Ordering q-rung Picture Fuzzy Numbers by Possible Grading Technique and its Utilization in Decision-Making Problem

R. Chitra and K. Prabakaran

***Abstract*—An essential aspect in the decision-making scenario is ordering and ranking $\tilde{q}$-rung picture fuzzy data. This paper presents an approach to order the $\tilde{q}$-rung picture fuzzy numbers through possible grading procedure is presented by extending the method of possibility degree for grading intuitionistic fuzzy numbers. Significant properties of the suggested grading measurement is also analyzed. Also, $\tilde{q}$-rung picture fuzzy ordered Frank weighted arithmetic and geometric accumulation operators have been suggested together with their properties. Further, based on possible grading procedure and those two operators, we devise an algorithm to make decisions. Ultimately, a decision-making problem with several attributes is being discussed to demonstrate the significance of the suggested method. The existing and suggested accumulation operators are compared to expose the pliability and reliability of our approach.**

***Index Terms*—Ordering, Accumulation operators, Possible grading, q-rung picture fuzzy, Several attributes.**

I. Introduction

IT would be hard for decision-makers to convey their opinions to evaluate the alternatives in precise figures due to the complexities of decision-making (DM) problems and the uncertainty in the information. Thus in order to deal with uncertain data, Zadeh [29] introduced the idea of fuzzy set (FS) that figure out the belongingness of an element to a given set with the help of membership degree (grade). Gong [13] established DM procedure on the basis of TODIM (TOmada de Decisao Interativa Multicriterio) and best-worst method in the framework of interval fuzzy sets of type-2 which is an extension of fuzzy set. Later on, the novel concept called intuitionistic fuzzy set (IFS) was developed by Atanassov [3] as an extension of FS. An intuitionistic fuzzy number (IFN) is an ordered pair that represents the element's membership (positive) as well as non-membership (negative) grades respectively. In IFSs, the degree of hesitation is obvious and obtained readily by subtracting the sum of both the membership and non-membership grades from 1. However, in certain circumstances, IFSs are inadequate and inappropriate for expressing the decision-maker's point of view. Therefore, Cuong [4] expanded on the traditional IFS and suggested the notion called picture fuzzy set (PFS), which contains grades of membership, neutral, and non-membership respectively. Furthermore, the notion of neutrosophic set (NSS) was initiated and introduced by Smarandache [24] which contains grades of a truth, an indeterminacy and a falsity respectively. As a result, multi attribute decision-making (MADM) problems based on PFSs and NSSs have emerged as an exciting study area. Garg [11] proposed few accumulation (aggregation) operators for picture fuzzy sets to apply in Multi-Criteria decision-making (MCDM). Jana et al. [15] suggested and used fuzzy Dombi accumulation operators for PFS to choose the most appropriate developing business solution. Qiyas et al. [21] developed Yager accumulation operators for picture fuzzy sets and employed them to make a decision with regard to the selection of emergency program. To resolve MADM issues under neutrosophic environment, Xu et al. [26] utilized enhanced TODIM procedure whereas Liu and Xu [19] employed new TOPSIS (Technique for Order Performance by Similarity to Ideal Solution) procedure.

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However, a picture fuzzy number (PFN) must adhere to the restriction that the total of the positive, neutral, and negative grades of membership cannot be greater than one. Therefore in real-world MADM issues, this limitation is not always met. Suppose the decision-maker (conclusion-maker), for instance, gives the grades of 0.9 for positive, 0.5 for the neutral and 0.3 for negative then their total sum will exceed one. So the assessment value (0.9, 0.5, 0.3) can no longer be expressed as picture fuzzy number. This leads to an idea of $\tilde{q}$-rung picture fuzzy sets ($\tilde{q}$-RPFS) developed by Li et al. [16] to successfully handle such a circumstance, which was prompted by Yager's $\tilde{q}$-rung orthopair fuzzy set ($\tilde{q}$-ROFS) [28]. Since $\tilde{q}$-ROFS meet the requirement that the addition of the $\tilde{q}^{th}$ power of positive and negative grades must lies within or equal to 1, so they are considered to be more helpful for decision-makers to provide the values after assessing the alternatives in MADM issues [17], [18], [1], [7] and [23]. As a result, $\tilde{q}$-RPFSs meet the similar requirement as $\tilde{q}$-ROFSs, i.e., addition of the $\tilde{q}^{th}$ power of positive, neutral and negative grades must lies within or equal to 1. Further, He et al. [14], Akram et al. [2], Chitra & Prabakaran [5] handled MADM issues under $\tilde{q}$-RPF environment.

While solving MADM issue, constructing the suitable function that combines various preference values of a decision-maker into an accumulated value is quite crucial. Then devising suitable function for measuring the combined information so as to find out the best choice. The Frank triangular norm (FT) and Frank triangular conorm (FTC) is a generic t-norm and t-conorm [8]. The benefits of the

FT & FTC is that it provides flexibility in the procedure of data accumulation. Based on FT & FTC Zhang et al. [30], Seikh & Mandal [22] established accumulation operators in the intuitionistic and picture fuzzy environment respectively. Xu and Da [27] developed the possibility degree approach for grading the numbers in the form of interval data. A novel generalised enhanced score function was proposed by Garg [10] to grade the various interval-valued intuitionistic fuzzy numbers. The possibility degree measurement for IFNs was introduced by Wei and Tang [25]. Gao [9] & Dammak et al. [6] given description about the possible measurement of interval-valued intuitionistic fuzzy sets and how it had been used in MCDM situations. The studies mentioned above shows that the researchers had been grading the various intuitionistic fuzzy numbers using various kind of measurements including score, accuracy, and possibility measurements. On the other hand, it is discovered and noted that while comparing two $\tilde{q}$-rung picture fuzzy numbers, we might obtain the same value for existing score and accuracy functions [20]. Since $\tilde{q}$-rung picture fuzzy decision-making relies on the ranking of $\tilde{q}$-rung picture fuzzy sets, so by drawing an inspiration from the enhanced possibility degree procedure for grading the IFNs suggested by Garg and Kumar [12], our current work of possible grading technique therefore helps us to compare and grade $\tilde{q}$-rung picture fuzzy numbers ($\tilde{q}$-RPFNs) that addresses the inadequacies of the existing score function.

The motivation behind this article:

- The paper aims to suggest the ordered frank accumulation operators for $\tilde{q}$-RPFS based on FT and FTC.
- To present the enhanced possible grading procedure to order and grade the $\tilde{q}$-RPFNs appropriately.
- To analyze the properties of enhanced possible grading measurement.
- Novel MADM approach is devised with the help of suggested possible grading method and ordered Frank operators.

The paper is oriented in the manner shown below. Section 2 comprises required definitions and notions. Ordered Frank accumulation operators for $\tilde{q}$-Rung picture fuzzy sets are established under Section 3. Inside Section 4, an enhanced approach for comparing and ordering $\tilde{q}$-RPFNs is discussed in detail. Under section 5, Mathematical formulation is designed to handle MADM issue based on possible grading procedure and operators respectively. The effectiveness of the suggested strategy in solving a MADM issue is demonstrated within Section 6. In Section 7, the influence of the variable $\tilde{q}$ with regard to making a decision is thoroughly examined. The comparison has been made to our suggested operators with contemporary operators in Section 8. Remarks on the conclusion are provided in Section 9.

II. Preliminaries

This section provides a quick overview of certain definitions and operating rules of $\tilde{q}$-RPFNs.

Definition 2.1 : [28]. Let $\tilde{U}$ represents the universe of discourse. The following is a definition of a $\tilde{q}$-rung orthopair fuzzy set τ' defined on $\tilde{U}$:
τ'=$\{(u, \mu_{\tau'}(u), \nu_{\tau'}(u))$: $u \in \tilde{U}\}$, where the positive grade of τ' expressed by $\mu_{\tau'}(u)$ and the negative grade of τ' expressed as $\nu_{\tau'}(u)$ such that it meets the requirement, $0 \leq (\mu_{\tau'}(u))^{\tilde{q}}+(\nu_{\tau'}(u))^{\tilde{q}} \leq 1$, $\tilde{q}$ is a positive integer.
Also the hesitation grade of u in τ' is obtained by, $(1 - (\mu_{\tau'}(u))^{\tilde{q}} - (\nu_{\tau'}(u))^{\tilde{q}})^{1/\tilde{q}}$. Now a single $\tilde{q}$-rung orthopair fuzzy number indicated simply as $\tau' = (\mu_{\tau'}, \nu_{\tau'})$ as mentioned in [17].

Definition 2.2 : [16]. Let $\tilde{U}$ represents the universe of discourse. The following is a definition of a $\tilde{q}$-rung picture fuzzy set τ defined on $\tilde{U}$:
$\tau = \{(u, \mu_\tau(u), \eta_\tau(u), \nu_\tau(u)) : u \in \tilde{U}\}$, where the positive grade of τ expressed by $\mu_\tau(u)$, the neutral grade of τ expressed as $\eta_\tau(u)$ and the negative grade of τ expressed as $\nu_\tau(u)$ such that it meets the requirement, $0 \leq (\mu_\tau(u))^{\tilde{q}} + (\eta_\tau(u))^{\tilde{q}} + (\nu_\tau(u))^{\tilde{q}} \leq 1$, $\tilde{q}$ is a positive integer.
Also the refusal grade of u in τ is obtained by, $(1 - (\mu_\tau(u))^{\tilde{q}} - (\eta_\tau(u))^{\tilde{q}} - (\nu_\tau(u))^{\tilde{q}}))^{1/\tilde{q}}$. Now a single $\tilde{q}$-rung picture fuzzy number indicated simply as $\tau = (\mu_\tau, \eta_\tau, \nu_\tau)$.

Definition 2.3 : [20]. Assume that $\tau = (\mu_\tau, \eta_\tau, \nu_\tau)$ is a $\tilde{q}$-RPFN. Its score (S_τ) and accuracy functions(A_τ) are given below,

$$S_\tau = \mu_\tau^{\tilde{q}} - \eta_\tau^{\tilde{q}} - \nu_\tau^{\tilde{q}} \tag{1}$$

$$A_\tau = \mu_\tau^{\tilde{q}} + \eta_\tau^{\tilde{q}} + \nu_\tau^{\tilde{q}} \tag{2}$$

where $S_\tau \in [-1, 1]$ and $A_\tau \in [0, 1]$.

Let us consider two $\tilde{q}$-RPFNs, $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$. Then out of equations (1) and (2),

1) If $S_{\tau_1} \succ S_{\tau_2}$ then $\tau_1 \succ \tau_2$
2) If $S_{\tau_1} = S_{\tau_2}$ then
   (i) if $A_{\tau_1} \succ A_{\tau_2},\ then\ \tau_1 \succ \tau_2$
   (ii) if $A_{\tau_1} = A_{\tau_2},\ then\ \tau_1 = \tau_2$.

Definition 2.4 : [8]. Suppose g and h be any two real numbers then the Frank triangular norm and Frank triangular conorm between them is described by,

$F(g, h) = log_{\tilde{r}}(1 + \frac{(\tilde{r}^g-1)(\tilde{r}^h-1)}{\tilde{r}-1})$.

$F'(g, h) = 1 - log_{\tilde{r}}(1 + \frac{(\tilde{r}^{1-g}-1)(\tilde{r}^{1-h}-1)}{\tilde{r}-1})$.

where $\tilde{r} \neq 1$ and $(g, h) \in [0, 1] \times [0, 1]$.

A. Operational laws

[17]. Let $\tau = (\mu_\tau, \eta_\tau, \nu_\tau)$, $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ are the three $\tilde{q}$-RPFNs (with $\lambda \succ 0$) agrees the following operational laws:

- $\tau_1 \cup \tau_2 = (\{\mu_{\tau_1} \cup \mu_{\tau_2}\}, \{\eta_{\tau_1} \cap \eta_{\tau_2}\}, \{\nu_{\tau_1} \cap \nu_{\tau_2}\})$.
- $\tau_1 \cap \tau_2 = (\{\mu_{\tau_1} \cap \mu_{\tau_2}\}, \{\eta_{\tau_1} \cup \eta_{\tau_2}\}, \{\nu_{\tau_1} \cup \nu_{\tau_2}\})$.
- $\tau^c = (\nu_\tau, \eta_\tau, \mu_\tau)$.
- $\tau_1 \oplus \tau_2 = ((\mu_{\tau_1}^{\tilde{q}} + \mu_{\tau_2}^{\tilde{q}} - \mu_{\tau_1}^{\tilde{q}}\mu_{\tau_2}^{\tilde{q}})^{1/\tilde{q}}, \eta_{\tau_1}\eta_{\tau_2}, \nu_{\tau_1}\nu_{\tau_2})$.

- $\tau_1 \otimes \tau_2 = (\mu_{\tau_1}\mu_{\tau_2}, (\eta^{\tilde{q}}_{\tau_1} + \eta^{\tilde{q}}_{\tau_2} - \eta^{\tilde{q}}_{\tau_1}\eta^{\tilde{q}}_{\tau_2})^{1/\tilde{q}},$ $(\nu^{\tilde{q}}_{\tau_1} + \nu^{\tilde{q}}_{\tau_2} - \nu^{\tilde{q}}_{\tau_1}\nu^{\tilde{q}}_{\tau_2})^{1/\tilde{q}})$.
- $\tilde{\lambda}\tau = ((1-(1-\mu^{\tilde{q}}_{\tau})^{\tilde{\lambda}})^{1/\tilde{q}}, \eta^{\tilde{\lambda}}_{\tau}, \nu^{\tilde{\lambda}}_{\tau})$.
- $\tau^{\tilde{\lambda}} = (\mu^{\tilde{\lambda}}_{\tau}, (1-(1-\eta^{\tilde{q}}_{\tau})^{\tilde{\lambda}})^{1/\tilde{q}}, (1-(1-\nu^{\tilde{q}}_{\tau})^{\tilde{\lambda}})^{1/\tilde{q}})$.

III. $\tilde{q}$-RUNG PICTURE FUZZY FRANK ACCUMULATION OPERATORS

To combine the $\tilde{q}$-RPF data, Chitra & Prabakaran [5] presented operational rules, weighted arithmetic and geometric operators depending on FT and FTC. Theorems related to those two operators and their properties have already been discussed there. Now, in this section we are going to focus mainly on two functions namely ordered Frank weighted arithmetic and geometric operators for accumulating the $\tilde{q}$-rung picture fuzzy data.

A. $\tilde{q}$-Rung Picture Fuzzy Ordered Frank Weighted Arithmetic($\tilde{q}$-RPFOFWA) Operator

Consider the $\tilde{q}$-RPFN collection, $\tau_z = (\mu_{\tau_z}, \eta_{\tau_z}, \nu_{\tau_z})$ and the weights $\tilde{\omega}_z$ (z varies from 1 to f), satisfying that total sum of weights should be one. Then the operator $\tilde{q}$-*RPFOFWA* : $\tau^f \rightarrow \tau$ is given by,

$$\tilde{q}\text{-}RPFOFWA(\tau_1, \tau_2, ..., \tau_f) = \bigoplus_{z=1}^{f} \tilde{\omega}_z \tau_{\phi(z)}$$
$$= \{(1 - log_{\tilde{r}}(1 + \prod_{z=1}^{f}(\tilde{r}^{1-\mu^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}},$$
$$(log_{\tilde{r}}(1 + \prod_{z=1}^{f}(\tilde{r}^{\eta^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}}, \tag{3}$$
$$(log_{\tilde{r}}(1 + \prod_{z=1}^{f}(\tilde{r}^{\nu^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}}\}$$

B. $\tilde{q}$-Rung Picture Fuzzy Ordered Frank Weighted Geometric($\tilde{q}$-RPFOFWG) Operator

Consider the $\tilde{q}$-RPFN collection, $\tau_z = (\mu_{\tau_z}, \eta_{\tau_z}, \nu_{\tau_z})$ and the weights $\tilde{\omega}_z$ (z varies from 1 to f), satisfies that total sum of weights should be one. Then the operator $\tilde{q}$-*RPFOFWG* : $\tau^f \rightarrow \tau$ is specified as,

$$\tilde{q}\text{-}RPFOFWG(\tau_1, \tau_2, ..., \tau_f) = \bigotimes_{z=1}^{f} (\tau_{\phi(z)})^{\tilde{\omega}_z}$$
$$= \{(log_{\tilde{r}}(1 + \prod_{z=1}^{f}(\tilde{r}^{\mu^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}},$$
$$(1 - log_{\tilde{r}}(1 + \prod_{z=1}^{f}(\tilde{r}^{1-\eta^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}}, \tag{4}$$
$$(1 - log_{\tilde{r}}(1 + \prod_{z=1}^{f}(\tilde{r}^{1-\nu^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}}\}$$

As discussed in [5], theorems related to suggested operators and their properties such as idempotency, boundedness and monotonicity hold for $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators too and can readily be proved.

IV. POSSIBLE GRADING TECHNIQUE

Within this section, we have suggested a technique for ordering $\tilde{q}$-rung picture fuzzy numbers through the possible grading measurement which is an extension of the method of possibility degree to rank intuitionistic fuzzy numbers by [12].

Definition 4.1 : Let us consider two $\tilde{q}$-RPFNs, $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ then the possible grading of $\tau_1 \geq \tau_2$ is denoted as $P^*(\tau_1 \geq \tau_2)$ and it is defined by,

$$P^*(\tau_1 \geq \tau_2) =$$
$$min\left(max\left(\frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}, 0\right), 1\right) \tag{5}$$

In the above equation, any one of the terms in the denominator ρ_{τ_1} or ρ_{τ_2} or η_{τ_1} or η_{τ_2} should not be equal to zero. Alternatively, if $\rho_{\tau_1} = \rho_{\tau_2} = \eta_{\tau_1} = \eta_{\tau_2} = 0$, we denote $P^*(\tau_1 \geq \tau_2)$ as follows:

$$P^*(\tau_1 \geq \tau_2) = \begin{cases} 1 & \text{if } \mu_{\tau_1} > \mu_{\tau_2} \\ 0 & \text{if } \mu_{\tau_1} < \mu_{\tau_2} \\ 0.5 & \text{if } \mu_{\tau_1} = \mu_{\tau_2} \end{cases} \tag{6}$$

Theorem 4.1 : Let $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ be $\tilde{q}$-RPFNs then
(i) $0 \leq P^*(\tau_1 \geq \tau_2) \leq 1$,
(ii) $P^*(\tau_1 \geq \tau_2) = 0.5$ if $\tau_1 = \tau_2$,
(iii) $P^*(\tau_1 \geq \tau_2) + P^*(\tau_2 \geq \tau_1) = 1$.

Proof :
(i)Let us assume $a = \frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}$.

Now the subsequent cases will be occuring:

case 1 : If $a \geq 1$ then

$$\begin{aligned} P^*(\tau_1 \geq \tau_2) &= min(max(a, 0), 1) \\ &= min(a, 1) \\ &= 1. \end{aligned}$$

case 2 : If $0 < a < 1$ then

$$\begin{aligned} P^*(\tau_1 \geq \tau_2) &= min(max(a, 0), 1) \\ &= min(a, 1) \\ &= a. \end{aligned}$$

case 3 : If $a \leq 0$ then

$$\begin{aligned} P^*(\tau_1 \geq \tau_2) &= min(max(a, 0), 1) \\ &= min(0, 1) \\ &= 0. \end{aligned}$$

Therefore for every instances, $0 \leq P^*(\tau_1 \geq \tau_2) \leq 1$ obtained.

(ii) Let $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ be $\tilde{q}$-RPFNs. Suppose, $\tau_1 = \tau_2$ it suggests that $\mu_{\tau_1} = \mu_{\tau_2}$; $\eta_{\tau_1} =$

η_{τ_2} and $\nu_{\tau_1} = \nu_{\tau_2}$. Then eqn.(5) becomes,

$$\begin{aligned}
P^*(\tau_1 \geq \tau_2) &= min\left[max\left[\frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}, 0\right], 1\right] \\
&= min\left[max\left[\frac{\mu_{\tau_1} - \mu_{\tau_1} + \eta_{\tau_1} + \rho_{\tau_1}}{(\eta_{\tau_1} + \eta_{\tau_1}) + (\rho_{\tau_1} + \rho_{\tau_1})}, 0\right], 1\right] \\
&= min\left[max\left[\frac{\eta_{\tau_1} + \rho_{\tau_1}}{2(\eta_{\tau_1} + \rho_{\tau_1})}, 0\right], 1\right] \\
&= min[max[0.5, 0], 1] \\
&= min[0.5, 1]
\end{aligned}$$

$P^*(\tau_1 \geq \tau_2) = 0.5.$

Hence the proof.

(iii) Consider $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ to be two $\tilde{q}$-RPFNs. Let us assume

$$a = \frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})},$$

$$b = \frac{\mu_{\tau_2} - \mu_{\tau_1} + \eta_{\tau_2} + \rho_{\tau_1}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}$$

Thus,

$$\begin{aligned}
a + b &= \frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2} + \mu_{\tau_2} - \mu_{\tau_1} + \eta_{\tau_2} + \rho_{\tau_1}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&= \frac{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&= 1.
\end{aligned}$$

Now the three subsequent cases will be occurring:

case 1 : If $a \leq 0$ and $b \geq 1$ then

$$\begin{aligned}
&P^*(\tau_1 \geq \tau_2) + P^*(\tau_2 \geq \tau_1) \\
&= min(max(a, 0), 1) + min(max(b, 0), 1) \\
&= min(0, 1) + min(b, 1) \\
&= 0 + 1. \\
&= 1.
\end{aligned}$$

case 2 : If $0 < a, b < 1$ then

$$\begin{aligned}
&P^*(\tau_1 \geq \tau_2) + P^*(\tau_2 \geq \tau_1) \\
&= min(max(a, 0), 1) + min(max(b, 0), 1) \\
&= min(a, 1) + min(b, 1) \\
&= a + b \\
&= 1.
\end{aligned}$$

case 3 : If $a \geq 1$ and $b \leq 0$ then

$$\begin{aligned}
&P^*(\tau_1 \geq \tau_2) + P^*(\tau_2 \geq \tau_1) \\
&= min(max(a, 0), 1) + min(max(b, 0), 1) \\
&= min(a, 1) + min(0, 1) \\
&= 1 + 0. \\
&= 1.
\end{aligned}$$

In all the instances, we obtain,
$P^*(\tau_1 \geq \tau_2) + P^*(\tau_2 \geq \tau_1) = 1.$

Theorem 4.2 : Let $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ be $\tilde{q}$-RPFNs then the suggested possible grading measurement $P^*(\tau_1 \geq \tau_2)$ fulfills the required criteria such that $(\tau_1 \neq \tau_2)$.

(i) $P^*(\tau_1 \geq \tau_2) = 1$ if $\mu_{\tau_1} - \rho_{\tau_1} \geq \mu_{\tau_2} + \eta_{\tau_2}$ and
(ii) $P^*(\tau_1 \geq \tau_2) = 0$ if $\mu_{\tau_2} - \rho_{\tau_2} \geq \mu_{\tau_1} + \eta_{\tau_1}$

Proof : Given $\tau_1 = (\mu_{\tau_1}, \eta_{\tau_1}, \nu_{\tau_1})$ and $\tau_2 = (\mu_{\tau_2}, \eta_{\tau_2}, \nu_{\tau_2})$ be two $\tilde{q}$-RPFNs then
(i) If $\mu_{\tau_1} - \rho_{\tau_1} \geq \mu_{\tau_2} + \eta_{\tau_2}$ then

$$\begin{aligned}
&\frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&\geq \frac{\mu_{\tau_2} + \eta_{\tau_2} + \rho_{\tau_1} + \rho_{\tau_2} + \eta_{\tau_1} - \mu_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&\geq \frac{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&= 1.
\end{aligned}$$

Thus, $min\left(max\left(\frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}, 0\right), 1\right) = 1.$

Therefore $P^*(\tau_1 \geq \tau_2) = 1.$

(ii) If $\mu_{\tau_2} - \rho_{\tau_2} \geq \mu_{\tau_1} + \eta_{\tau_1}$ then

$$\begin{aligned}
&\frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&= \frac{\rho_{\tau_2} - \mu_{\tau_2} + \mu_{\tau_1} + \eta_{\tau_1}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&\leq \frac{-(\mu_{\tau_1} + \eta_{\tau_1}) + (\mu_{\tau_1} + \eta_{\tau_1})}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})} \\
&= 0.
\end{aligned}$$

So, $min\left(max\left(\frac{\mu_{\tau_1} - \mu_{\tau_2} + \eta_{\tau_1} + \rho_{\tau_2}}{(\eta_{\tau_1} + \eta_{\tau_2}) + (\rho_{\tau_1} + \rho_{\tau_2})}, 0\right), 1\right) = 0.$

Therefore $P^*(\tau_1 \geq \tau_2) = 0$. Hence the proof.

Thus, in general

$$P^*(\tau_i \geq \tau_j) = \begin{cases} 1 & \text{if } \mu_{\tau_i} - \rho_{\tau_i} \geq \mu_{\tau_j} + \eta_{\tau_j} \\ 0 & \text{if } \mu_{\tau_j} - \rho_{\tau_j} \geq \mu_{\tau_i} + \eta_{\tau_i} \\ 0.5 & \text{if } \tau_i = \tau_j \\ P^*_{ij} & otherwise \end{cases} \tag{7}$$

where, $P^*_{ij} = min\left(max\left(\frac{\mu_{\tau_i} - \mu_{\tau_j} + \eta_{\tau_i} + \rho_{\tau_j}}{(\eta_{\tau_i} + \eta_{\tau_j}) + (\rho_{\tau_i} + \rho_{\tau_j})}, 0\right), 1\right).$

Furthermore, in order to rank the various $\tilde{q}$-RPFNs, the relative possibility of $\tilde{q}$-RPFNs with $(\tau_x \geq \tau_y) \ni x, y \in 1, 2, ..., f$ is given by $p^*(\tau_x \geq \tau_y)$, and whose associated possible grading matrix is represented by
$P^* = (p^*_{xy})_{f \times f}$, where $p^*_{xy} = p^*(\tau_x \geq \tau_y)$, $x, y \in 1, 2, ..., f$ denoted by,

$$P^* = \begin{pmatrix} p^*_{11} & p^*_{12} & . & . & . & p^*_{1f} \\ p^*_{21} & p^*_{22} & . & . & . & p^*_{2f} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ p^*_{f1} & p^*_{f2} & . & . & . & p^*_{ff} \end{pmatrix}$$

Hence for all the $\tilde{q}$-RPFNs $\tau_x (x = 1, 2, ..., f)$, the value of the rank is defined as given in [12],

$$r_x = \frac{1}{f(f-1)}\left(\sum_{y=1}^{f} p^*_{xy} + \frac{f}{2} - 1\right). \tag{8}$$

Consequently the alternative's ranking order will be determined based on the decreasing sequence of r_x values and thereby we will be able to select the optimal alternative.

Example 4.1 : Suppose $\tau_1 = (0.3, 0.2, 0.2)$, $\tau_2 = (0.4, 0.2, 0.1)$, $\tau_3 = (0.6, 0.1, 0.2)$ and $\tau_4 = (0.4, 0.1, 0.2)$ be four $\tilde{q}$-RPFNs and $\tilde{q}$-*RPFOFWA* operator has accompanied by weight vector $\tilde{\omega} = (0.112, 0.304, 0.348, 0.236)$ then these numbers can be ordered and accumulated as follows,

The associated score values of the $\tilde{q}$-RPFNs (with $\tilde{q}$ = 2) are determined using equation (1),
$S_{\tau_1} = 0.01$; $S_{\tau_2} = 0.11$; $S_{\tau_3} = 0.31$ and $S_{\tau_4} = 0.11$. Since, $S_{\tau_2} = S_{\tau_4}$ and their corresponding accuracy values are also same, so we cannot compare which number is bigger between the two, based upon existing score function. But we need to order these $\tilde{q}$-RPFNs in order to apply $\tilde{q}$-*RPFOFWA* operator. So using possible grading procedure we will order them as follows:
From equation (7), we have $P^*(\tau_1 \geq \tau_1)$ = P^*_{11} = 0.5.
Similarly, P^*_{22} = P^*_{33} = P^*_{44} = 0.5.

$$\begin{aligned} P^*(\tau_1 \geq \tau_2) = &\ P^*_{12} \\ = &\ min\left(max\left(\frac{\mu_{\tau_1}-\mu_{\tau_2}+\eta_{\tau_1}+\rho_{\tau_2}}{(\eta_{\tau_1}+\eta_{\tau_2})+(\rho_{\tau_1}+\rho_{\tau_2})}, 0\right), 1\right) \\ = &\ min\left(max\left(\frac{0.3-0.4+0.2+0.3}{1}, 0\right), 1\right) \\ = &\ min(max(0.4, 0), 1) \\ = &\ min(0.4, 1) \\ P^*_{12} = &\ 0.4. \end{aligned}$$

$\Rightarrow$ by condition (iii) of **theorem 4.1**,
P^*_{21} = 1 - P^*_{12} = 0.6.

Now consider,

$$\begin{aligned} P^*(\tau_2 \geq \tau_3) = &\ P^*_{23} \\ = &\ min\left(max\left(\frac{\mu_{\tau_2}-\mu_{\tau_3}+\eta_{\tau_2}+\rho_{\tau_3}}{(\eta_{\tau_2}+\eta_{\tau_3})+(\rho_{\tau_2}+\rho_{\tau_3})}, 0\right), 1\right) \\ = &\ min\left(max\left(\frac{0.4-0.6+0.2+0.1}{0.7}, 0\right), 1\right) \\ = &\ min(max(0.14, 0), 1) \\ = &\ min(0.14, 1) \\ P^*_{23} = &\ 0.14. \end{aligned}$$

$\Rightarrow P^*_{32}$ = 1 - P^*_{23} = 0.86.

In a similar manner we obtain,
P^*_{14} = 0.44 $\Rightarrow P^*_{41}$ = 1 - P^*_{14} = 0.56.

P^*_{24} = 0.56 $\Rightarrow P^*_{42}$ = 1 - P^*_{24} = 0.44.

By condition (ii) of **theorem 4.2** we get,
P^*_{13} = 0 $\Rightarrow P^*_{31}$ = 1 - P^*_{13} = 1.

By condition (i) of **theorem 4.2** we get,
P^*_{34} = 1 $\Rightarrow P^*_{43}$ = 1 - P^*_{34} = 0.

Thus the possible grade measurement matrix will be,

$$P^* = \begin{pmatrix} 0.5 & 0.4 & 0.0 & 0.44 \\ 0.6 & 0.5 & 0.14 & 0.56 \\ 1.0 & 0.86 & 0.5 & 1.0 \\ 0.56 & 0.44 & 0.0 & 0.5 \end{pmatrix}$$

Using equation (8) we can now rank and order the $\tilde{q}$-RPFNs as follows:

$$r_x = \frac{1}{f(f-1)}\left(\sum_{y=1}^{f} p^*_{xy} + \frac{f}{2} - 1\right)$$

since f = 4 we obtain,
r_1 = 0.195; r_2 = 0.2333; r_3 = 0.3633 and r_4 = 0.2083.
$\Rightarrow r_3 > r_2 > r_4 > r_1$

$\Rightarrow P^*(\tau_3) > P^*(\tau_2) > P^*(\tau_4) > P^*(\tau_1)$.

$\Rightarrow \tau_3 > \tau_2 > \tau_4 > \tau_1$.

Hence, $\tau_{\phi(1)} = \tau_3 = (0.6, 0.1, 0.2)$;
$\tau_{\phi(2)} = \tau_2 = (0.4, 0.2, 0.1)$;
$\tau_{\phi(3)} = \tau_4 = (0.4, 0.1, 0.2)$ and
$\tau_{\phi(4)} = \tau_1 = (0.3, 0.2, 0.2)$.

Now, we will be able to combine the ordered $\tilde{q}$-RPFNs by the ordered operator with $\tilde{q}$ = 2 and $\tilde{r}$ = 3.

$$\tilde{q}\text{-}RPFOFWA(\tau_{\phi(1)}, \tau_{\phi(2)}, \tau_{\phi(3)}, \tau_{\phi(4)}) = \bigoplus_{z=1}^{4} \tilde{\omega}_z \tau_{\phi(z)}$$

$$= \{(1 - log_{\tilde{r}}(1 + \prod_{z=1}^{4}(\tilde{r}^{1-\mu^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}},$$

$$(log_{\tilde{r}}(1 + \prod_{z=1}^{4}(\tilde{r}^{\eta^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}},$$

$$(log_{\tilde{r}}(1 + \prod_{z=1}^{4}(\tilde{r}^{\nu^{\tilde{q}}_{\tau_{\phi(z)}}} - 1)^{\tilde{\omega}_z}))^{1/\tilde{q}}\}.$$

= (0.4101,0.1456,0.1627).

V. Mathematical approach to solve Multiple Attribute Decision Making Problems using Possible Grading Technique

Here, we discuss a method for making decisions that uses the suggested operators and possible grading technique to solve the issues with MADM in the circumstance of $\tilde{q}$-RPFNs.

Let $V = \{V_1, V_2, ..., V_k\}$ be a limited set of alternatives (choices). Let $T = \{T_1, T_2, ..., T_l\}$ be a limited set of criteria and whose corresponding weights $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, ..., \tilde{\omega}_l)^T$ satisfies that total sum of weights must be one. The decision matrix D_m, specified by $D_m = (\delta_{ef})_{k\times l} = (\mu_{ef}, \eta_{ef}, \nu_{ef})_{k\times l}$. The decision-maker must use a $\tilde{q}$-RPFN to convey the evaluation value that they believe best represents the situation is specified as, (δ_{ef}) such that $(\mu_{ef})^{\tilde{q}}$ + $(\eta_{ef})^{\tilde{q}}$ + $(\nu_{ef})^{\tilde{q}} \leq 1$ and $\mu_{ef} \in [0, 1]$, $\eta_{ef} \in [0, 1]$ and $\nu_{ef} \in [0, 1]$. Then, on the basis of suggested possible grading procedure, a Mathematical formulation had been designed for decision-making problem with several attributes to obtain the optimal choice(s), is explained below:

Step 1: With respect to conclusion-maker's preference values for the choices, we obtain the decision matrix D_m in terms of $\tilde{q}$-RPF information.

Step 2: If there are distinct attributes in a given DM issue, regarding beneficial $(T_{\tilde{b}})$ and an attribute regarding cost $(T_{\tilde{c}})$, then we can use the condition provided below to convert the given matrix D_m into a normalised $\tilde{q}$-RPF matrix, $D'_m = (\delta'_{ef})_{k\times l}$.

$$\delta'_{ef} = \begin{cases} \delta_{ef} & \text{if } f \in T_{\tilde{c}} \\ {\delta_{ef}}^{c} & \text{if } f \in T_{\tilde{b}} \end{cases} \tag{9}$$

where $(\delta_{ef})^c = (\nu_{ef}, \eta_{ef}, \mu_{ef})$. It should be noted that the above process can be avoided in case of uniform category of attributes.

Step 3: We need to order the $\tilde{q}$-RPFNs for each of the alternatives V_1, V_2, V_3 and V_4 using score function given in equation(1). If at all we are not able to order the $\tilde{q}$-RPFNs using score function then we can readily order them using equations (7) and (8) as given in possible grading technique of section 3.

Step 4: Obtain the ordered normalised $\tilde{q}$-RPF matrix $D'_{\phi(m)}$ at once the ordering of $\tilde{q}$-RPFNs is done. Then utilizing $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators, the accumulated value ζ_e regarding the choice $V_e(e = 1, 2, ..., k)$ can be evaluted with the help of equations (3) & (4).

Step 5: Using (1), the values of score S_{ζ_e} concerning each combined value $\zeta_e(e = 1, 2, ..., k)$ of choices can be obtained respectively.

Step 6: By grading the options in accordance with the descending value of the scores, we can choose which among the options is the most desired one.

VI. Numerical Illustration

To show the advantage of suggested operators that utilizes possible grading procedure, it would be fascinating to provide a numerical example relating to the identification of optimal Marketplaces in Asia for an investment through MADM [11].

Assume that MNC in India is devoloping its finance plan for the coming year in accordance with the group strategic target. After the preliminary filtering, the four choices are acquired and are categorized as: V_1: investment at the “South Asian Marketplaces”; V_2: investment in the “East Asian Marketplaces”; V_3: investment at the “North Asian Marketplaces”; and V_4: investment at the “Local Marketplaces”. This evaluation is based on the four aspects, specifically T_1: “Assessment on the growth”, T_2: “Assessment on the the risk”, T_3: “Assessment on the political and social effect”, and T_4: “Assessment on the environmental effect”. The weights provided by decision maker for each criteria are $\tilde{\omega}_1 = 0.2, \tilde{\omega}_2 = 0.3, \tilde{\omega}_3 = 0.1, \tilde{\omega}_4 = 0.4$. Utilizing the $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators, we now find out the optimal Asian Markertplace to make an investment.

Step 1: Below is the matrix D_m, indicates an assessment of alternatives V_1, V_2, V_3 and V_4 with respect to T_1, T_2, T_3 and T_4 provided by decision-maker in terms of $\tilde{q}$-RPF data.

$$\begin{array}{c c} & \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{pmatrix} (0.2,0.1,0.6) & (0.5,0.3,0.1) & (0.5,0.1,0.3) & (0.4,0.3,0.2) \\ (0.1,0.4,0.4) & (0.6,0.3,0.1) & (0.5,0.2,0.2) & (0.2,0.1,0.7) \\ (0.3,0.2,0.2) & (0.6,0.2,0.1) & (0.4,0.1,0.3) & (0.3,0.3,0.4) \\ (0.3,0.1,0.6) & (0.1,0.2,0.6) & (0.1,0.3,0.5) & (0.2,0.3,0.2) \end{pmatrix} \end{array}$$

Step 2: From the given attributes, T_2 and T_3 are considered to be of cost type whereas the attributes T_1 and T_4 are considered to be benefit type. Since there are various kinds of criteria, the above given matrix can be normalised as $D'_m = (\delta'_{ef})_{k\times l} = (\mu'_{ef}, \eta'_{ef}, \nu'_{ef})_{k\times l}$ by using Eq.(9) is shown below:

$$\begin{array}{c c} & \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{pmatrix} (0.6,0.1,0.2) & (0.5,0.3,0.1) & (0.5,0.1,0.3) & (0.2,0.3,0.4) \\ (0.4,0.4,0.1) & (0.6,0.3,0.1) & (0.5,0.2,0.2) & (0.7,0.1,0.2) \\ (0.2,0.2,0.3) & (0.6,0.2,0.1) & (0.4,0.1,0.3) & (0.4,0.3,0.3) \\ (0.6,0.1,0.3) & (0.1,0.2,0.6) & (0.1,0.3,0.5) & (0.2,0.3,0.2) \end{pmatrix} \end{array}$$

Step 3: We need to order the $\tilde{q}$-RPFNs of each of the alternatives V_1, V_2, V_3 and V_4 respectively before we apply the $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators. Note that while ordering $\tilde{q}$-RPFNs $\delta'_{11} = (0.6, 0.1, 0.2)$, $\delta'_{12} = (0.5, 0.3, 0.1)$, $\delta'_{13} = (0.5, 0.1, 0.3)$ and $\delta'_{14} = (0.2, 0.3, 0.4)$ of V_1, for $\tilde{q} = 2$ using score function (given in equation(1)) we cannot compare which is greater between δ'_{12} and δ'_{13}. Since the score and accuracy values are happened to be one and the same for both the numbers δ'_{12} and δ'_{13}.

To compare these two numbers, we shall now apply our possible grading method so as to order the $\tilde{q}$-RPFNs of alternative V_1.
Given, $\delta'_{11} = (0.6, 0.1, 0.2)$, $\delta'_{12} = (0.5, 0.3, 0.1)$, $\delta'_{13} = (0.5, 0.1, 0.3)$ and $\delta'_{14} = (0.2, 0.3, 0.4)$.
From equation(7), It is clear that,

$P^*(\delta'_{11} \geq \delta'_{11})$ = P^*_{11} = 0.5.
Similary, P^*_{22} = P^*_{33} = P^*_{44} = 0.5.

$$\begin{aligned} P^*(\delta'_{11} \geq \delta'_{12}) &= P^*_{12} \\ &= min\left(max\left(\frac{\mu_{\delta_{11}} - \mu_{\delta_{12}} + \eta_{\delta_{11}} + \rho_{\delta_{12}}}{(\eta_{\delta_{11}} + \eta_{\delta_{12}}) + (\rho_{\delta_{11}} + \rho_{\delta_{12}})}, 0\right), 1\right) \\ &= min\left(max\left(\frac{0.6 - 0.5 + 0.1 + 0.1}{0.6}, 0\right), 1\right) \\ &= min(max(0.5, 0), 1) \\ &= min(0.5, 1) \end{aligned}$$

$P^*_{12} = 0.5$.

$\Rightarrow$ by condition (iii) of **theorem 4.1**,
P^*_{21} = 1 - P^*_{12} = 0.5.
Now, $P^*(\delta'_{11} \geq \delta'_{13})$ = P^*_{13}

$$= min\left(max\left(\frac{\mu_{\delta'_{11}} - \mu_{\delta'_{13}} + \eta_{\delta'_{11}} + \rho_{\delta'_{13}}}{(\eta_{\delta'_{11}} + \eta_{\delta'_{13}}) + (\rho_{\delta'_{11}} + \rho_{\delta'_{13}})}, 0\right), 1\right) = 0.75.$$

$\Rightarrow$ P^*_{31} = 1 - P^*_{13}= 0.25.

By condition (i) of **theorem 4.2** we get,
$P^*(\delta'_{11} \geq \delta'_{14})$ = P^*_{14} = 1.
$\Rightarrow$ P^*_{41} = 1 - P^*_{14}= 0.

Now, $P^*(\delta'_{12} \geq \delta'_{13}) = P^*_{23}$
$= min\left(max\left(\frac{\mu_{\delta'_{12}} - \mu_{\delta'_{13}} + \eta_{\delta'_{12}} + \rho_{\delta'_{13}}}{(\eta_{\delta'_{12}} + \eta_{\delta'_{13}}) + (\rho_{\delta'_{12}} + \rho_{\delta'_{13}})}, 0\right), 1\right)$ = 0.67.
$\Rightarrow P^*_{32}$ = 1 - P^*_{23}= 0.33.

$P^*(\delta'_{12} \geq \delta'_{14}) = P^*_{24}$
$= min\left(max\left(\frac{\mu_{\delta'_{12}} - \mu_{\delta'_{14}} + \eta_{\delta'_{12}} + \rho_{\delta'_{14}}}{(\eta_{\delta'_{12}} + \eta_{\delta'_{14}}) + (\rho_{\delta'_{12}} + \rho_{\delta'_{14}})}, 0\right), 1\right)$ = 0.875.
$\Rightarrow P^*_{42}$ = 1 - P^*_{24}= 0.125.

$P^*(\delta'_{13} \geq \delta'_{14}) = P^*_{34}$
$= min\left(max\left(\frac{\mu_{\delta'_{13}} - \mu_{\delta'_{14}} + \eta_{\delta'_{13}} + \rho_{\delta'_{14}}}{(\eta_{\delta'_{13}} + \eta_{\delta'_{14}}) + (\rho_{\delta'_{13}} + \rho_{\delta'_{14}})}, 0\right), 1\right)$ = 0.83.
$\Rightarrow P^*_{43}$ = 1 - P^*_{34}= 0.17.
Since $P^*_{23} > P^*_{32} \Rightarrow \delta'_{12} \geq \delta'_{13}$.

Hence the possible grade measurement matrix is

$$P^* = \begin{pmatrix} 0.5 & 0.5 & 0.75 & 1.0 \\ 0.5 & 0.5 & 0.67 & 0.875 \\ 0.25 & 0.33 & 0.5 & 0.83 \\ 0.0 & 0.125 & 0.17 & 0.5 \end{pmatrix}$$

Now we can rank and order the $\tilde{q}$-RPFNs of V_1 using equation(8) as follows:
From (8), we have

$$r_x = \frac{1}{f(f-1)}\left(\sum_{y=1}^{f} p^*_{xy} + \frac{f}{2} - 1\right)$$

where, $(x, y$ = 1,2,3,4) and here f = 4
Thus, r_1 = 0.3125; r_2 = 0.2954; r_3 = 0.2425 and r_4 = 0.1496.
$\Rightarrow r_1 > r_2 > r_3 > r_4$
$\Rightarrow P^*(\delta'_{11}) > P^*(\delta'_{12}) > P^*(\delta'_{13}) > P^*(\delta'_{14})$.

Therefore,
$\delta'_{\phi(1)} = \delta'_{11} = (0.6, 0.1, 0.2)$;
$\delta'_{\phi(2)} = \delta'_{12} = (0.5, 0.3, 0.1)$;
$\delta'_{\phi(3)} = \delta'_{13} = (0.5, 0.1, 0.3)$ and
$\delta'_{\phi(4)} = \delta'_{14} = (0.2, 0.3, 0.4)$.

Similarly we can order the $\tilde{q}$-RPFNs of rest of the alternatives.

Step 4: After ordering the $\tilde{q}$-RPFNs with respect to each alternatives, we obtain the ordered normalized $\tilde{q}$-RPF matrix $D'_{\phi(m)}$ as,

$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad T_1 \quad & \quad T_2 \quad & \quad T_3 \quad & \quad T_4 \quad \end{array} \\ \begin{pmatrix} (0.6, 0.1, 0.2) & (0.5, 0.3, 0.1) & (0.5, 0.1, 0.3) & (0.2, 0.3, 0.4) \\ (0.7, 0.1, 0.2) & (0.6, 0.3, 0.1) & (0.5, 0.2, 0.2) & (0.4, 0.4, 0.1) \\ (0.6, 0.2, 0.1) & (0.4, 0.1, 0.3) & (0.4, 0.3, 0.3) & (0.2, 0.2, 0.3) \\ (0.6, 0.1, 0.3) & (0.2, 0.3, 0.2) & (0.1, 0.3, 0.5) & (0.1, 0.2, 0.6) \end{pmatrix} \end{array}$$

We then employ $\tilde{q}$-*RPFOFWA* (keeping $\tilde{r}$ = 1.5 and $\tilde{q}$ = 2) and $\tilde{q}$-*RPFOFWG* (keeping $\tilde{r}$ = 5 and $\tilde{q}$ = 2) operators to find out the combined values $\zeta_e(e = 1, 2, 3, 4)$ for each choices, . Table I displays that performance evaluation of each choices using $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators.

TABLE I
COMBINED VALUES OF EACH ALTERNATIVES

Accumulation operator	Over all rating values
$\tilde{q}$-*RPFOFWA*	ζ_1 = (0.4432,0.2162,0.2240)
	ζ_2 = (0.5528,0.2602,0.1232)
	ζ_3 = (0.3993,0.1693,0.2412)
	ζ_4 = (0.3146,0.2050,0.3712)
$\tilde{q}$-*RPFOFWG*	ζ_1 = (0.3678,0.2575,0.2916)
	ζ_2 = (0.5221,0.3128,0.138)
	ζ_3 = (0.3345,0.1900,0.2725)
	ζ_4 = (0.1805,0.2329,0.4533)

TABLE II
ORDERING THE ALTERNATIVES IN ACCORDANCE WITH SCORES

Accumulation Operator	Score values	Ranking
$\tilde{q}$-*RPFOFWA*	S_{ζ_1} = 0.0996	$V_2 > V_1 > V_3 > V_4$
	S_{ζ_2} = 0.2227	
	S_{ζ_3} = 0.0726	
	S_{ζ_4} = -0.0808	
$\tilde{q}$-*RPFOFWG*	S_{ζ_1} = -0.0161	$V_2 > V_3 > V_1 > V_4$
	S_{ζ_2} = 0.1557	
	S_{ζ_3} = 0.0016	
	S_{ζ_4} = -0.2271	

Step 5: With the help of equation (1), calculate the values to put score $S_{\zeta_e}(e = 1, 2, 3, 4)$ for the given choices. Table II exhibits the alternatives scores and rankings respectively.

Step 6: Hence we will conclude that V_2 is the best option.

VII. EXAMINING THE RESULTS OF DECISION-MAKING PROBLEM BASED ON VARIABLES $\tilde{q}$ AND $\tilde{r}$

The outcome of decision-framing problem with regard to varied values of $\tilde{q}$ and $\tilde{r}$ will be analysed.

A. The impact of parameter $\tilde{q}$

While using the operators $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG*, we will give different values for $\tilde{q}$ (treating $\tilde{r}$ as constant) in order to scrutinize the outcomes of a conclusion-making problem. Thus for calculating purpose, considering $\tilde{r} = 3$ and variations in parameter $\tilde{q} = 4, 5, 7, 9, 12, 15, 20$, we determine the values of score for every alternative. Through Table III, we could see the ranking order obtained as $V_2 \succ V_1 \succ V_3 \succ V_4$ with respect to $\tilde{q}$-*RPFOFWA* operator and $V_2 \succ V_3 \succ V_1 \succ V_4$ based on $\tilde{q}$-*RPFOFWG* operator for varied values of $\tilde{q}$. Thus the optimal choice is always V_2, regardless of distinct values of $\tilde{q}$. The graphs of Fig. 1 and 2 depict the alternative's score values against the varied $\tilde{q}$ values obtained by $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators respectively.

B. The impact of parameter $\tilde{r}$

While using the operators $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG*, we will give different values for $\tilde{r}$ (treating $\tilde{q}$ as constant) in order to scrutinize the outcomes of a conclusion-making problem. Thus for calculating purpose, considering $\tilde{q} = 3$ and variations in parameter $\tilde{r} = 3, 5, 8, 11, 15, 20$, we determine the values of score for every alternative. Through Table IV,

TABLE III
INFLUENCE OF A VARIABLE $\tilde{q}$ UPON THE RESULTS OF DM UTILIZING $\tilde{q}$-*RPFOFWA* & $\tilde{q}$-*RPFOFWG* OPERATORS

$\tilde{q}$	Accumulation operator	$S(\zeta_1)$	$S(\zeta_2)$	$S(\zeta_3)$	$S(\zeta_4)$	Ranking order	Optimal choice
4	$\tilde{q}$-*RPFOFWA*	0.0476	0.1009	0.0333	0.0066	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-0.0001	0.0591	0.0035	-0.0642	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
5	$\tilde{q}$-*RPFOFWA*	0.0274	0.0641	0.0191	0.0086	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-4×10^{-5}	0.0324	0.0015	-0.0361	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
7	$\tilde{q}$-*RPFOFWA*	0.0087	0.0265	0.0062	0.0047	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-5×10^{-5}	0.0092	0.0002	-0.0122	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
9	$\tilde{q}$-*RPFOFWA*	0.0028	0.0115	0.0021	0.0019	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-2×10^{-5}	0.0025	3×10^{-5}	-0.0042	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
12	$\tilde{q}$-*RPFOFWA*	0.0005	0.0035	4.42×10^{-4}	4.29×10^{-4}	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-2×10^{-6}	3.55×10^{-4}	1×10^{-6}	-0.0009	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
15	$\tilde{q}$-*RPFOFWA*	0.00011	0.0011	9.45×10^{-5}	9.37×10^{-5}	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-2×10^{-7}	5×10^{-5}	4×10^{-8}	-2×10^{-4}	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
20	$\tilde{q}$-*RPFOFWA*	8×10^{-6}	0.00017	7.32×10^{-6}	7.31×10^{-6}	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-3×10^{-9}	2×10^{-6}	2×10^{-10}	-1×10^{-5}	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2

TABLE IV
INFLUENCE OF A VARIABLE $\tilde{r}$ UPON THE RESULTS OF DM UTILIZING $\tilde{q}$-*RPFOFWA* & $\tilde{q}$-*RPFOFWG* OPERATORS

$\tilde{r}$	Accumulation operator	$S(\zeta_1)$	$S(\zeta_2)$	$S(\zeta_3)$	$S(\zeta_4)$	Ranking order	Optimal choice
3	$\tilde{q}$-*RPFOFWA*	0.0771	0.1564	0.0551	-0.0119	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-0.0020	0.1024	0.0066	-0.1184	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
5	$\tilde{q}$-*RPFOFWA*	0.0766	0.1554	0.0546	-0.0133	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-0.0013	0.1036	0.0070	-0.1176	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
8	$\tilde{q}$-*RPFOFWA*	0.0761	0.1546	0.0542	-0.0144	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-0.0007	0.1047	0.0073	-0.1170	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
11	$\tilde{q}$-*RPFOFWA*	0.0759	0.1541	0.0540	-0.0151	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	-0.0002	0.1054	0.0075	-0.1166	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
15	$\tilde{q}$-*RPFOFWA*	0.0756	0.1537	0.0538	-0.0158	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	0.0001	0.1061	0.0077	-0.1163	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
20	$\tilde{q}$-*RPFOFWA*	0.0755	0.1534	0.0536	-0.0164	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
	$\tilde{q}$-*RPFOFWG*	0.0005	0.1068	0.0079	-0.1161	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2

TABLE V
COMPARING $\tilde{q}$-*RPFOFWA* AND $\tilde{q}$-*RPFOFWG* WITH CURRENT OPERATORS

$Operators$	S_{ζ_1}	S_{ζ_2}	S_{ζ_3}	S_{ζ_4}	Ranking order	Optimal choice
PFWA operator[11]	-0.0161	0.2666	0.0020	-0.2738	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
PFHA operator[11]	-0.1114	0.1892	-0.0423	-0.3684	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
q-RPFEOWA operator	0.0934	0.2172	0.0675	-0.091	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
q-RPFEOWG operator	-0.0170	0.1551	0.0006	-0.2255	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2
$\tilde{q}$-*RPFOFWA operator*$(Suggested)$	0.0996	0.2227	0.0726	-0.0808	$V_2 \succ V_1 \succ V_3 \succ V_4$	V_2
$\tilde{q}$-*RPFOFWG operator*$(Suggested)$	-0.0161	0.1557	0.0016	-0.2271	$V_2 \succ V_3 \succ V_1 \succ V_4$	V_2

we could see the ranking order obtained as $V_2 \succ V_1 \succ V_3 \succ V_4$ with respect to $\tilde{q}$-*RPFOFWA* operator and $V_2 \succ V_3 \succ V_1 \succ V_4$ based on $\tilde{q}$-*RPFOFWG* operator for varied values of $\tilde{r}$. Thus the optimal choice is always V_2, regardless of distinct values of $\tilde{r}$. The graphs of Fig. 3 and 4 depict the alternative's score values against the varied $\tilde{r}$ values obtained by $\tilde{q}$-*RPFOFWA* and $\tilde{q}$-*RPFOFWG* operators respectively.

VIII. COMPARATIVE ANALYSIS

We can show the effectiveness and superiority of our suggested technique by comparing it with some existing operators. It is clear that V_2 is the best option out of the available options when using our suggested technique. So when we compare our ordered operators with existing q-rung picture fuzzy Einstein ordered weighted averaging (q-*RPFEOWA*) and geometric (q-*RPFEOWG*) operators [2], we get the same conclusion with better scores for each choices. This indicates how our recommended paradigm is consistent. It can be seen from the Table V that the ranking order and an optimal choice had been found using the suggested accumulation operators $\tilde{q}$-*RPFOFWA*, $\tilde{q}$-*RPFOFWG* and the current operators.

IX. CONCLUSION

In this study, we devised a possible grading method to compare and order the $\tilde{q}$-RPFNs. The striking characteristics of the suggested method have been analyzed. By making use of possible grading method we employed the Frank accumu-

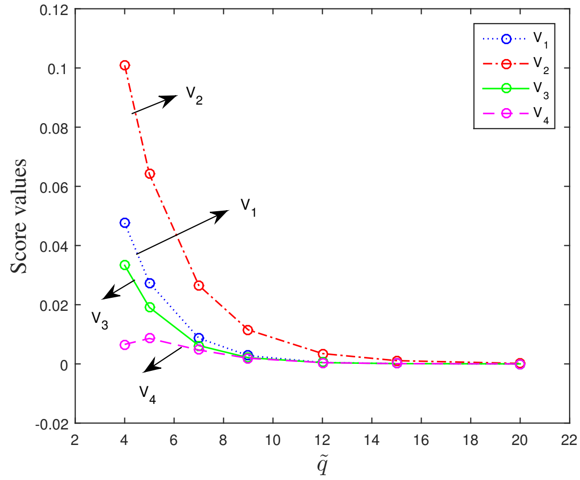

Fig. 1. Scores of alternatives by $\tilde{q}$-*RPFOFWA* operator against $\tilde{q}$

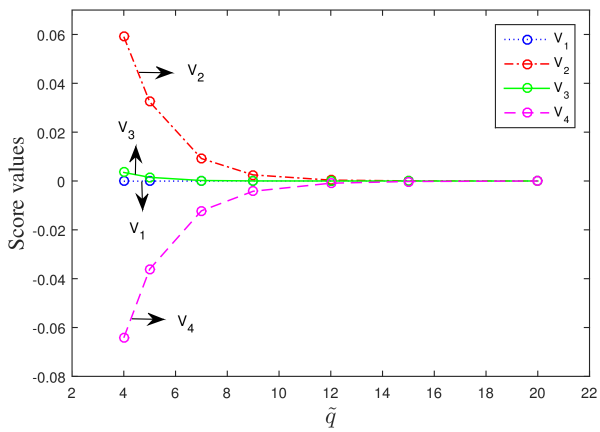

Fig. 2. Scores of alternatives by $\tilde{q}$-*RPFOFWG* operator against $\tilde{q}$

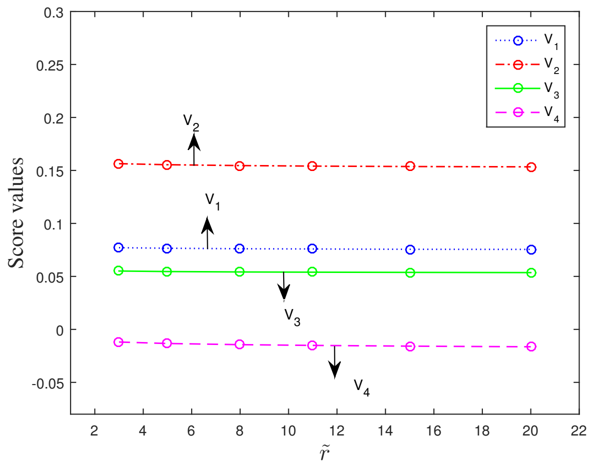

Fig. 3. Alternative's scores by $\tilde{q}$-*RPFOFWA* operator against $\tilde{r}$

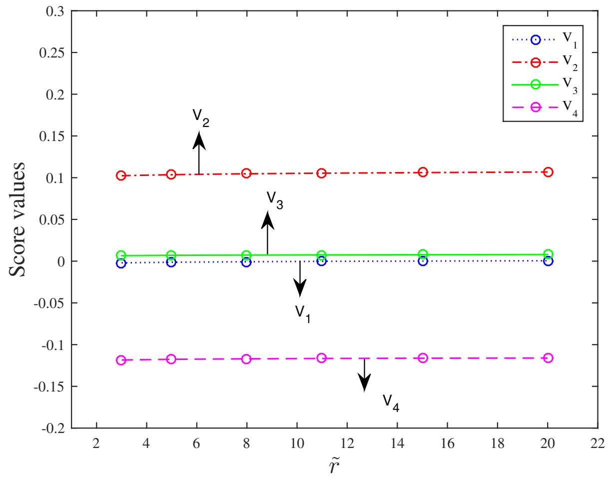

Fig. 4. Alternative's scores by $\tilde{q}$-*RPFOFWG* operator against $\tilde{r}$

lation operators, namely $\tilde{q}$-*RPFOFWA* & $\tilde{q}$-*RPFOFWG*, in order to explore the MADM issues in the $\tilde{q}$-Rung picture fuzzy environment. Additionally, the effectiveness of the parameters $\tilde{q}$ and $\tilde{r}$ in the decision-making issue's outcomes using both the ordered operators point of view had been examined. To demonstrate the adaptability and efficacy of our strategy, we ultimately compared our suggested procedure with the current procedures.

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