Stability of the SIRS Epidemic Models with General Transmission Rate Functions

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Abstract—In epidemiological models, a transmission rate function plays an important role in the transmission of diseases. The transmission rate is usually affected by many factors, such as media coverage, density of population and life style. In this work, we study the SIRS epidemic models with general functions of transmission rate. Firstly, we establish conditions for the existence and uniqueness of disease-free and endemic equilibria under general conditions on the transmission rate functions. The disease-free equilibrium is investigated to calculate the basic reproduction number. Next, we analyze the local stability of each equilibrium. In addition, the Lyapunov's functions are used to prove global stability of the model's equilibria. Finally, various classes of numerical simulations are illustrated to support the analytical results with some examples of the transmission rate functions, such as bilinear, saturated and media coverage functions.

Index Terms—SIRS model, basic reproduction number, transmission rate function, Lyapunov function, global stability.

I. INTRODUCTION

EPIDEMIC models of outbreak and transmission of many diseases, such as dengue fever, malaria, influenza and HIV/AIDS, have been developed and studied by many scholars. Many mathematical models for the transmission of infectious diseases are based on the classical susceptible-infectious-recovered (SIR) model of Kermack and McKendrick [1]. In the SIR model, a population is divided into three groups, which are a susceptible group (S) who can become infected with a disease through contact with an infectious person, an infectious group (I) who have the disease and can infect a susceptible person through contact, and a recovered group (R) who have recovered from the disease. The SIR model assumes that a person who has recovered from the disease has acquired lifelong immunity to the disease. For some classes of diseases, the infected individuals can acquire only temporary immunity on recovery and then become susceptible to infection again. In this case, an SIRS model is used. One of the basic SIRS epidemic models is described by

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - G(I)S - \mu S + \delta R, \\ \frac{dI}{dt} &= G(I)S - (\mu + \gamma)I, \\ \frac{dR}{dt} &= \gamma I - (\mu + \delta)R. \end{aligned} \quad (1)$$

In (1), $G(I)$ is the infectious force function which plays an important role in the transmission of diseases as it represents the rate at which one susceptible individual can be infected by contact with the infectious population $I(t)$. The linear infectious force function $G(I) = \beta I$ was the original function used by Kermack and McKendrick [1] and it is still frequently used in many literatures (see, e.g. Anderson and May [2]).

The variables and parameters which are used in (1) are summarized in Table I and Table II, respectively. Note that all variables are assumed to be non-negative and all parameters are assumed to be positive.

TABLE I
DEFINITIONS OF MODEL'S VARIABLES.

Variables	Unit	Definition
t	day	time
$S(t)$	person	The number of susceptible individuals at time t
$I(t)$	person	The number of infectious individuals at time t
$R(t)$	person	The number of recovered individuals at time t

TABLE II
DEFINITIONS OF MODEL'S PARAMETERS.

Parameter	Unit	Meaning
Λ	person· day^{-1}	The recruitment rate of susceptible individuals
μ	day^{-1}	The natural death rate
γ	day^{-1}	The recovery rate of infectious individuals
δ	day^{-1}	The rate of loss of immunity by recovered individuals

Many researchers are now interested in cases of $G(I)$ as the formulation of nonlinear infectious force functions. Capasso and Serio [3] introduced an infectious force function $G(I)$ into epidemic SIR models, which saturates at high levels of I. This general force function can be used to explain psychological effects which occur when a very large number of infectious people in a region might cause susceptible people to avoid crowded areas, such as shopping centers and schools. In addition, the infectious force function $G(I) = \beta I/N$ has been used in [4]-[5] as a model for extreme case, where a high percentage of the population is infected.

Liu *et al.* [6]-[7] analyzed the SIRS models with the infectious force function

$$G(I) = \frac{kI^l}{1 + \alpha I^h}. \quad (2)$$

Ruan and Wang [8] studied the SIRS epidemic model with the special case of (2), where $l = h = 2$. In addition, Xiao and Ruan [9] studied the SIRS epidemic model with the special case of (2) $l = 1$ and $h = 2$. Moreover, Li *et al.*

[10] also studied the SIR model with the special case when $l = h = 1$ in (2).

In real life, the infectious force function $G(I)$ may be affected by many factors, such as media coverages, density of population, life styles, and behavior changing of people when a disease enters a population. There are variety of force functions that have been discussed in preview literatures.

From many previous works, it can be seen that I is a factor of $G(I)$ function. Hence, we can write the infectious force function as $G(I) = g(I)I$, where $g(I)$ is the rate of infection of a susceptible individual through contact with an infectious individual.

For examples, Cui and Tao [11] studied the SIS model with a general awareness-induced incidence $f(I)$ to investigate effects of media coverage of an epidemic on the transmission dynamics using the force function $G(I) = g(I)I = (\beta_1 - \beta_2 f(I))I$, where β_1 is the contact rate before the media alerts, and the term $\beta_2 f(I)$ represents the reduction in the contact rate when the media reports the epidemic. Korobeinikov [12], Mei and Fuqin [13] studied the SIR models with general incidence rate $G(I, S)$ under the constant population size assumption. Moreover, Udomchalermpat *et al.* [14] analyzed dynamics of the generalized tumor-virotherapy model with general classes of transmission rate functions.

In our study, we study some properties of the SIRS model with a general transmission rate function as a function of infectious individual I. The model is adapted from Miaochan and Huitao [15] as

$$\frac{dS}{dt} = \Lambda - g(I)IS - \mu S + \delta R, \qquad (3)$$

$$\frac{dI}{dt} = g(I)IS - (\mu + \gamma)I, \qquad (4)$$

$$\frac{dR}{dt} = \gamma I - (\mu + \delta)R, \qquad (5)$$

where the total population is $N(t) = S(t) + I(t) + R(t)$ and

$$\frac{dN}{dt} = \Lambda - \mu N. \qquad (6)$$

The assumptions for the transmission rate function $g(I)$ are stated as follows.

Assumption 1: We make the assumptions for the transmission rate function $g(I)$ as follows :

(A1) the function $g(I) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and differentiable function,

(A2) $g(0) = g_0 > 0$ and $g(I) > 0$ for all $I > 0$,

(A3) $g'(I) \leq 0$ for all $I \geq 0$.

The rest of this paper is organized as follows. In Section II, we show the non-negativity and boundedness of the solutions of (3)-(5). In Section III, we investigate the basic reproduction number, $\mathcal{R}_0$. Moreover, we find all equilibria of the model and investigate conditions for the existence of unique endemic equilibrium. In Section IV, we analyze local stability of the disease-free equilibrium and the endemic equilibrium. In Section V, we analyze global stability of all equilibria by using the Lyapunov's direct method. In Section VI, we provide some numerical simulations to support our analytical results. Finally, we provide brief discussion and summary of the main results.

II. NON-NEGATIVITY AND BOUNDEDNESS OF SOLUTIONS

In this section, we study the positivity and boundedness of solutions for model (3)-(5), which is subjected to positive initial conditions:

$$S(0) = S_0 > 0, \qquad I(0) = I_0 > 0, \qquad R(0) = R_0 > 0. \qquad (7)$$

The solutions of (3)-(5) with initial conditions (7) are in the non-negative bounded set Γ defined by

$$\Gamma = \left\{ (S, I, R) \in \mathbb{R}_+^3 \cup \{\mathbf{0}\} \,|\, 0 \leq S + I + R \leq \frac{\Lambda}{\mu} \right\} \subset \mathbb{R}_+^3. \quad (8)$$

The set Γ is closed and bounded, and hence it is a compact set. Next, we will show that all solutions of (3)-(5) are non-negative and bounded in $\mathbb{R}_+^3$.

Theorem 2.1: Solutions of the SIRS model (3)-(5) with initial conditions (7) are non-negative for all $t \geq 0$.

Proof: Let $(S(t), I(t), R(t))$ be a solution of (3)-(5) with initial conditions (7).

From (4), it follows that

$$I(t) = I_0 e^{\int_0^t \{g(I(\tau))S(\tau) - (\mu+\gamma)\} d\tau} \geq 0, \qquad (9)$$

for all $t \geq 0$ since $I_0 \geq 0$.

From (5), we have

$$\frac{dR}{dt} = \gamma I - (\mu + \delta)R \geq -(\mu + \gamma)R.$$

Because $R_0 \geq 0$, then it can be shown that

$$R(t) \geq R_0 e^{-(\mu+\gamma)t}, \qquad (10)$$

for all $t \geq 0$. According to (3), since $\Lambda > 0$ and $R \geq 0$, then we obtain that

$$\frac{dS}{dt} = \Lambda - g(I)IS - \mu S + \delta R \geq -(g(I)I - \mu)S.$$

Therefore, we have

$$S(t) \geq S_0 e^{-\int_0^t (g(I(\tau))I(\tau) + \mu) d\tau} \geq 0, \qquad (11)$$

for all $t \geq 0$ since $S_0 > 0$.

As the results, it can be concluded that $S(t), I(t)$ and $R(t)$ are non-negative for all $t \geq 0$. ∎

Theorem 2.2: If the set Γ is defined in (8), then it is a positive invariant set and all solutions of the SIRS model (3)-(5) with the initial conditions (7) are ultimately bounded.

Proof: From Theorem 2.1, solutions of (3)-(5) with the initial conditions (7) are positive for all $t \geq 0$.

Using the related rates in (3)-(5), it follows the relation (6) :

$$\frac{dN}{dt} = \Lambda - \mu N,$$

which is a linear differential equation and the solution is

$$N(t) = \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu} \right) e^{-\mu t}, \qquad (12)$$

where $N(0) = N_0$. From (6) and (8), $N(0) = S(0) + I(0) + R(0) \leq \frac{\Lambda}{\mu}$. Therefore, $N(t) < \frac{\Lambda}{\mu}$ for all $t > 0$. Thus $S(t), I(t)$ and $R(t)$ are ultimately bounded. The proof is complete. ∎

III. Existence of Equilibria

In this section, we investigate the basic reproduction number $\mathcal{R}_0$ and find conditions for an existence of unique positive equilibrium of (3)-(5). Suppose that a disease-free equilibrium denoted by $\mathbf{E}_0^*$ is the steady-state solution where there is no infection, and an endemic equilibrium denoted by $\mathbf{E}_+^*$ is a positive steady-state solution where the disease persists in the population.

A. The disease-free equilibrium

The equilibria of (3)-(5) can be found by assuming that all derivative terms are zero. Let $\mathbf{E}^* = (S^*, I^*, R^*)$ be an equilibrium of the model. Then the system of nonlinear algebraic equations for finding equililbria are given by

$$\Lambda - g(I^*)I^*S^* - \mu S^* + \delta R^* = 0, \qquad (13)$$
$$I^*[g(I^*)S^* - (\mu + \gamma)] = 0, \qquad (14)$$
$$\gamma I^* - (\mu + \delta)R^* = 0. \qquad (15)$$

We solve this system by first writing S^* and R^* as functions of I^*. From (15), we have

$$R^* = \frac{\gamma I^*}{\mu + \delta}. \qquad (16)$$

From (14), we have

$$I^* = 0, \qquad \text{or} \quad S^* = \frac{\mu + \gamma}{g(I^*)}. \qquad (17)$$

For the case that $I^* = 0$, it is not difficult to show that the DFE is

$$\mathbf{E}_0^* = (\frac{\Lambda}{\mu}, 0, 0). \qquad (18)$$

B. The basic reproduction number

For epidemic models, we usually find the basic reproduction number, $\mathcal{R}_0$, which is the most fundamental parameter used by epidemiologists. In epidemic models, $\mathcal{R}_0$ is defined as the average number of secondary cases caused by an infectious individual in a completely susceptible population [2]. It is an important parameter that gives us whether an infection will spread through the population. To obtain $\mathcal{R}_0$, we use the next-generation matrix technique described in [16]. Firstly let $\mathbf{x} = [I^*, R^*]$, then Eqs. (3)-(5) can be written as $\frac{d\mathbf{x}}{dt} = \mathcal{F}(\mathbf{x}) - \mathcal{V}(\mathbf{x})$, where

$$\mathcal{F}(\mathbf{x}) = \begin{bmatrix} g(I)SI \\ 0 \end{bmatrix} \text{ and } \mathcal{V}(\mathbf{x}) = \begin{bmatrix} (\mu + \gamma)I \\ -\gamma I + (\mu + \delta)R \end{bmatrix}.$$

Then we linearize (3)-(5) about the disease-free equilibrium, $\mathbf{E}_0^* = (\frac{\Lambda}{\mu}, 0, 0)$, by taking the first partial derivatives of $\mathcal{F}$ and $\mathcal{V}$ at $\mathbf{E}_0^*$ to obtain the matrices $\mathbf{F}$ and $\mathbf{V}$ respectively, where

$$\mathbf{F} = \begin{bmatrix} \frac{g_0\Lambda}{\mu} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} \mu + \gamma & 0 \\ -\gamma & \mu + \delta \end{bmatrix}.$$

Then the product of $\mathbf{F}$ and $\mathbf{V}^{-1}$ is

$$\mathbf{F}\mathbf{V}^{-1} = \begin{bmatrix} \frac{g_0\Lambda}{\mu(\mu+\gamma)} & 0 \\ 0 & 0 \end{bmatrix}.$$

The basic reproduction number $\mathcal{R}_0$ is the spectral radius of $\mathbf{F}\mathbf{V}^{-1}$. Hence, the basic reproduction number of (3)-(5) is

$$\mathcal{R}_0 = \frac{g_0\Lambda}{\mu(\mu+\gamma)}. \qquad (19)$$

C. The endemic equilibrium

The basic reproduction number $\mathcal{R}_0$ in (19) is used to analyze for existency of the endemic equilibrium. Firstly, we state the following theorem.

Theorem 3.1: If all conditions in Assumption 1 are satisfied and $\mathcal{R}_0 < 1$, then model (3)-(5) has no endemic equilibrium. On the other hand, if $\mathcal{R}_0 > 1$, then model (3)-(5) has unique endemic equilibrium, which is

$$\mathbf{E}_+^* = (S_+^*, I_+^*, R_+^*) = \left(\frac{\mu + \gamma}{g(I_+^*)}, I_+^*, \frac{\gamma I_+^*}{\mu + \delta} \right), \qquad (20)$$

where I_+^* is the solution of

$$g(I) = \frac{\mu(\mu + \delta)(\mu + \gamma)}{\Lambda(\mu + \delta) - \mu(\mu + \gamma + \delta)I}. \qquad (21)$$

Proof: In (17), if $I^* \neq 0$ and $S^* = \frac{\mu+\gamma}{g(I^*)}$, then we have $R^* = \frac{\gamma I^*}{\mu+\delta}$. Substituting S^* and R^* into (13), we can solve for I^*. Then I^* must satisfies

$$g(I^*) = \frac{\mu(\mu + \delta)(\mu + \gamma)}{\Lambda(\mu + \delta) - \mu(\mu + \gamma + \delta)I^*}. \qquad (22)$$

Let $\mathbf{E}^* = (S^*, I^*, R^*)$ be an equilibrium and I^* can be found by solving (22). We will prove that (22) has unique solution of I^*. To find I^*, let the right-hand side of (22) be $h(I)$, namely

$$h(I) = \frac{\mu(\mu + \delta)(\mu + \gamma)}{\Lambda(\mu + \delta) - \mu(\mu + \gamma + \delta)I}, \qquad (23)$$

where $I \neq \frac{\Lambda(\mu + \delta)}{\mu(\mu + \gamma + \delta)}$.

Suppose that I^* is the root of (22), then I^* must satisfy $g(I^*) = h(I^*)$. At $I^* = 0$, we have $g(0) = g_0 > 0$ and $h(0) = \frac{\mu(\mu+\gamma)}{\Lambda} > 0$. Find derivatives of $g(I)$ and $h(I)$. From (A3), $g'(I) < 0$, i.e., $g(I)$ is non-increasing function for all $I \geq 0$. Then we have

$$h'(I) = \frac{\mu^2(\mu + \delta)(\mu + \gamma)(\mu + \delta + \gamma)}{[\Lambda(\mu + \delta) - \mu(\mu + \delta + \gamma)I]^2} > 0, \quad (24)$$

where $I \neq \frac{\Lambda(\mu + \delta)}{\mu(\mu + \gamma + \delta)}$.

Hence, $h(I)$ is a monotonically increasing function. If $g(I)$ is a non-increasing function with the initial condition $g_0 > h_0$, which related to $\mathcal{R}_0 = \frac{g_0\Lambda}{\mu(\mu+\gamma)} > 1$, then two curves $g(I)$ and $h(I)$ of (22) have a unique positive intersection on $I \in [0, \frac{\Lambda}{\mu}]$, which is a unique solution where $I = I^*$. We denote that $I^* = I_+^* \neq 0$ and $\mathbf{E}_+^* = (S_+^*, I_+^*, R_+^*)$ is only a unique positive endemic equilibrium of (3)-(5). It can also be seen that if $g_0 < h_0 = \frac{\mu(\mu+\gamma)}{\Lambda}$ or $\mathcal{R}_0 = \frac{g_0\Lambda}{\mu(\mu+\gamma)} < 1$, then $g(I) = h(I)$ cannot have a positive solution for I^*, i.e., an endemic equilibrium does not exist.

As the results, the endemic equilibrium is $\mathbf{E}_+^* = (S_+^*, I_+^*, R_+^*)$, where

$$S_+^* = \frac{\mu + \gamma}{g(I_+^*)}, \qquad R_+^* = \frac{\gamma I_+^*}{\mu + \delta}, \qquad (25)$$
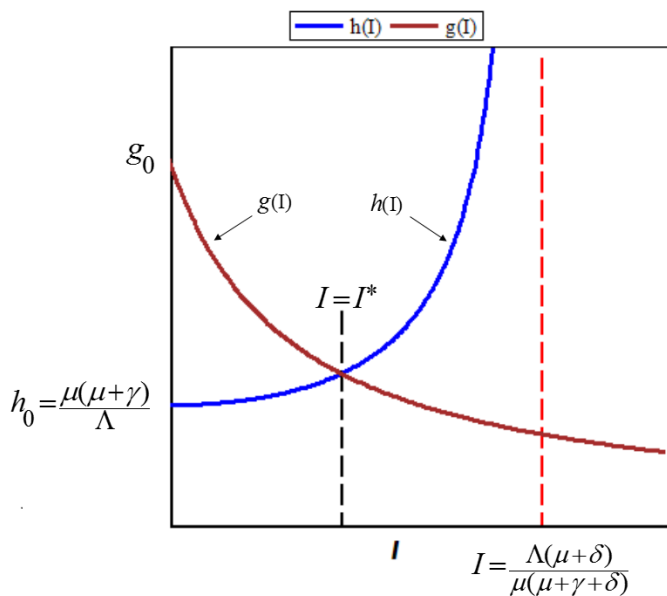
and I_+^* is the solution of (22) ∎

Fig. 1. The intersection of the curves $g(I)$ and $h(I)$ indicate the existence of I^*.

IV. LOCAL STABILITY OF EQUILIBRIA

In this section, we analyze local stability properties of the disease-free equilibrium $(\mathbf{E}_0^*)$ and the endemic equilibrium $(\mathbf{E}_+^*)$ using the linearization method. We prove that the disease-free equilibrium $\mathbf{E}_0^*$ is locally asymptotically stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$. Moreover, we also show that the endemic equilibrium $\mathbf{E}_+^*$ is locally asymptotically stable when $\mathcal{R}_0 > 1$.

Firstly, rewrite the right-hand side of (3)-(5) with functions F_1, F_2 and F_3, respectively. Then change the variables by defining

$$S = S^* + s, \quad I = I^* + i \quad \text{and} \quad R = R^* + r, \quad (26)$$

where s, i and r are deviations from the equilibrium.

By the linearization method, we obtain the Jacobian matrix of (3)-(5) at the equilibrium $\mathbf{E}^* = (S^*, I^*, R^*)$ is

$$J(\mathbf{E}^*) =$$
$$\begin{bmatrix} -g(I^*)I^* - \mu & -g(I^*)S^* - g'(I^*)I^*S^* & \delta \\ g(I^*)I^* & g(I^*)S^* + g'(I^*)I^*S^* - (\mu + \gamma) & 0 \\ 0 & \gamma & -(\mu + \delta) \end{bmatrix}.$$
$$(27)$$

The eigenvalues are the solutions of the characteristic equation

$$J(\mathbf{E}^*) = \det(J - \lambda I) = 0, \quad (28)$$

where I is the identity matrix.

A. Asymptotic stability of the disease-free equilibrium

In this part, we analyze the asymptotic stability of the disease-free equilibrium $\mathbf{E}_0^* = (\frac{\Lambda}{\mu}, 0, 0)$. The aim is to find the conditions in which all real parts of eigenvalues of the Jacobian matrix are negative.

Theorem 4.1: Suppose that all conditions in Assumption 1 holds. The disease-free equilibrium $\mathbf{E}_0^*$ is locally asymptotically stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$.

Proof: From (27), the Jacobian matrix of (3)-(5) about $\mathbf{E}_0^* = (\frac{\Lambda}{\mu}, 0, 0)$ is

$$J(\mathbf{E}_0^*) = \begin{bmatrix} -\mu & -\frac{g_0\Lambda}{\mu} & \delta \\ 0 & \frac{g_0\Lambda}{\mu} - (\mu + \gamma) & 0 \\ 0 & \gamma & -(\mu + \delta) \end{bmatrix}. \quad (29)$$

The characteristic equation of (29) is

$$(\lambda + \mu)(\lambda + \mu + \delta)\left(\lambda - \frac{g_0\Lambda}{\mu} + \mu + \gamma\right) = 0. \quad (30)$$

From (30), we have two negative eigenvalues which are $\lambda_1 = -\mu$ and $\lambda_2 = -(\mu + \delta)$. The third eigenvalue is $\lambda_3 = \frac{g_0\Lambda}{\mu} - (\mu + \gamma) = -(\mu + \gamma)(1 - \mathcal{R}_0)$. Note that λ_3 is negative if $\mathcal{R}_0 < 1$ and it is positive if $\mathcal{R}_0 > 1$. Therefore, if $\mathcal{R}_0 < 1$, then all eigenvalues of (29) are negative. Hence, it can be concluded that the disease-free equilibrium $\mathbf{E}_0^*$ is locally asymptotically stable.

On the contrary, if $\mathcal{R}_0 > 1$, then λ_3 is positive. Thus, in this case, the disease-free equilibrium $\mathbf{E}_0^*$ is unstable. ∎

B. Asymptotic stability of the endemic equilibrium

From Theorem 3.1, if $\mathcal{R}_0 > 1$ and all conditions in Assumption 1 holds, then the endemic equilibrium $\mathbf{E}_+^* = (S_+^*, I_+^*, R_+^*)$ of (3)-(5) exists. In this part, we will analyze the local asymptotic stability of the endemic equilibrium $\mathbf{E}_+^*$ defined in (20). The local asymptotic stability is proved by using the Routh-Hurwitz criteria. We will investigate the conditions in which all eigenvalues of Jacobian matrix (27) at $\mathbf{E}_+^*$ have negative real parts.

Theorem 4.2: If the endemic equilibrium $\mathbf{E}_+^*$ exists and satisfies all conditions in Assumption 1, then $\mathbf{E}_+^*$ is locally asymptotically stable.

Proof: We have already proved in Theorem 3.1 that $\mathbf{E}_+^*$ exists if and only if $\mathcal{R}_0 > 1$.

Substitute the endemic equilibrium $\mathbf{E}_+^*$ defined in (20) into (27). From (27) the Jacobian matrix about $\mathbf{E}_+^*$ is

$$J(\mathbf{E}_+^*) =$$
$$\begin{bmatrix} -g(I^*)I^* - \mu & -(\mu + \gamma) - \frac{(\mu+\gamma)g'(I^*)I^*}{g(I^*)} & \delta \\ g(I^*)I^* & \frac{(\mu+\gamma)g'(I^*)I^*}{g(I^*)} & 0 \\ 0 & \gamma & -(\mu + \delta) \end{bmatrix}.$$
$$(31)$$

The eigenvalues, λ, of (31) can be evaluated by solving the characteristic equation

$$P(\lambda) = A_0\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \quad (32)$$

where

$$\begin{aligned} A_0 &= g(I^*), \\ A_1 &= (2\mu + \delta + g(I^*)I^*)g(I^*) - (\mu + \gamma)g'(I^*)I^*, \\ A_2 &= (2\mu + \gamma + \delta)(g(I^*))^2I^* + \mu(\mu + \delta)g(I^*) \\ &\quad -(\mu + \gamma)(2\mu + \delta)g'(I^*)I^*, \\ A_3 &= \mu(\mu + \gamma + \delta)(g(I^*))^2I^* - \mu(\mu + \gamma)(\mu + \delta)g'(I^*)I^*. \end{aligned}$$
$$(33)$$

According to the condition A_1 in Assumption 1, $g'(I^*) < 0$, it can be seen that the values of A_0, A_1, A_2 and A_3 are positive. By the Descarte's rule of signs [17], the characteristic equation (32) can not have any positive real roots. Also, zero is also not a root of (32).

Next, we apply the Routh-Hurwitz criterion [18] to prove that all roots of $P(\lambda)$ in (32) have negative real parts. Three Hurwitz matrices corresponding to the coefficients of (32) are given by

$$H_1 = [A_1], \qquad H_2 = \begin{bmatrix} A_1 & A_0 \\ A_3 & A_2 \end{bmatrix},$$

$$\text{and} \qquad H_3 = \begin{bmatrix} A_1 & A_0 & 0 \\ A_3 & A_2 & A_1 \\ 0 & 0 & A_3 \end{bmatrix}.$$

From the Routh-Hurwitz conditions, all real parts of the eigenvalues of $J(\mathbf{E}_*^+)$ defined in (31) are negative if the following three conditions are satisfied:

1. $\det(H_1) = A_1 = g(I^*) \qquad > 0$,
2. $\det(H_2) = A_1 A_2 - A_0 A_3$
$$= \left[(2\mu + \delta + g(I^*)I^*)g(I^*) - (\mu + \gamma)g'(I^*)I^*\right]$$
$$\times \left[(2\mu + \gamma + \delta)(g(I^*))^2 I^* + \mu(\mu + \delta)g(I^*)\right.$$
$$\left. -(\mu + \gamma)(2\mu + \delta)g'(I^*)I^*\right]$$
$$-g(I^*)\left[\mu(\mu + \gamma + \delta)(g(I^*))^2 I^* - \mu(\mu + \gamma)(\mu + \delta)g'(I^*)I^*\right]$$
$$\geq \left[\mu g(I^*) - (\mu + \gamma)g'(I^*)I^* + (g(I^*))^2 I^*\right]$$
$$\times \left[\gamma(g(I^*))^2 I^* + \mu(\mu + \delta)g(I^*)\right]$$
$$+\mu(\mu + \gamma)(\mu + \delta)g'(I^*)g(I^*)I^* - \mu(\mu + \gamma + \delta)(g(I^*))^3 I^*$$
$$= \mu^2(\mu + \delta)(g(I^*))^2 - \gamma(\mu + \gamma)(g(I^*))^2(I^*)^2 g'(I^*)$$
$$+\gamma(g(I^*))^4(I^*)^2$$
$$> 0,$$
3. $\det(H_3) = A_3 \det(H_2) \quad > 0.$ \hfill (34)

The proof is complete. ∎

V. Global stability of equilibria

In this section, We prove the global stability of the disease-free equilibrium $\mathbf{E}_0^*$ and the endemic equilibrium $\mathbf{E}_+^*$. The aim is to analyze the global stability by constructing the Lyapunov functions releted to the Lyapunov theorem [19].

A. Global stability analysis of the disease-free equilibrium

To prove global stability of the disease-free equilibrium $\mathbf{E}_0^*$ in (18), we firstly reduce parameter by setting $\Lambda = \mu S_0^*$. Then, substitute Λ into (3)-(5) and rearrange system. We obtain the reduced system as

$$\begin{aligned} \frac{dS}{dt} &= -\mu(S - S_0^*) - g(I)IS + \delta R, \\ \frac{dI}{dt} &= g(I)IS - (\mu + \gamma)I, \\ \frac{dR}{dt} &= \gamma I - (\mu + \delta)R. \end{aligned} \qquad (35)$$

Theorem 5.1: Consider the function

$$V(S, I, R) = S - S^* - S^* \ln\left(\frac{S}{S^*}\right) + I + mR, \qquad (36)$$

There exists a positive value of a constant m for which the function $V(S, I, R)$ is a Lyapunov function with unique minimum at $(S, I, R) = (S_0^*, I_0^*, R_0^*)$. The disease-free equilibrium $\mathbf{E}_0^*$ of (3)-(5) is therefore globally asymptotically stable.

Proof: It is clear that for all $m > 0$, the function $V(S, I, R)$ satisfies two conditions of the Lyapunov function, i.e.,

1) function V is a continuous function.

2) function $V \geq 0$ for all $t \geq 0$ with the unique minimum value $V(S_0^*, I_0^*, R_0^*) = 0$ at the disease-free equilibrium $\mathbf{E}_0^*$, because all variables are nonnegative and all parameters are positive.

Next, we will prove that V satisfies the third condition of the Lyapunov theorem by proving that $\frac{dV}{dt} \leq 0$ for all $t \geq 0$ for a selected positive value of the constant m.

The derivative of V along the solutions of (36) is

$$\begin{aligned} \frac{dV}{dt} &= \left(1 - \frac{S_0^*}{S}\right)\frac{dS}{dt} + \frac{dI}{dt} + m\frac{dR}{dt} \\ &= -\frac{\mu}{S}(S - S_0^*)^2 - g(I)I(S - S_0^*) + \delta\left(1 - \frac{S_0^*}{S}\right)R \\ &\quad +g(I)IS - (\mu + \gamma)I + m\gamma I - m(\mu + \delta)R \\ &= -\frac{\mu}{S}(S - S_0^*)^2 + \delta\left(1 - \frac{S_0^*}{S}\right)R - m(\mu + \delta)R \\ &\quad +I[-(\mu + \gamma) + m\gamma + g(I)S_0^*]. \end{aligned}$$

Because $I \geq 0$ and $g(I) \leq g_0$, we have that

$$\begin{aligned} \frac{dV}{dt} &\leq -\frac{\mu}{S}(S - S_0^*)^2 + \delta\left(1 - \frac{S_0^*}{S}\right)R - m(\mu + \delta)R \\ &\quad -I[\mu + \gamma - m\gamma - g_0 S_0^*]. \end{aligned}$$

Substitute $S_0^* = \frac{\Lambda}{\mu}$ and $\mathcal{R}_0 = \frac{g_0 \Lambda}{\mu(\mu + \gamma)}$, then we obtain that

$$\begin{aligned} \frac{dV}{dt} &\leq -\frac{\mu}{S}(S - S_0^*)^2 + \delta\left(1 - \frac{S_0^*}{S}\right)R - m(\mu + \delta)R \\ &\quad -I(-m\gamma + (\mu + \gamma)(1 - \mathcal{R}_0)). \end{aligned} \qquad (37)$$

For $\mathcal{R}_0 < 1$, let $m = \frac{(\mu + \gamma)(1 - \mathcal{R}_0)}{\gamma} > 0$. Because $S \leq S_0^*$ and $R \geq 0$, the inequality (37) becomes

$$\begin{aligned} \frac{dV}{dt} &\leq -\frac{\mu}{S}(S - S_0^*)^2 + \delta\left(1 - \frac{S_0^*}{S}\right)R \\ &\quad -\frac{(\mu + \gamma)(\mu + \delta)}{\gamma}(1 - \mathcal{R}_0)R \\ &\leq 0, \end{aligned} \qquad (38)$$

This satisfies the third condition of the Lyapunov function, therefore the disease-free equilibrium $\mathbf{E}_0^*$ is globally stable.

We also note that $\frac{dV}{dt} = 0$ only at $\mathbf{E}_0^*$, i.e., at $S = S_0^*, I = 0$ and $R = 0$. Therefore $\mathbf{E}_0^*$ is globally asymptotically stable. The proof is complete. ∎

B. Global stability analysis of the endemic equilibrium

Let $\mathbf{E}_+^* = (S_+^*, I_+^*, R_+^*)$ be the endemic equilibrium defined in (20). From Theorem 3.1, the endemic equilibrium of (3)-(5) exists and it is unique if all conditions in Assumption 1 holds. From (13)-(15), we can reduce parameters in term of the endemic equilibrium as

$$\mu + \gamma = g(I_+^*)S_+^*, \qquad \mu + \delta = \frac{\gamma I_+^*}{R_+^*}$$
$$\Lambda = g(I_+^*)S_+^* I_+^* + \mu S_+^* - \delta R_+^* \quad \text{or} \quad \Lambda = \mu(S_+^* + I_+^* + R_+^*). \qquad (39)$$

Substitute (39) into (3)-(5) and rearrange the system, then we obtain that

$$\begin{aligned} \frac{dS}{dt} &= -(\mu + g(I)I)(S - S_+^*) + \delta(R - R_+^*) \\ &\quad -(g(I)I - g(I_+^*)I_+^*)S_+^*, \\ \frac{dI}{dt} &= I[(g(I) - g(I_+^*))S + g(I_+^*)(S - S_+^*)] \\ \frac{dR}{dt} &= \gamma(I - I_+^*) - \frac{\gamma I_+^*}{R_+^*}(R - R_+^*). \end{aligned} \qquad (40)$$

It follows from (40) that

$$\frac{dS}{dt} + \frac{dI}{dt} = -\mu(S - S_+^* + I - I_+^*) + \delta(R - R_+^*) - \gamma(I - I_+^*)$$

$$\frac{(I - I_+^*)}{I} \frac{dI}{dt} = \frac{(g(I) - g(I_+^*))}{I - I_+^*} S(I - I_+^*)^2$$
$$+ g(I_+^*)(S - S_+^*)(I - I_+^*). \quad (41)$$

Theorem 5.2: Consider the function

$$V(S, I, R) = \frac{1}{2}\left(S - S_+^* + I - I_+^* + R - R_+^*\right)^2$$
$$+ m_1(S - S_+^* + I - I_+^*)^2 + m_2(R - R_+^*)^2$$
$$+ 2m_3\left(I - I_+^* - I_+^* \ln \frac{I}{I_+^*}\right). \quad (42)$$

There exist positive values of the constants m_1, m_2 and m_3 for which the function $V(S, I, R)$ is a Lyapunov function with unique minimum at $(S, I, R) = (S_+^*, I_+^*, R_+^*)$ and the endemic equilibrium $\mathbf{E}_+^*$ of (3)-(5) is globally asymptotically stable.

Proof: We have already proved in Theorems 2.1 and 2.2 that all populations are positive and bounded above if $\mathcal{R}_0 > 1$. Now, we will prove that function V satisfies two conditions of the Lyapunov function, i.e.,

1) function V is clearly a continuous function,
2) function $V \geq 0$ for all $t \geq 0$ with unique minimum value $V(S_{*+}, I_{*+}, R_{*+}) = 0$ at the endemic equilibrium $\mathbf{E}_+^*$ because all variables are nonnegative and all parameters are positive.

Next, we will prove the third condition of the Lyapunov function, i.e. the derivative $\frac{dV}{dt} \leq 0$ along the solutions of (40) to find the values of the constants m_1, m_2 and m_3.

The derivative of V along the solutions of (42) is

$$\frac{dV}{dt} = (S - S_+^* + I - I_+^* + R - R_+^*)\left(\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}\right)$$
$$+ 2m_1(S - S_+^* + I - I_+^*)\left(\frac{dS}{dt} + \frac{dI}{dt}\right) + 2m_2(R - R_+^*)\frac{dR}{dt}$$
$$+ 2m_3\frac{(I - I_+^*)}{I}\frac{dI}{dt}$$
$$= -\mu(S - S_+^* + I - I_+^* + R - R_+^*)^2 - 2\mu m_1(S - S_+^* + I - I_+^*)^2$$
$$+ 2m_1(S - S_+^* + I - I_+^*)\{\delta(R - R_+^*) - \gamma(I - I_+^*)\}$$
$$+ 2\gamma m_2(I - I_+^*)(R - R_+^*) - 2m_2\frac{\gamma I_+^*}{R_+^*}(R - R_+^*)^2$$
$$+ 2m_3\frac{(g(I) - g(I_+^*))}{I - I_+^*} S(I - I_+^*)^2 + 2m_3 g(I_+^*)(S - S_+^*)(I - I_+^*)$$
$$= -(\mu + 2\mu m_1)(S - S_+^*)^2 - \left(\mu + 2m_2\frac{\gamma I_+^*}{R_+^*}\right)(R - R_+^*)^2$$
$$- (\mu + 2\gamma m_1 + 2\mu m_1)(I - I_+^*)^2$$
$$+ 2(\delta m_1 - \mu)(S - S_+^*)(R - R_+^*)$$
$$- 2\left(\mu + \gamma m_1 + 2\mu m_1 - m_3 g(I_+^*)\right)(S - S_+^*)(I - I_+^*)$$
$$+ 2(\gamma m_2 + \delta m_1 - \mu)(I - I_+^*)(R - R_+^*)$$
$$+ 2m_3\frac{(g(I) - g(I_+^*))}{I - I_+^*} S(I - I_+^*)^2.$$

Let $m_1 = \frac{\mu}{\delta} > 0$ and $m_3 = \frac{\mu + (\gamma + 2\mu)m_1}{g(I_+^*)} > 0$. It can be seen that $2(m_1\delta - \mu)(S - S_+^*)(R - R_+^*) = 0$ and $2\left(\mu + \gamma m_1 + 2\mu m_1 - m_3 g(I_+^*)\right)(S - S_+^*)(I - I_+^*) = 0$.

Then, we obtain that

$$\frac{dV}{dt} = -(\mu + 2\mu m_1)(S - S_+^*)^2 - (\mu + 2\gamma m_1)(I - I_+^*)^2$$
$$- \mu(R - R_+^*)^2 + 2m_3\frac{(g(I) - g(I_+^*))}{I - I_+^*} S(I - I_+^*)^2$$
$$- 2\gamma m_2\left(\frac{I_+^*}{R_+^*} - \frac{\gamma m_2}{4\mu m_1}\right)(R - R_+^*)^2$$
$$- 2\mu m_1\left\{I - I_+^* - \frac{\gamma m_2}{2\mu m_1}(R - R_+^*)\right\}^2.$$

Setting $m_2 = \frac{4\mu m_1 I_+^*}{\gamma R_+^*}$, then we have

$$2\gamma m_2\left(\frac{I_+^*}{R_+^*} - \frac{\gamma m_2}{4\mu m_1}\right)(R - R_+^*)^2 = 0.$$

From Assumption $(A3)$, the function $g(I)$ is nonincreasing for $I \geq 0$. Therefore,

$$\text{if} \quad I \leq I_+^*, \quad \frac{g(I) - g(I_+^*)}{I - I_+^*} \leq 0$$

and $\quad$ if $\quad I > I_+^*, \quad \dfrac{g(I) - g(I_+^*)}{I - I_+^*} < 0,$

then

$$2m_3\frac{(g(I) - g(I_+^*))}{I - I_+^*} S(I - I_+^*)^2 \leq 0.$$

We then have that

$$\frac{dV}{dt} = -(\mu + 2\mu m_1)(S - S_+^*)^2 - (\mu + 2\gamma m_1)(I - I_+^*)^2$$
$$- \mu(R - R_+^*)^2 - 2\mu m_1\left\{I - I_+^* - \frac{\gamma \mu_2}{2\mu m_1}(R - R_+^*)\right\}^2$$
$$+ 2m_3\frac{(g(I) - g(I_+^*))}{I - I_+^*} S(I - I_+^*)^2$$
$$\leq 0. \quad (43)$$

Clearly, we have $\frac{dV}{dt} = 0$, only if $S = S_+^*$, $I = I_+^*$ and $R = R_+^*$.

Now, we have already proved that $V(S, I, R)$ is a Lyapunov function and then the endemic equilibrium is globally asymptotically stable. ∎

VI. Numerical Results

In general, there are many different incidence rate functions $G(I)S = g(I)IS$. In this section, we will show some numerical results of the solutions for system (1) with three examples of $g(I)IS$ as follows

$$\textit{Bilinear function} \qquad : g(I)IS = \beta IS,$$
$$\textit{Saturated function} \qquad : g(I)IS = \frac{\beta I}{(\alpha + I)}S,$$
$$\textit{Media coverage function} : g(I)IS = \left(\beta_1 - \frac{\beta_2 I}{(\alpha + I)}\right)IS. \quad (44)$$

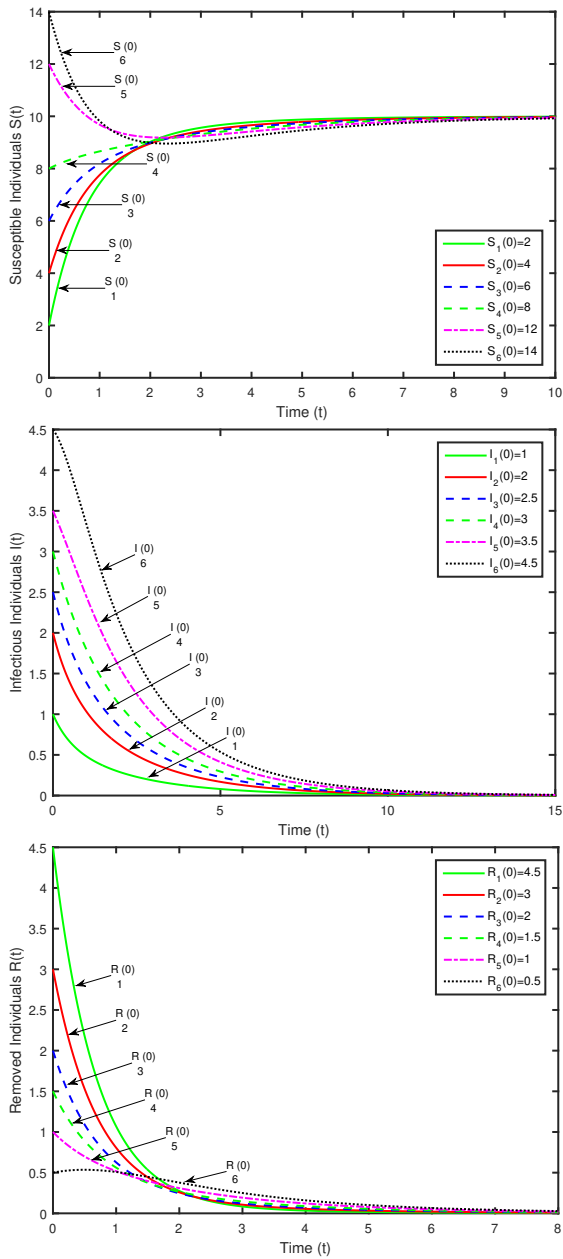It is clear that all functions defined in (44) satisfy all conditions in Assumption 1.

Fig. 2.   All solutions of (45) converge to $\mathbf{E}_0^* = (10, 0, 0)$.
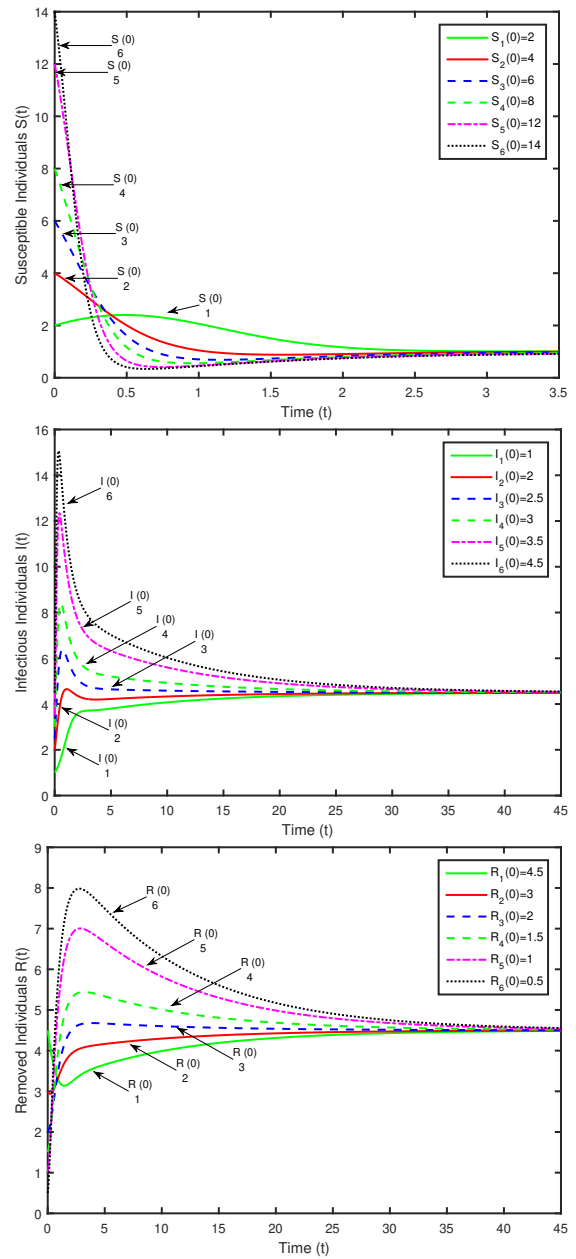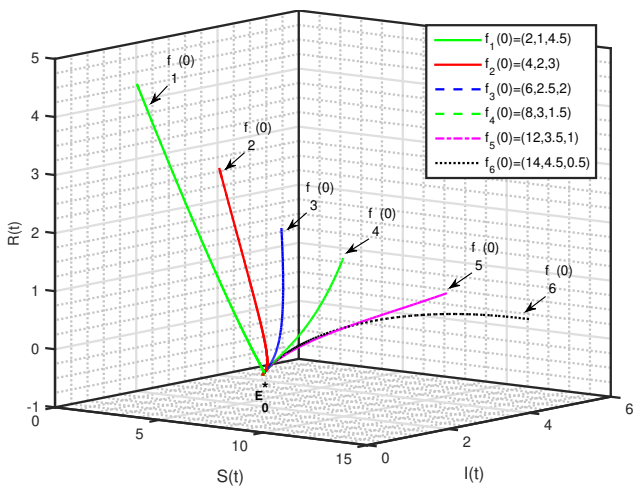

Fig. 4.   All solutions of (45) converge to $\mathbf{E}_+^* = (1, 4.5, 4.5)$.

Fig. 3.   All trajectories of (45) for different initial conditions in (46), when $\mathcal{R}_0 = 0.6364 < 1$, converge to $\mathbf{E}_0^* = (10, 0, 0)$.
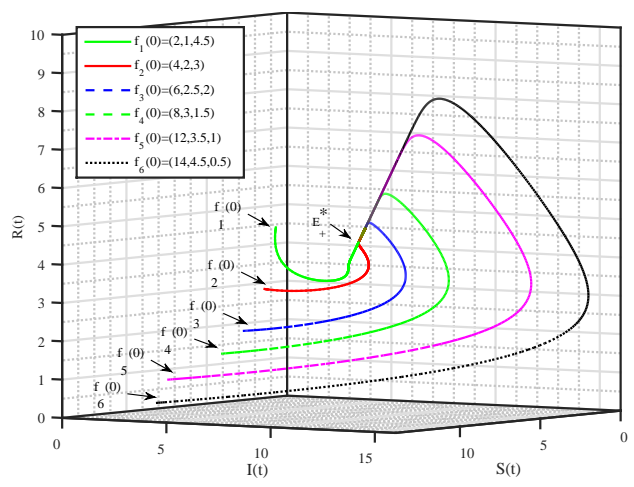

Fig. 5.   All trajectories of (45) for different initial conditions in (46), when $\mathcal{R}_0 = 10 > 1$, converge to $\mathbf{E}_+^* = (1, 4.5, 4.5)$.

A. *The bilinear incidence function*

For the SIRS model (1) with the bilinear incidence function $G(I)S = g(I)IS = \beta IS$, where β is the transmission rate. The model (1) becomes

$$\frac{dS}{dt} = \Lambda - \beta IS - \mu S + \delta R,$$
$$\frac{dI}{dt} = \beta IS - (\mu + \gamma)I, \qquad (45)$$
$$\frac{dR}{dt} = \gamma I - (\mu + \delta)R.$$

TABLE III
PARAMETERS FOR THE SIRS MODEL WITH THE BILINEAR INCIDENCE FUNCTION

Parameter	Values used		Unit	Reference
	DFE	EE		
Λ	9	1	day^{-1}	Estimated
μ	0.9	0.1	day^{-1}	[20]
δ	0.6	0.5	day^{-1}	[20]
γ	0.2	0.6	day^{-1}	[20]
β	0.07	0.7	day^{-1}	Estimated

We apply the parameter values in Table III to examine dynamical behaviors of the disease-free equilibrium and the endemic equilibrium of (45)
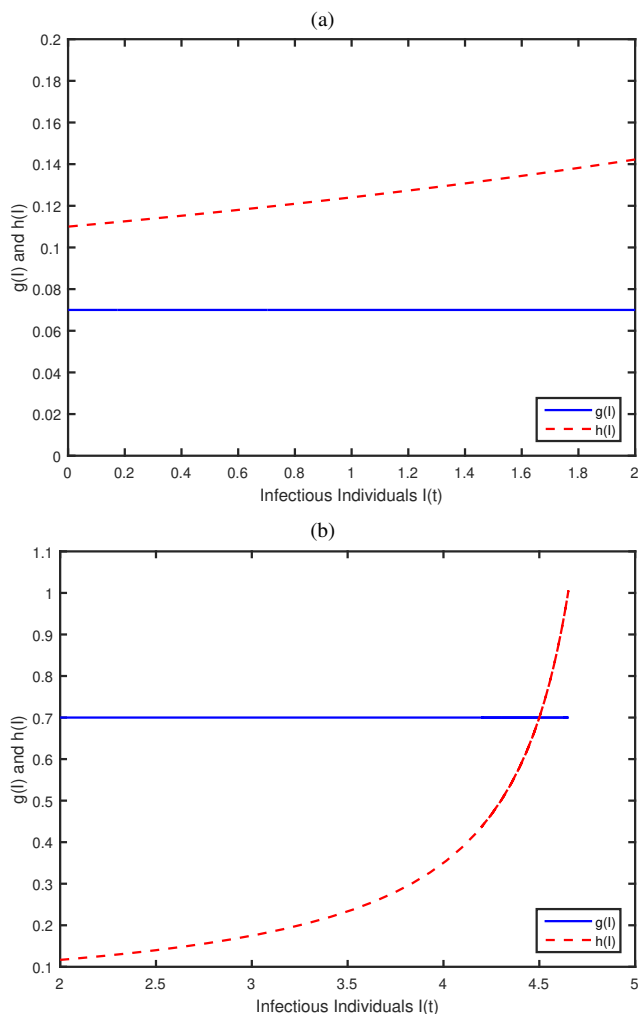

Fig. 6. Two possible cases of the intersection of the curves, $g(I)$ and $h(I)$ to show the existence of positive equilibria : (a) for $\mathbf{E}_0^*$, $\mathcal{R}_0 = 0.6364 < 1$, (b) for $\mathbf{E}_+^*$, $\mathcal{R}_0 = 10 > 1$.

Substitute all parameters from Table III into (21), we obtain that functions $g(I)$ and $h(I)$ are depended on the infected population I. Fig. 6 shows two possible cases of the intersection of the curves $g(I)$ and $h(I)$. It is obvious to see that $h(I)$ is a increasing function and $g(I)$ is non-increasing function. Fig 6(a) shows that two curves do not intersect with initial condition $g^0 < h^0$ when $\mathcal{R}_0 = 0.6364 < 1$. It implies that (3)-(5) has no endemic equilibrium. Fig 6(b) shows that two curves intersect at only one positive values in $[0, \frac{\Lambda}{\mu}]$ with initial condition $g^0 > h^0$ when $\mathcal{R}_0 = 10 > 1$. It implies that (3)-(5) has unique endemic equilibrium.

Next, we show the numerical simulations of (45) using the parameter values given in Table III. Here, we obtain that the disease-free equilibrium is $\mathbf{E}_0^* = (10, 0, 0)$ and the basic reproduction number is $\mathcal{R}_0 = 0.6364 < 1$. The eigenvalues of the characteristic equation at $\mathbf{E}_0^*$ are

$$\lambda_1 = -0.9 < 0, \qquad \lambda_2 = -1.5 < 0, \qquad \lambda_3 = -0.4 < 0.$$

It can be seen that all eigenvalues are negative. Therefore $\mathbf{E}_0^*$ is asymptotically stable.

According to the parameter values in Table III, the endemic equilibrium is $\mathbf{E}_+^* = (1, 4.5, 4.5)$, and the basic reproduction number is $\mathcal{R}_0 = 10 > 1$. The eigenvalues are

$$\lambda_1 = -0.1000 < 0, \qquad \lambda_2 = -1.8750 - 0.5142i,$$
$$\lambda_3 = -1.8750 + 0.5142i.$$

It can be seen that all eigenvalues have negative real parts. Therefore $\mathbf{E}_+^*$ is asymptotically stable.

The numerical solutions of the SIRS model of (45) with different initial conditions in (46),

$$
\begin{aligned}
f_1(0) &= (S_1(0), I_1(0), R_1(0)) = (2, 1, 4.5), \\
f_2(0) &= (S_2(0), I_2(0), R_2(0)) = (4, 2, 3), \\
f_3(0) &= (S_3(0), I_3(0), R_3(0)) = (6, 2.5, 2), \qquad (46) \\
f_4(0) &= (S_4(0), I_4(0), R_4(0)) = (8, 3, 1.5), \\
f_5(0) &= (S_5(0), I_5(0), R_5(0)) = (12, 3.5, 1), \\
f_6(0) &= (S_6(0), I_6(0), R_6(0)) = (14, 4.5, 0.5),
\end{aligned}
$$

are shown in Fig.2 to Fig.5. It can be seen in Fig. 2 that all solutions converge to the disease-free equilibrium $\mathbf{E}_0^* = (10, 0, 0)$. Moreover, Fig. 4 shows that all solutions converge to the endemic equilibrium $\mathbf{E}_+^* = (1, 4.5, 4.5)$ with different initial conditions in (46). Fig.3 and Fig.5 shows the relations between $S(t)$, $I(t)$ and $R(t)$ with different initial conditions in (46). In Fig.3 if $\mathcal{R}_0 = 0.6364 < 1$, then all solutions $(S(t), I(t), R(t))$ converge to the disease-free equilibrium $\mathbf{E}_0^* = (10, 0, 0)$. On the other hand, in Fig.5 if $\mathcal{R}_0 = 10 > 1$, then all solutions $(S(t), I(t), R(t))$ converge to the endemic equilibrium $\mathbf{E}_+^* = (1, 4.5, 4.5)$. The results suggest that under the appropriate conditions and the parameter values provided in Table III, if $\mathcal{R}_0 < 1$, then the disease-free equilibrium $\mathbf{E}_0^*$ is globally asymptotically stable. However if $\mathcal{R}_0 > 1$, then the endemic equilibrium $\mathbf{E}_+^*$ becomes a globally asymptotic stability.

B. *The saturated incidence function*

For the SIRS model (1) with the saturated incidence function $G(I) = g(I)IS = \beta I/(\alpha + I)S$, where β is the proportionality constant and α is the parameter affected by the psychological or inhibitory effect. Replacing $G(I)S$ as
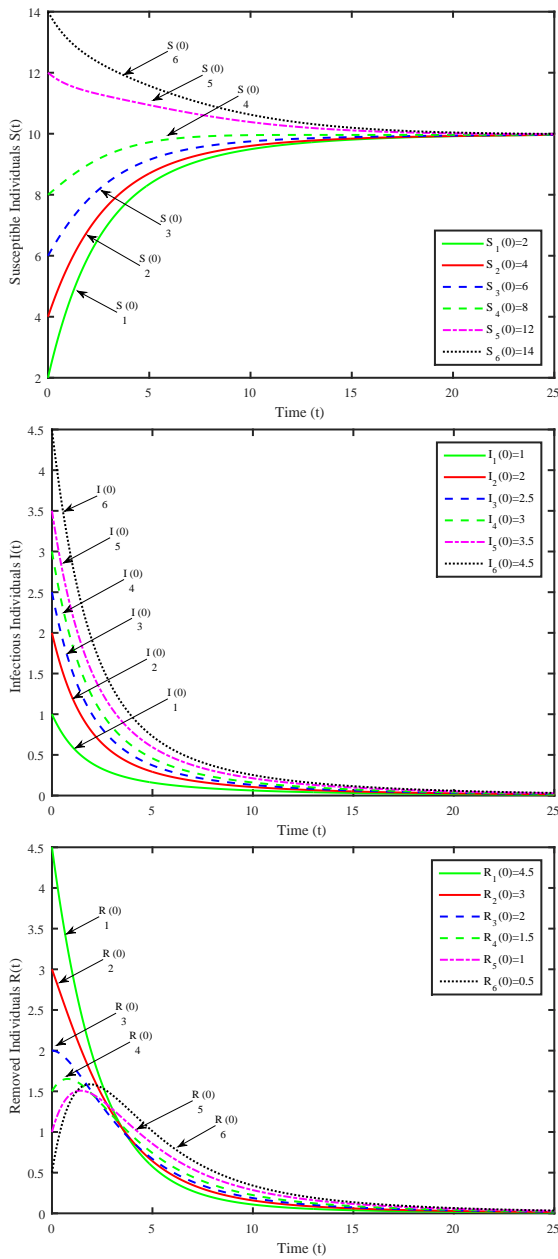
Fig. 7. All solutions of (47) converge to $\mathbf{E}_0^* = (10, 0, 0)$.
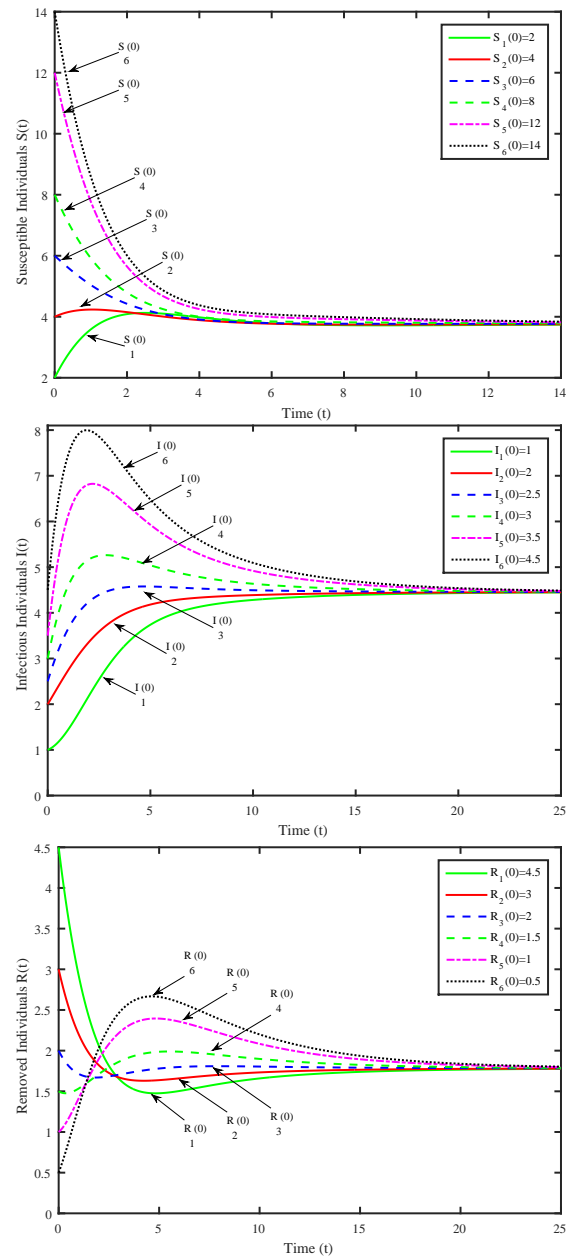

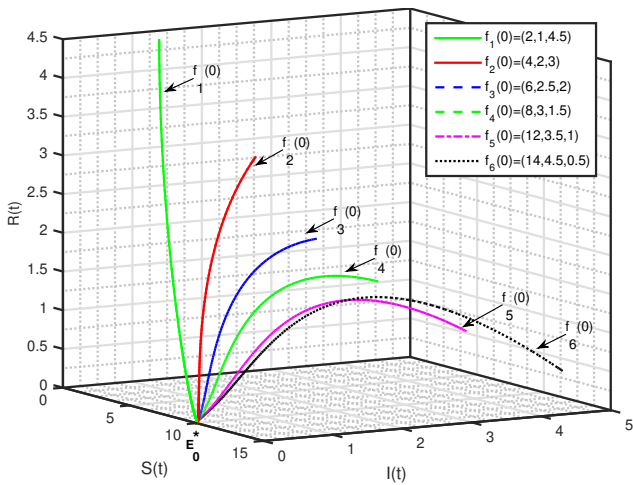Fig. 9. All solutions of (47) converge to $\mathbf{E}_+^* = (3.7624, 4.4554, 1.7822)$.

Fig. 8. All trajectories of (47) for different initial conditions in (46), when $\mathcal{R}_0 = 0.833333 < 1$, converge to $\mathbf{E}_0^* = (10, 0, 0)$.
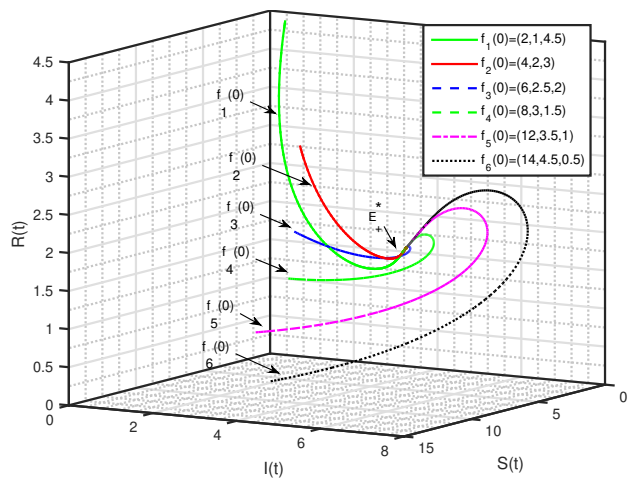

Fig. 10. All trajectories of (47) for different initial conditions in (46), when $\mathcal{R}_0 = 14.5 > 1$, converge to $\mathbf{E}_+^* = (3.7624, 4.4554, 1.7822)$.

the saturated incidence function, model (1) becomes

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta I}{(\alpha + I)} S - \mu S + \delta R, \\
\frac{dI}{dt} &= \frac{\beta I}{(\alpha + I)} S - (\mu + \gamma) I, \qquad (47) \\
\frac{dR}{dt} &= \gamma I - (\mu + \delta) R.
\end{aligned}
$$

We illustrate solutions of (47) with the parameter values provided in Table IV to examine dynamical behaviors of both disease-free and endemic equilibria of (47).

TABLE IV
PARAMETERS FOR THE SIRS MODEL WITH THE SATURATED INCIDENCE
FUNCTION

Parameter	Values used		Unit	Reference
	DFE	EE		
Λ	2	2	day^{-1}	Estimated
μ	0.2	0.2	day^{-1}	[21]
δ	0.3	0.3	day^{-1}	[21]
γ	0.4	0.2	day^{-1}	Estimated
β	0.05	0.58	day^{-1}	[21]
α	1	1	day^{-1}	[21]

In this case, we obtain that the disease-free equilibrium $\mathbf{E}_0^* = (10, 0, 0)$ and the basic reproduction number is $\mathcal{R}_0 = 0.833333 < 1$. The eigenvalues of the characteristic equation at $\mathbf{E}_0^*$ are

$$\lambda_1 = -0.2 < 0, \qquad \lambda_2 = -0.1 < 0, \qquad \lambda_3 = -0.5 < 0.$$

It can be seen that all eigenvalues are negative. Therefore $\mathbf{E}_0^*$ is asymptotically stable. For the endemic equilibrium, $\mathbf{E}_+^* = (3.7624, 4.4554, 1.7822)$, the basic reproduction number is $\mathcal{R}_0 = 14.5 > 1$. The eigenvalues are

$$
\begin{aligned}
\lambda_1 &= -0.6502 + 0.2687i, \qquad \lambda_2 = -0.1999 < 0, \\
\lambda_3 &= -0.6502 - 0.2687i.
\end{aligned}
$$

It can be seen that all eigenvalues have negative real parts. Therefore $\mathbf{E}_+^*$ is asymptotically stable.

The numerical simulations of (47) are shown from Fig. 7 to Fig. 10. In Fig. 7, with different initial conditions in (46), all numerical solutions converge to $\mathbf{E}_0^* = (10, 0, 0)$. Fig. 8 shows the trajectories of the solutions, when $\mathcal{R}_0 < 1$. It can be seen that all solutions $(S(t), I(t), R(t))$ converge to $\mathbf{E}_0^*$. In addition, Fig. 9 shows that all numerical solutions converge to the endemic equilibrium $\mathbf{E}_+^* = (3.7624, 4.4554, 1.7822)$. Note that, from Fig. 10, all trajectories converge to $\mathbf{E}_+^*$. The result suggests that each equilibrium is globally asymptotically stable under the appropriate conditions of parameters.

C. Media coverage

For the SIRS model (1), we investigate effects of media coverage on the transmission incidence function $g(I)IS = \left(\beta_1 - \frac{\beta_2 I}{\alpha + I} \right) IS$, where $\beta_1 > 0$ is the contact rate before media alerts and the term $\beta_2 I / (\alpha + I)$ measure the effect of reduction of the contact rate when infectious individuals are reported in the media. Because the coverage report cannot prevent disease from spreading completely, we

have $\beta_1 \geq \beta_2 > 0$ [15]. Model (1) becomes

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \left(\beta_1 - \frac{\beta_2 I}{\alpha + I} \right) IS - \mu S + \delta R, \\
\frac{dI}{dt} &= \left(\beta_1 - \frac{\beta_2 I}{\alpha + I} \right) IS - (\mu + \gamma) I, \qquad (48) \\
\frac{dR}{dt} &= \gamma I - (\mu + \delta) R.
\end{aligned}
$$

Use the parameter values in Table V, the disease-free equilibrium is $\mathbf{E}_0^* = (10, 0, 0)$ and the endemic equilibrium is $\mathbf{E}_+^* = (4.5913, 3.4419, 1.9668)$.

TABLE V
PARAMETERS FOR THE SIRS MODEL WITH MEDIA COVERAGE
INCIDENCE FUNCTION

Parameter	Values used		Unit	Reference
	DFE	EE		
Λ	2	2.5	day^{-1}	Estimated
μ	0.2	0.25	day^{-1}	[22]
δ	0.2	0.1	day^{-1}	[22]
γ	0.1	0.2	day^{-1}	[22]
β_1	0.02	0.16	day^{-1}	Estimated
β_2	0.008	0.08	day^{-1}	Estimated
α	1	1	day^{-1}	[22]

The numerical solutions of (48) for the disease-free equilibrium $\mathbf{E}_0^* = (10, 0, 0)$, with the basic reproduction number $\mathcal{R}_0 = 0.6667 < 1$. The eigenvalues of the characteristic equation at $\mathbf{E}_0^*$ are

$$\lambda_1 = -0.4 < 0, \qquad \lambda_2 = -0.2 < 0, \qquad \lambda_3 = -0.1 < 0.$$

It can be seen that all eigenvalues are negative. Therefore $\mathbf{E}_0^*$ is asymptotically stable.

On the other hand, for the endemic equilibrium $\mathbf{E}_+^* = (4.5913, 3.4419, 1.9668)$, the basic reproduction number $\mathcal{R}_0 = 3.5556 > 1$. The eigenvalues are

$$
\begin{aligned}
\lambda_1 &= -0.3757 + 0.2585i, \qquad \lambda_2 = -0.2499 < 0, \\
\lambda_3 &= -0.3757 - 0.2585i.
\end{aligned}
$$

It can be seen that all eigenvalues have negative real parts. Therefore $\mathbf{E}_+^*$ is asymptotically stable.

The numerical solutions of the SIRS model (48) which are shown in Fig. 11 to Fig. 14. For the disease-free equilibrium, Fig. 11 and Fig. 12 show that, with different initial conditions in (46), all numerical solutions converge to the disease-free equilibrium $\mathbf{E}_0^* = (10, 0, 0)$. On the contrary, Fig. 13 and Fig. 14 show that all numerical solutions converge to the endemic equilibrium $\mathbf{E}_+^* = (4.5913, 3.4419, 1.9668)$. The result suggests that if $\mathcal{R}_0 < 1$, then $\mathbf{E}_0^*$ is asymptotically stable. However if $\mathcal{R}_0 > 1$, then $\mathbf{E}_+^*$ becomes asymptotically stable.

VII. CONCLUSION

In this work, we have developed and studied the SIRS epidemic model (1) with general incidence function $G(I)S$. In the analytical part, we proved the positivity and boundedness of the model's solutions in Theorems 2.1 and 2.2. Next, we computed the basic reproduction number $\mathcal{R}_0$ by using the next generation method. The result in Theorem 3.1 shows conditions for the existence of the model's equilibria and the conditions for biological meaning of each equilibrium. Next, we analyze the conditions of function $g(I)$. If $g(I)$ is a non-increasing function with $\mathcal{R}_0 > 1$, then the model has unique
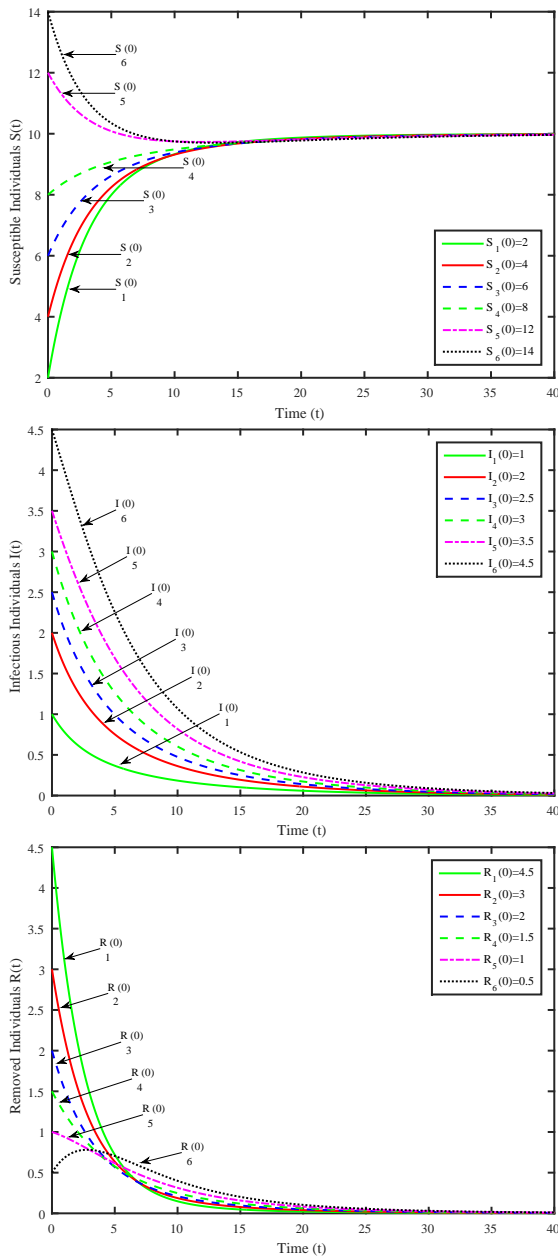
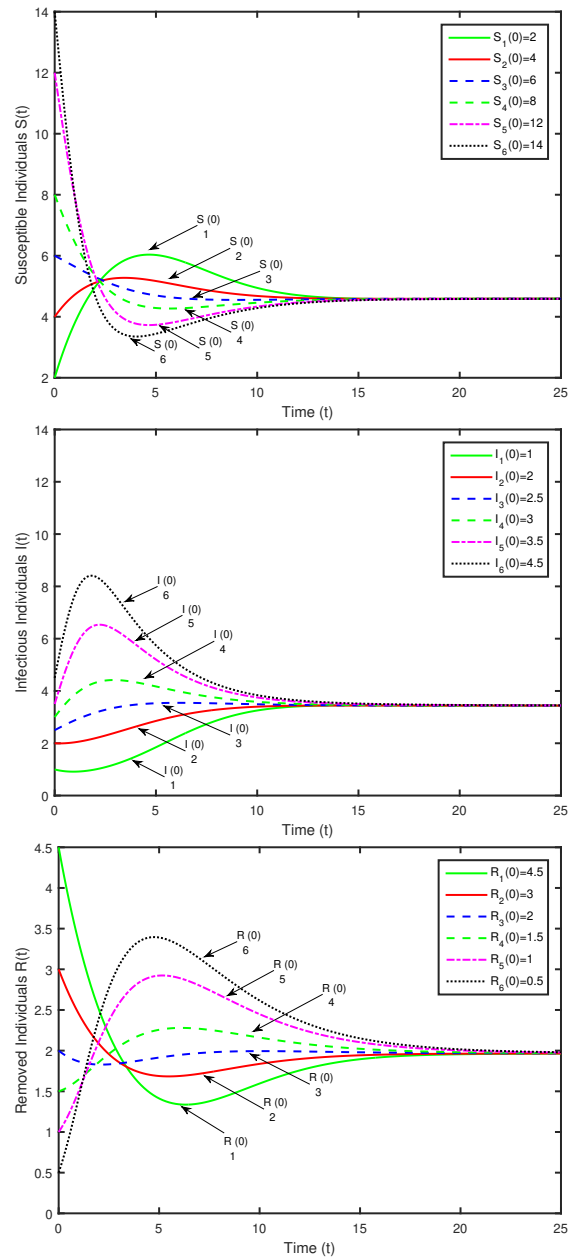Fig. 11. All solutions of (48) converge to $\mathbf{E}_0^* = (10, 0, 0)$.

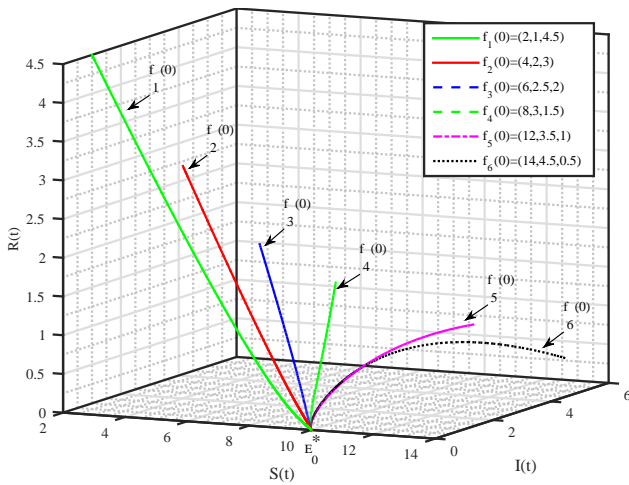Fig. 13. All solutions of (48) converge to $\mathbf{E}_+^* = (4.5913, 3.4419, 1.9668)$.

Fig. 12. A trajectories of (48) for different initial conditions in (46), when $\mathcal{R}_0 = 0.6667 < 1$, converge to $\mathbf{E}_0^* = (10, 0, 0)$.
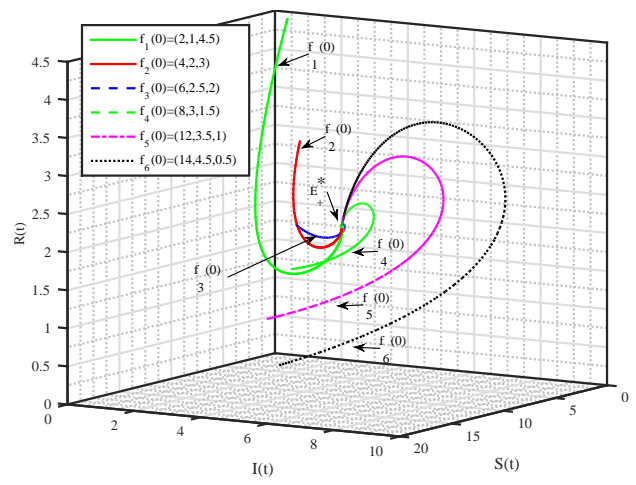

Fig. 14. All trajectories of (48) for different initial conditions in (46), when $\mathcal{R}_0 = 3.5556 > 1$, converge to $\mathbf{E}_+^* = (4.5913, 3.4419, 1.9668)$.

endemic equilibrium. The theorems for locally asymptotic stability of equilibria are stated in Theorem 4.1 and Theorem 4.2. Finally, we constructed the Lyapunov functions to show globally asymptotic stability of all equilibria in Theorem 5.1 and Theorem 5.2.

In the part of numerical simulations, we apply the analytical results to the bilinear, saturated and media coverage incidence functions. For the bilinear function, we present numerical simulations with two sets of parameter values in Table III. The results are shown in Fig.2 to Fig.4. For the saturated function, the numerical results with two different sets of parameter values in Table IV are shown in Fig.7 to Fig.10. Finally, for the media coverage function, we show the numerical results with two sets of parameter values in Table V. The results are shown in Fig.11 and Fig.14. For all classes of the incidence functions, all solutions converge to the equilibrium, which related to the analytical and theoretical results.

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