1+r+s) + n_2)\lambda^3 + ((2n_1+1)(rs + (r+s)n_2))\lambda^2 + 2n_1(n_2(r+s) + rs)\lambda - 4n_1n_2rs]$.

Proof: From Lemma 2.11, we note that 0 is the eigenvalue of G with multiplicity at least $n-6$. The partition $D : \cup_{i=1}^{6} U_i$ induces an equitable partition of G with $U_1, U_2 \in V_{11}, U_3, U_4 \in V_{12}, U_5 = V_{21}$ and $U_6 = V_{22}$. Here $|U_1| = |U_2| = 1, |U_3| = r$ and $|U_4| = s$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & 1 & 0 & s & n_1 & n_2 \\ 1 & 0 & r & 0 & n_1 & n_2 \\ 0 & 1 & 0 & 0 & n_1 & 0 \\ 1 & 0 & 0 & 0 & n_1 & 0 \\ 1 & 1 & r & s & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ n-6 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \ldots, \lambda_6$ are the roots of $|A_D - \lambda I|$. ∎

Theorem 4.8: Let $G = PCG(m_1, m_2; n_1, n_2)$ be a partial chain graph on $n \geq 5$ vertices, with $\langle V_1 \rangle \cong DNG(1; m_1 - 1)$ with a dominating vertex being the dominating vertex of $DNG(1; m_1 - 1)$. Then, $E(G) = \sum_{i=1}^{5} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_5$ are the roots of the polynomial,
$[-\lambda^5 + (m_1 + m_2 + m_1n_1 + m_2n_1 + m_1n_2 - 1)\lambda^3 + 2(m_1n_1 + m_2n_1 - n_1 - n_2 + m_1n_2)\lambda^2 + m_2n_2(1 - m_1 - m_1n_1)\lambda + 2m_2n_1n_2(1 - m_1)]$.

Proof: From Lemma 2.11, we note that 0 is the eigenvalue of G with multiplicity at least $n-5$. The partition $D : U_1 \cup U_2 \cup \ldots \cup U_5$, with $U_1, U_2 \in V_{11}, U_3 = V_{12}, U_3 = V_{21}$ and $U_5 = V_{22}$ induces an equitable partition of G. Here, $|U_1| = 1, |U_2| = m_1 - 1$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_1-1 & m_2 & n_1 & n_2 \\ 1 & 0 & 0 & n_1 & n_2 \\ 1 & 0 & 0 & n_1 & 0 \\ 1 & m_1-1 & m_2 & 0 & 0 \\ 1 & m_1-1 & 0 & 0 & 0 \end{bmatrix}$$

Then, $Spec(G) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ n-5 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \ldots, \lambda_5$ are the roots of $|A_D - \lambda I|$. ■

Theorem 4.9: Let $G = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a partial chain graph on $n, \geq 2h$ vertices, with $\langle V_1 \rangle \cong K_{m_1,m_2,\ldots,m_h}$. Then, Then, $E(G) = \sum_{i=1}^{2h} |\lambda_i|$ where $\lambda_1, \ldots, \lambda_{2h}$ are the eigenvalues of A_D, where A_D is the divisor matrix of G of order $2h$ which is given by

$$\begin{bmatrix} 0 & m_2 & m_3 & \cdots & m_h & n_1 & n_2 & \cdots & n_{h-1} & n_h \\ m_1 & 0 & m_3 & \cdots & m_h & n_1 & n_2 & \cdots & n_{h-1} & 0 \\ m_1 & m_2 & 0 & \cdots & m_h & n_1 & n_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ m_1 & m_2 & m_3 & \cdots & 0 & n_1 & 0 & \cdots & 0 & 0 \\ m_1 & m_2 & m_3 & \cdots & m_h & 0 & 0 & \cdots & 0 & 0 \\ m_1 & m_2 & m_3 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1 & m_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ m_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Proof: From Lemma 2.11, we note that 0 is the eigenvalue of G with multiplicity at least $n - 2h$. The partition $D : V_{11} \cup V_{12} \cup \ldots \cup V_{1h} \cup V_{21} \cup \ldots \cup V_{2h}$ induces an equitable partition of G. The divisor matrix A_D of D can be obtained. Therefore, $Spec(G) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2h} \\ n-2h & 1 & 1 & \cdots & 1 \end{pmatrix}$, where $\lambda_1, \lambda_2, \ldots, \lambda_{2h}$ are the eigenvalues of the divisor matrix A_D. ■

Theorem 4.10: Let $G = PCG(m_1, m_2; n_1, n_2)$ be a partial chain graph on n vertices, with $\langle V_1 \rangle \cong K_{m_1} \cup K_{m_2}$. Then, $E(G) = \sum_{i=1}^{4} |\lambda_i| + m_1 + m_2 - 2$, where $\lambda_1, \ldots, \lambda_4$ are the roots of the polynomial,
$[\lambda^4 + (2 - m_1 - m_2)\lambda^3 + (1 - m_1 - m_2 + m_1 m_2 - m_1 n_1 - m_2 n_1 - m_1 n_2)\lambda^2 + (2m_1 m_2 n_1 - m_1 n_1 - m_2 n_1 - m_1 n_2 + m_1 m_2 n_2)\lambda + m_1 m_2 n_1 n_2]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $n_1 + n_2 - 2$ and $m_1 + m_2 - 2$. The partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ induces an equitable partition of G. The divisor matrix A_D of D is given by,

$$\begin{bmatrix} m_1-1 & 0 & n_1 & n_2 \\ 0 & m_2-1 & n_1 & 0 \\ m_1 & m_2 & 0 & 0 \\ m_1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ m_1+m_2-2 & n_1+n_2-2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \ldots, \lambda_5$ are the roots of $|A_D - \lambda I|$. ■

Theorem 4.11: Let $G = PCG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ be a partial chain graph on $n \geq 2h$ vertices, with $\langle V_1 \rangle \cong K_{m_1} \cup K_{m_2} \cup \ldots \cup K_{m_h}$. Then, $E(G) = \sum_{i=1} 2h |\lambda_i| + \sum_{i=1}^{h} m_i - h$, where $\lambda_1, \ldots, \lambda_{2h}$ are the eigenvalues of A_D, where A_D is the divisor matrix of G of order $2h$ which is given by,

$$\begin{bmatrix} m_1-1 & 0 & \cdots & 0 & n_1 & n_2 & \cdots & n_{h-1} & n_h \\ 0 & m_2-1 & \cdots & 0 & n_1 & n_2 & \cdots & n_{h-1} & 0 \\ 0 & 0 & \cdots & 0 & n_1 & n_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_h-1 & n_1 & 0 & \cdots & 0 & 0 \\ m_1 & m_2 & \cdots & m_h & 0 & 0 & \cdots & 0 & 0 \\ m_1 & m_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1 & m_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ m_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $\sum_{i=1}^{h} n_i - h$ and $\sum_{i=1}^{h} m_i - h$. The partition $D : V_{11} \cup V_{12} \cup \ldots \cup V_{1h} \cup V_{21} \cup \ldots \cup V_{2h}$ induces an equitable partition of G. The divisor matrix A_D of D can be obtained.
Therefore, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2h} \\ \sum_{i=1}^{h} m_i - h & \sum_{i=1}^{h} n_i - h & 1 & 1 & \cdots & 1 \end{pmatrix}$ where $\lambda_1, \lambda_2, \ldots, \lambda_{2h}$ are the eigenvalues of the divisor matrix A_D. ■

Theorem 4.12: Let $G = PCG(m_1, m_2; n_1, n_2)$ be a partial chain graph on $n \geq 6$ vertices, with $\langle V_1 \rangle \cong NSG(1, m_1 - 1; 1, m_2 - 1)$ with partition $V_1 = V_{11} \cup V_{12}$, where V_{11} induces a clique and V_{12} induces a co-clique. Then, $E(G) = \sum_{i=1}^{6} |\lambda_i| + m_1 - 2$, where $\lambda_1, \ldots, \lambda_6$ are the roots of the polynomial,
$[\lambda^6 + (2 - m_1)\lambda^5 + (2 - 2m_1 - m_2 - m_1 n_1 - m_2 n_1 - m_1 n_2)\lambda^4 + (2 - 2m_1 - 2m_2 + m_1 m_2 + 2n_1 - 3m_1 n_1 - 4m_2 n_1 + m_1 m_2 n_1 - m_1 n_2)\lambda^3 + (1 - m_1 - m_2 + m_1 m_2 + 4n_1 - 4m_1 n_1 - 5m_2 n_1 + 3m_1 m_2 n_1 + n_2 - m_1 n_2 - m_2 n_2 + m_1 m_2 n_2 + m_1 m_2 n_1 n_2)\lambda^2 + n_1(2 - 2m_1 - 2m_2 + 2m_1 m_2 + m_1 m_2 n_2)\lambda + n_1 n_2 (m_1 m_2 - 1 - m_1 m_2)]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $n - m_1 - 4$ and $m_1 - 2$ respectively. The partition $D : \cup_{i=1}^{6} U_i$ with $U_1, U_2 \in V_{11}, U_3, U_4 \in V_{12}, U_5 = V_{21}$ and $U_6 = V_{22}$ induces an equitable partition of G. Here $|U_1| = |U_3| = 1, |U_2| = m_1 - 1$ and $|U_4| = m_2 - 1$. The divisor matrix A_D of D is given by,

$$A_D = \begin{bmatrix} 0 & m_1-1 & 1 & m_2-1 & n_1 & n_2 \\ 1 & m_1-2 & 1 & 0 & n_1 & n_2 \\ 1 & m_1-1 & 0 & 0 & n_1 & 0 \\ 1 & 0 & 0 & 0 & n_1 & 0 \\ 1 & m_1-1 & 1 & m_2-1 & 0 & 0 \\ 1 & m_1-1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ m_1-2 & n_1+n_2-2 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, where $\lambda_1, \ldots, \lambda_6$ are the roots of $|A_D - \lambda I|$. ■

Theorem 4.13: Let $G = PCG(m_1, m_2; n_1, n_2)$ be a partial chain graph on $n \geq 4$ vertices, with $\langle V_1 \rangle \cong NSG(m_1; m_2)$ with partition $V_1 = V_{11} \cup V_{12}$, where V_{11} induces a clique and V_{12} induces a co-clique. Then, $E(G) = \sum_{i=1}^{4} |\lambda_i| + m_1 - 1$, where $\lambda_1, \ldots, \lambda_4$ are the roots

of the polynomial,
$[\lambda^4 + (1-m_1)\lambda^3 - (m_1m_2 + m_1n_1 + m_2n_1 + m_1n_2)\lambda^2 - m_2n_1(1+m_1)\lambda + m_1m_2n_1n_2]$.

Proof: From Lemma 2.11, we note that 0 and -1 are the eigenvalues of G with multiplicity at least $n-m_1-3$ and m_1-1 respectively. The partition $D : V_{11} \cup V_{12} \cup V_{21} \cup V_{22}$ induces an equitable partition of G. The divisor matrix of D is given by,

$$A_D = \begin{bmatrix} m_1-1 & m_2 & n_1 & n_2 \\ m_1 & 0 & n_1 & 0 \\ m_1 & m_2 & 0 & 0 \\ m_1 & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$Spec(G) = \begin{pmatrix} -1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ m_1-1 & n-m_1-3 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_4$ are the roots of $|A_D - \lambda I|$. ■

V. Conclusion

Recently, the authors of the article [11] introduced the concept of k-nested graphs by extending the nesting property from bipatiteness to k-partite graphs. One can obtain the spectral properties of a k-nested graph.

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