Fuzzy Topological Graphs in Bipolar and Related Settings

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***Abstract*—Bipolar objects are widespread in nature, and their two attributes describe the opposition and unity of the things. Motivated by characterizing fuzzy topological structures by means of fuzzy graphs, we propose a bipolar fuzzy topological graph to measure the features of bipolar fuzzy systems. The topological characteristics in bipolar neutrosophic and bipolar interval-valued fuzzy settings are discussed as well. Several instances are manifested to clarify new definitions and conclusions. Furthermore, the properties of edge calculating and graph isomorphic are determined.**

Keywords: bipolar fuzzy topological graph, neutrosophic set, bipolar interval-valued fuzzy topological, homomorphism, isomorphism

I. Introduction

Fuzzy graph research has witnessed many years of contributions from scholars. Kauffman [1] first defined the framework of fuzzy graph which was later expanded by Rosenfield [2]. Akram [3] argued a bipolar fuzzy graph, and several properties and applications were obtained in [4] and [5]. Yang et al. [6] suggested a generalized version of bipolar fuzzy graphs. Furthermore, fuzzy line graph, fuzzy tree, fuzzy block, fuzzy planner graph, and fuzzy incidence graph were introduced by Mordeson [7], Sunitha and Vijayakumar [8, 9], Samanta and Pal [10] and Dinesh [11] respectively. Fuzzy graph models have been prevalent and widely used in various decision-making algorithms and applications in recent years (Sitara et al. [12], Jia et al. [13], Karaaslan [14], and Akram et al. [15], [16] and [17]).

The topological indices of graph structures are widely investigated (see Gao et al. [18], [19] and [20], Anuradha et al. [21], Azeem et al. [22] and Mondal et al. [23]). Notably, the topological indices of various fuzzy graphs in distinct settings are studied. Authors in [24] revised the concepts of fuzzy graphs (FGs), and they defined and researched fuzzy connectivity and distance-based indices for fuzzy graph setting (see [25], [26] and [27]), including the bipolar fuzzy connectivity index and bipolar fuzzy Wiener index. Binu [28] applied the fuzzy Wiener index to illegal immigration networks, and Ali et al. [29] considered the fuzzy graphs with the Hamiltonian cycle.

Recently, Atef et al. [30] characterized the fuzzy topological (FT) structures of fuzzy sets (FSs) by means of FGs and applied them in smart cities. Fueled by this contribution, we aim to feature the fuzzy topological structures of bipolar fuzzy set (BFS), neutrosophic set (NS), bipolar neutrosophic set (BNS), interval-valued fuzzy set (IVFS), and bipolar interval-valued fuzzy set (BIVFS) in light of FGs called a bipolar fuzzy topological graph (BFTG), neutrosophic topology graph (NTG), bipolar neutrosophic topology graph (BNTG), interval-valued fuzzy topology graph (IVFTG) and bipolar interval-valued fuzzy topology graph (BIVFTG), respectively.

The following parts of this work are built as follows: concepts and notations are introduced first; then the parallel classes and their fuzzy graphs in various settings are determined; new algebraic operations on fuzzy topological graphs in different settings are presented subsequently. The main conclusions are manifested in Section III and Section IV, and each section is divided into several subsections corresponding to different settings. The arguments are stated in this paper along with some examples to clearly explain the connotation of contents.

Note that the bipolar fuzzy graph in this article has analogous definitions from the bipolar fuzzy graph in [14-17, 24-28], but it still involves tiny differences. The bipolar fuzzy graph in other articles are fuzzy graph structure itself, and its edge membership function (MF) and vertex set MF are determined by the fuzzy data itself. However, the bipolar fuzzy graph (BFG) in this article is determined by the relationship of bipolar sets in special settings, and a vertex in the BFG represents a BFS. This essential difference is also applied to neutrosophic graphs (NGs), bipolar neutrosophic graphs (BNGs), etc.

II. Preliminaries

The purpose here is to review the concepts and terminologies of FSs, NSs, BNSs, IVFSs and BIVFSs.

Let V be a universal set (US) with at least one element, and $K=\{(v,\mu_K^P(v),\mu_K^N(v)):v\in V\}$ be a BFS in V if two maps satisfy $\mu_K^P:V\to[0,1]$ and $\mu_K^N:V\to[-1,0]$. $\varnothing_V$ or $\varnothing$ in short is a null BFS on V such that

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$\mu^N_{\varnothing}(v)=\mu^P_{\varnothing}(v)=0$ for any $v\in V$. $\mathbb{V}$ is absolute BFS on V if $\mu^P_{\mathbb{V}}(v)=1$ and $\mu^N_{\mathbb{V}}(v)=-1$ for all $v\in V$.

Let $S_1=\{(v,\mu^+_{S_1}(v),\mu^-_{S_1}(v))\}$ and $S_2=\{(v,\mu^+_{S_2}(v),\mu^-_{S_2}(v))\}$ be two BFSs on V. If $\mu^+_{S_1}(v)\le\mu^+_{S_2}(v)$ and $\mu^-_{S_1}(v)\ge\mu^-_{S_2}(v)$ hold for any $v\in V$, then we say S_1 is a bipolar fuzzy subset of S_2, denoted by $S_1\subseteq S_2$. If S_1 is a part of S_2, then we say S_1 is a bipolar fuzzy partial subset of S_2.

Let G and G' be two fuzzy graphs (bipolar fuzzy graph, interval-valued fuzzy graph or others). The two graphs are isomorphic $G\cong G'$, if there is a bijective $f:V\to V'$ to establish a corresponding one-to-one relationship for the vertex and edge membership functions.

Tehrim [31] introduced the bipolar fuzzy topology as follows: let V be a universal set, $BF(V)$ be the family of all bipolar fuzzy sets on V, $X=\{(v,1,-1),\ v\in V\}\in BF(V)$ be an absolute bipolar fuzzy set, $\mathbb{BF}(X)$ be the class of all bipolar fuzzy subsets of X, and τ be the subclass of $\mathbb{BF}(X)$. Then τ is a bipolar fuzzy topology if (i) $\varnothing, X\in\tau$, where $\varnothing=\{(v,0,0),v\in V\}$; (ii) $S_1,S_2\in\tau\ \Rightarrow S_1\cap S_2\in\tau$; (iii) $S_l\in\tau$ where $l\in\psi\Rightarrow\bigcup_{l\in\psi}S_l\in\tau$. If τ is a bipolar fuzzy topological on X, then (V,τ) is a bipolar fuzzy topological space over X.

For a universal set V, a neutrosophic set is denoted by

$$K=\{(v,T_K(v),I_K(v),F_K(v)):v\in V\},$$

where $T_K,I_K,F_K\in[0,1]$ denotes the membership value of truthness, indeterminacy and falsity, respectively. Thus, $0\le T_K+I_K+F_K\le 3$. The basic operation of inclusion, equality, union, intersection and complement can be referred to Tang [32].

Let $\varnothing=\{(v,0,1,1)):v\in V\}$ and $X=\{(v,1,0,0)):v\in V\}$ be null neutrosophic set and absolute neutrosophic set respectively. Let V be a universal set, $N(V)$ be the family of all neutrosophic sets on V, $X\in N(V)$ be an absolute neutrosophic set, $\mathbb{N}(X)$ be the class of all neutrosophic subsets of X, and τ be the subclass of $\mathbb{N}(X)$. Then τ is called neutrosophic topology if (i) $\varnothing,X\in\tau$; (ii) $S_1,S_2\in\tau\Rightarrow S_1\cap S_2\in\tau$; (iii) $S_l\in\tau$ where $l\in\psi\Rightarrow\bigcup_{l\in\psi}S_l\in\tau$. If τ is a neutrosophic topological on X, then (V,τ) is the neutrosophic topological space over X.

The bipolar neutrosophic set of the universal set V is formulated by

$$K=\{(v,T^+_K(v),I^+_K(v),F^+_K(v),T^-_K(v),I^-_K(v),F^-_K(v)):v\in V\},$$

where $T^+_K(v),I^+_K(v),F^+_K(v)\in[0,1]$ are positive membership of truthness, indeterminacy and falsity respectively; and $T^-_K(v),I^-_K(v),F^-_K(v)\in[-1,0]$ are negative membership of truthness, indeterminacy and falsity respectively. The basic operations of inclusion, equality, union, intersection and complement can be referred to Zhu et al. [33] and Ali et al. [34].

Let $\varnothing=\{(v,0,1,1,0,-1,-1)):v\in V\}$ and $X=\{(v,1,0,0,-1,0,0)):v\in V\}$ be a null bipolar neutrosophic set and an absolute bipolar neutrosophic set respectively. Let V be a US, $BN(V)$ be the family of all bipolar neutrosophic sets on V, $X\in BN(V)$ be an absolute bipolar neutrosophic set, $\mathbb{BN}(X)$ be the class of all bipolar neutrosophic subsets of X, and τ be the subclass of $\mathbb{BN}(X)$. Then τ is called bipolar neutrosophic topology if (i) $\varnothing,X\in\tau$; (ii) $S_1,S_2\in\tau\Rightarrow S_1\cap S_2\in\tau$; (iii) $S_l\in\tau$ where $l\in\psi\Rightarrow\bigcup_{l\in\psi}S_l\in\tau$ (see Tehrim [31]). If τ is a bipolar neutrosophic topological on X, then (V,τ) is the bipolar neutrosophic topological space over X.

For the universal set V, an interval-valued fuzzy set is denoted by

$$K=\{(v,[\mu^{+l}_K(v),\mu^{+u}_K(v)]):v\in V\},$$

where $0\le\mu^{+l}_K(v)\le\mu^{+u}_K(v)\le 1$ and $[\mu^{+l}_K(v),\ \mu^{+u}_K(v)]$ is an interval in [0,1]. Let $\varnothing=\{(v,[0,0]))\ :v\in V\}$ and $X=\{(v,[1,1])):v\in V\}$ be a null IVFS and an absolute interval-valued fuzzy set respectively. Let V be a US, $IVF(V)$ be the family of all interval-valued fuzzy sets on V, $X\in IVF(V)$ be an absolute interval-valued fuzzy set, $\mathbb{IVF}(X)$ be the class of all interval-valued fuzzy subsets of X, and τ be the subclass of $\mathbb{IVF}(X)$. Then τ is called interval-valued fuzzy topology (IVFT) if (i) $\varnothing,X\in\tau$; (ii) $S_1,S_2\in\tau\Rightarrow S_1\cap S_2\in\tau$; (iii) $S_l\in\tau$ where $l\in\psi\Rightarrow\bigcup_{l\in\psi}S_l\in\tau$. If τ is an IVFT on X, then (V,τ) is interval-valued fuzzy topological space over X.

For a universal set V, a bipolar interval-valued fuzzy set is denoted by

$$K=\{(v,[\mu^{+l}_K(v),\mu^{+u}_K(v)],[\mu^{-l}_K(v),\mu^{-u}_K(v)]):v\in V\},$$

where $0\le\mu^{+l}_K(v)\le\mu^{+u}_K(v)\le 1$, $-1\le\mu^{-l}_K(v)\le\mu^{-u}_K(v)\le 0$, $[\mu^{+l}_K(v),\mu^{+u}_K(v)]$ is an interval in [0,1] and $[\mu^{-l}_K(v),\mu^{-u}_K(v)]$ is an interval in [-1,0]. Let $\varnothing=\{(v,[0,0],[0,0])):v\in V\}$ and $X=\{(v,[1,1],\ [-1,-1])):v\in V\}$ be a null bipolar interval-valued fuzzy set and an absolute bipolar interval-valued fuzzy set respectively. Let V be a US, $BIVF(V)$ be the family of all BIVFSs on V, $X\in BIVF(V)$ be an absolute bipolar interval-valued fuzzy set, $\mathbb{BIVF}(X)$ be the class of all bipolar interval-valued fuzzy subsets of X, and τ be the subclass of $\mathbb{BIVF}(X)$. Then τ is called bipolar interval-valued fuzzy topology (BIVFT) if (i) $\varnothing,X\in\tau$; (ii) $S_1,S_2\in\tau\Rightarrow S_1\cap S_2\in\tau$; (iii) $S_l\in\tau$ where $l\in\psi\Rightarrow\bigcup_{l\in\psi}S_l\in\tau$.

If τ is a BIVFT on X, then (V,τ) is bipolar interval-valued fuzzy topological space over X [35].

If two graphs G_1 and G_2 are obtained from the neutrosophic set, bipolar neutrosophic set, interval-valued fuzzy set or interval valued bipolar fuzzy set, then we can define the isomorphism $G_1 \cong G_2$ using the same fashion as in the bipolar setting.

III. PARALLEL CLASSES

We raise new concepts of parallel classes of bipolar fuzzy set, NSs, BNSs, interval-valued fuzzy sets and BIVFSs.

A. Parallel classes in bipolar fuzzy setting

Definition 1. Let *V* be a universal set, $BF(V)$ be the class of all bipolar fuzzy sets of *V*, and $C_1, C_2 \subseteq BF(V)$. We say C_1 is parallel to C_2 (denoted by $X \sim Y$, where X, Y are bipolar fuzzy sets in $BF(V)$, each element in C_1 is a bipolar fuzzy partial subset of *X*, and each element in C_2 is a bipolar fuzzy partial subset of *Y*), if there exists a bijective bipolar fuzzy mapping $F: X \to Y$ such that for any $c_1 \in C_1$, we have $F(c_1) = c_2$ and $c_2 \in C_2$. Let $X = Y$. In this case, C_1 is parallel to C_2 if there exists a bijective bipolar fuzzy mapping $F: X \to X$ satisfying $F(C_1) = C_2$.

Example 1. Let $V = \{v_1, v_2, v_3\}$ be a universal set, and $X = \{(v_1, 0.4, -0.2), (v_2, 0.5, -0.3), (v_3, 0.1, -0.8)\}$ be a bipolar fuzzy set on V. Set $C_1 = \{\{(v_3, 0.1, -0.8)\}$, $\{(v_1, 0.4, -0.2), (v_2, 0.5, -0.3)\}, \{(v_3, 0.1, -0.8), (v_1, 0.4, -0.2)\}\}$,

$C_2 = \{\{(v_1, 0.4, -0.2)\}, \{(v_2, 0.5, -0.3), (v_3, 0.1, -0.8)\},$
$\{(v_1, 0.4, -0.2), (v_2, 0.5, -0.3)\}\}$.

Clearly, all the elements in C_1 and C_2 are the bipolar fuzzy partial subsets of X. There is a bijective bipolar mapping $F: X \to X$ with

$$F(\{(v_3, 0.1, -0.8)\}) = \{(v_1, 0.4, -0.2)\},$$
$$F(\{(v_1, 0.4, -0.2), (v_2, 0.5, -0.3)\}) = \{(v_2, 0.5, -0.3), (v_3, 0.1, -0.8)\},$$
$$F(\{(v_3, 0.1, -0.8), (v_1, 0.4, -0.2)\}) = \{(v_1, 0.4, -0.2), (v_2, 0.5, -0.3)\}.$$

Therefore, C_1 and C_2 are parallel.

From Example 1, we know that the essence of parallelism is the one-to-one correspondence between the elements in the bipolar fuzzy set. In this example, we can see the correspondence between the following elements in *X*:

$$(v_1, 0.4, -0.2) \xrightarrow{F} (v_2, 0.5, -0.3),$$
$$(v_2, 0.5, -0.3) \xrightarrow{F} (v_3, 0.1, -0.8),$$
$$(v_3, 0.1, -0.8) \xrightarrow{F} (v_1, 0.4, -0.2).$$

Next, we argue that any class of BFSs is denoted by a BFG in view of operation $\wedge$ between bipolar fuzzy set classes. Let $S_1, \cdots, S_n$ be bipolar fuzzy sets, $C = \{S_1, \cdots, S_n\}$ be class of these bipolar fuzzy sets and *G* be a BFG corresponding to *C*. BFG *G* is constructed as follows: each vertex in *G* corresponds to a BFS among $S_1, \cdots, S_n$, and there are $|S_i \wedge S_j|$ edges between vertices S_i and S_j. The following example is applied to illustrate such a kind of BFG.

Example 2. The classes of BFSs

$C_1 = \{S_1 = \{(v_1, 0.2, -0.7)\}, S_2 = \{(v_2, 0.3, -0.6)\},$
$S_3 = \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6)\}\}$,
$C_2 = \{S_4 = \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6), (v_3, 0.8, -0.1)\},$
$S_5 = \{(v_4, 0.4, -0.6), (v_5, 0.9, -0.3)\}, S_6 = \{(v_3, 0.8, -0.1),$
$(v_4, 0.4, -0.6)\}\}$,
$C_3 = \{S_7 = \{(v_3, 0.8, -0.1), (v_6, 0.6, -0.6)\}, S_8 =$
$\{(v_7, 0.4, -0.9), (v_8, 0.3, -0.5), (v_9, 0.2, -0.7)\},$
$S_9 = \{(v_3, 0.8, -0.1), (v_9, 0.2, -0.7), (v_{10}, 0.4, -0.3)\}\}$.

represent the same bipolar fuzzy graphs which are depicted in Fig 1.

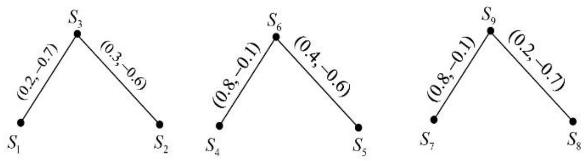

Fig 1. A bipolar fuzzy graph represents C_1, C_2 and C_3.

Definition 2. Let $C = \{C_i : i \in I\}$ be a collection of all classes of a BFS *X*, and hence C_i can be expressed by the same BFG *G* ($\{|\vee C_i| : i \in I\}$ which is formulated by the graph number of a BFG *G*).

Example 3. Consider C_1, C_2 and C_3 as defined in Example 2, we get

$\vee C_1 = \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6)\}$,
$\vee C_2 = \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6), (v_3, 0.8, -0.1),$
$(v_4, 0.4, -0.6), (v_5, 0.9, -0.3)\}$,
$\vee C_3 = \{(v_3, 0.8, -0.1), (v_6, 0.6, -0.6), (v_7, 0.4, -0.9),$
$(v_8, 0.3, -0.5), (v_9, 0.2, -0.7), (v_{10}, 0.4, -0.3)\}$.

The numbers of a BFG $|\vee C_1|$, $|\vee C_2|$ and $|\vee C_3|$ are 2, 5 and 6 respectively.

Next, we argue that any BFG *G* can be expressed by a class of BFSs. Notation $\wedge$ is re-formulated to an operator for vertices of BFGs.

Definition 3. Let *G* be a BFG and v_i, v_j be two vertices of *G*. Suppose that v_i and v_j correspond to bipolar fuzzy sets S_i and S_j respectively, then $N(v_i, v_j) = |S_i \wedge S_j|$. Note that $N(v_i, X) = |S_i|$ (resp. $N(v_j, X) = |S_j|$) if S_i (resp. S_j) is a bipolar fuzzy subset of *X*.

Example 4. If $S_i = \{(v_1, 0.2, -0.7)\}$ and $S_j = \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6)\}$, then $S_i \wedge S_j = \{(v_1, 0.2, \ -0.7)\}$ and thus $N(S_i, S_j) = |S_i \wedge S_j| = 1$.

Theorem 1. If G_1 and G_2 are two BFGs corresponding to two parallel classes C_1 and C_2, then $G_1 \cong G_2$.

Unfortunately, the converse of Theorem 1 may not establish in general since each BFG can express many classes, and we use the following example to illustrate it.

Example 5. Consider the classes of bipolar fuzzy sets

$C_1 = \{S_1 = \{(v_1, 0.2, -0.7)\}, S_2 = \{(v_2, 0.3, -0.6)\}, \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6)\}\}$,

$C_2 = \{S_4 = \{(v_1, 0.2, -0.7), (v_2, 0.3, -0.6), (v_3, 0.8, -0.1)\}, S_5 = \{(v_4, 0.4, -0.6), (v_5, 0.9, -0.3)\}, S_6 = \{(v_3, 0.8, -0.1), (v_4, 0.4, -0.6)\}\}$.

Hence, bipolar fuzzy graphs corresponding to C_1 and C_2 are isomorphic to each other as depicted in Fig 2, but C_1 and C_2 are not parallel.

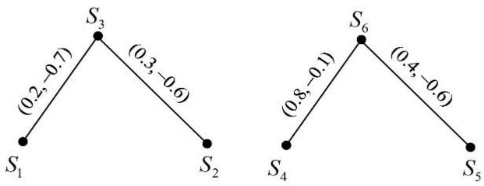

Fig 2. A bipolar fuzzy graph represents C_1 and C_2.

B. Parallel classes in neutrosophic setting

Definition 4. Let *V* be a US, $NF(V)$ be the class of all NSs of *V*, and $C_1, C_2 \subseteq NF(V)$. We say C_1 is parallel to C_2 (denoted by $X \sim Y$, where X, Y are neutrosophic sets in $NF(V)$, each element in C_1 is a neutrosophic partial subset of *X*, and each element in C_2 is a neutrosophic partial subset of *Y*), if there exists a bijective neutrosophic mapping $F: X \to Y$ such that for any $c_1 \in C_1$, we have $F(c_1) = c_2$ and $c_2 \in C_2$. Let $X = Y$. Then C_1 is parallel to C_2 if there exists a bijective neutrosophic mapping $F: X \to X$ satisfying $F(C_1) = C_2$.

Example 6. Let $V = \{v_1, v_2, v_3\}$ be a US, $X = \{(v_1, 0.3, 0.4, 0.2), (v_2, 0.4, 0.8, 0.3), (v_3, 0.1, 0.6, 0.8)\}$ be a NS on V. Set

$C_1 = \{\{(v_3, 0.1, 0.6, 0.8)\}, \{(v_1, 0.3, 0.4, 0.2), (v_2, 0.4, 0.8, 0.3)\}, \{(v_3, 0.1, 0.6, 0.8), (v_1, 0.3, 0.4, 0.2)\}\}$,

$C_2 = \{\{(v_1, 0.3, 0.4, 0.2)\}, \{(v_2, 0.4, 0.8, 0.3), (v_3, 0.1, 0.6, 0.8)\}, \{(v_1, 0.3, 0.4, 0.2), (v_2, 0.4, 0.8, 0.3)\}\}$.

Obviously, all the elements in C_1 and C_2 are the neutrosophic partial subsets of X. There is a bijective neutrosophic mapping $F: X \to X$ satisfying

$F(\{(v_3, 0.1, 0.6, 0.8)\}) = \{(v_1, 0.3, 0.4, 0.2)\}$,

$F(\{(v_1, 0.3, 0.4, 0.2), (v_2, 0.4, 0.8, 0.3)\}) = \{(v_2, 0.4, 0.8, 0.3), (v_3, 0.1, 0.6, 0.8)\}$,

$F(\{(v_3, 0.1, 0.6, 0.8), (v_1, 0.3, 0.4, 0.2)\}) = \{(v_1, 0.3, 0.4, 0.2), (v_2, 0.4, 0.8, 0.3)\}$.

Therefore, C_1 and C_2 are parallel.

In view of Example 6, it is clear that the essence of parallelism is the one-to-one correspondence between the elements in the neutrosophic set. In this instance, we can see the correspondence between the following elements in *X*:

$$(v_1, 0.3, 0.4, 0.2) \xrightarrow{F} (v_2, 0.4, 0.8, 0.3),$$
$$(v_2, 0.4, 0.8, 0.3) \xrightarrow{F} (v_3, 0.1, 0.6, 0.8),$$
$$(v_3, 0.1, 0.6, 0.8) \xrightarrow{F} (v_1, 0.3, 0.4, 0.2).$$

Next, it is presented that any class of NSs can be denoted by a general NG in view of operation $\wedge$ between neutrosophic set classes. Let $S_1, \cdots, S_n$ be neutrosophic sets, $C = \{S_1, \cdots, S_n\}$ be a class of these neutrosophic sets and *G* be a NG corresponding to *C*. Neutrosophic graph *G* is constructed as follows: each vertex in *G* corresponds to a neutrosophic set among $S_1, \cdots, S_n$, and there are $|S_i \wedge S_j|$ edges between vertices S_i and S_j. The below insance is applied to illustrate such a kind of neutrosophic graph.

Example 7. Consider the classes of neutrosophic sets

$C_1 = \{S_1 = \{(v_1, 0.4, 0.5, 0.2)\}, S_2 = \{(v_2, 0.5, 0.2, 0.9)\}, S_3 = \{(v_1, 0.4, 0.5, 0.2), (v_2, 0.5, 0.2, 0.9)\}\}$,

$C_2 = \{S_4 = \{(v_1, 0.4, 0.5, 0.2), (v_2, 0.5, 0.2, 0.9), (v_3, 0.8, 0.7, 0.3)\}, S_5 = \{(v_4, 0.1, 0.6, 0.9), (v_5, 0.9, 0.3, 0.2)\}, S_6 = \{(v_3, 0.8, 0.7, 0.3), (v_4, 0.1, 0.6, 0.9)\}\}$,

$C_3 = \{S_7 = \{(v_3, 0.8, 0.7, 0.3), (v_6, 0.3, 0.6, 0.9)\}, S_8 = \{(v_7, 0.7, 0.4, 0.1), (v_8, 0.2, 0.5, 0.6), (v_9, 0.3, 0.7, 0.7)\}, S_9 = \{(v_3, 0.8, 0.7, 0.3), (v_9, 0.3, 0.7, 0.7), (v_{10}, 0.6, 0.5, 0.5)\}\}$.

to represent the same NGs as determined in Fig 3.

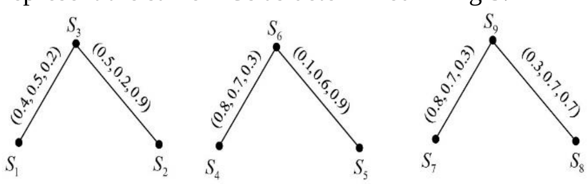

Fig 3. A neutrosophic graph represents C_1, C_2 and C_3.

Definition 5. Let $C = \{C_i : i \in I\}$ be a collection of all classes of a NS *X*, and thus C_i can be represented by the same neutrosophic graph *G* ($\{|\vee C_i| : i \in I\}$ which is formulated as the graph number of a NG *G*).

Example 8. Discuss C_1, C_2 and C_3 as given in Example 7, we yield

$\vee C_1=\{(v_1,0.4,0.5,0.2),(v_2,0.5,0.2,0.9)\}$,
$\vee C_2=\{(v_1,0.4,0.5,0.2),(v_2,0.5,0.2,0.9),(v_3,0.8,0.7,0.3),(v_4,0.1,0.6,0.9),(v_5,0.9,0.3,0.2)\}$,
$\vee C_3=\{(v_3,0.8,0.7,0.3),(v_6,0.3,0.6,0.9),(v_7,0.7,0.4,0.1),(v_8,0.2,0.5,0.6),(v_9,0.3,0.7,0.7),(v_{10},0.6,0.5,0.5)\}$.

The numbers of a neutrosophic graph $|\vee C_1|$, $|\vee C_2|$ and $|\vee C_3|$ are 2, 5 and 6 respectively.

Next, we show that any NG *G* can be expressed by a class of NSs. Similarly, $\wedge$ is denoted by an operator *N* for vertices of neutrosophic graphs.

Definition 6. Let *G* be a NG and v_i, v_j be two vertices of *G*. Suppose that v_i and v_j correspond to neutrosophic sets S_i and S_j respectively, then $N(v_i,v_j)=|S_i\wedge S_j|$. Note that $N(v_i,X)=|S_i|$ (resp. $N(v_j,X)=|S_j|$) if S_i (resp. S_j) is a neutrosophic subset of *X*.

Example 9. If $S_i=\{(v_1,0.4,0.5,0.2)\}$ and $S_j=\{(v_1,0.4,0.5,0.2),(v_2,0.5,0.2,0.9)\}$, then $S_i\wedge S_j=\{(v_1,0.4,0.5,0.2)\}$ and $N(S_i,S_j)=|S_i\wedge S_j|=1$.

Theorem 2. If G_1 and G_2 are two NGs corresponding to two parallel classes C_1 and C_2, then $G_1\cong G_2$.

Similar to Theorem 1, the converse of the above theorem may not true in general since each neutrosophic graph can express many classes, and the Example 10 is used to explain it.

Example 10. Consider the classes of neutrosophic sets
$C_1=\{S_1=\{(v_1,0.4,0.5,0.2)\},S_2=\{(v_2,0.5,0.2,0.9)\},S_3=\{(v_1,0.4,0.5,0.2),(v_2,0.5,0.2,0.9)\}\}$,
$C_2=\{S_4=\{(v_1,0.4,0.5,0.2),(v_2,0.5,0.2,0.9),(v_3,0.8,0.7,0.3)\},S_5=\{(v_4,0.1,0.6,0.9),(v_5,0.9,0.3,0.2)\},S_6=\{(v_3,0.8,0.7,0.3),(v_4,0.1,0.6,0.9)\}\}$.

Therefore, neutrosophic graphs corresponding to C_1 and C_2 are isomorphic to each other as manifested in Fig 4, but C_1 and C_2 are not parallel.

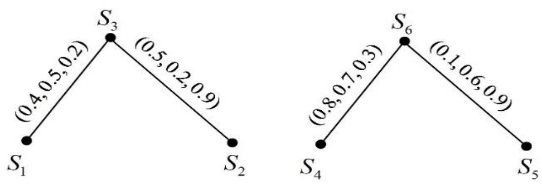

Fig 4. A neutrosophic graph represents C_1 and C_2.

C. Parallel classes in bipolar neutrosophic setting

Definition 7. Let *V* be a US, $BNF(V)$ be the class of all bipolar neutrosophic sets of *V*, and $C_1,C_2\subseteq BNF(V)$. We say C_1 is parallel to C_2 (denoted by $X\sim Y$, where X,Y are bipolar neutrosophic sets in $BNF(V)$, each element in C_1 is a bipolar neutrosophic partial subset (BNPS) of *X*, and each element in C_2 is a bipolar neutrosophic partial subset of *Y*), if there exists a bijective bipolar neutrosophic mapping $F:X\to Y$ such that for any $c_1\in C_1$, we have $F(c_1)=c_2$ and $c_2\in C_2$. Let $X=Y$. Then C_1 is parallel to C_2 if there exists a bijective bipolar neutrosophic mapping $F:X\to X$ such that $F(C_1)=C_2$.

Example 11. Let $V=\{v_1,v_2,v_3\}$ be a universal set, $X=\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3),(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9),(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)\}$ be a bipolar neutrosophic set on V, $C_1=\{\{(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)\},\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3),(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9)\},\{(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3),(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3)\}\}$ and $C_2=\{\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3)\},\{(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9),(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)\},\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3),(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9)\}\}$. Obviously, all the elements in C_1 and C_2 are the BNPS of *X*. There is a bijective bipolar neutrosophic mapping $F:X\to X$ such that

$F(\{(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)\})=\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3)\}$,

$F(\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3),(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9)\})$
$=\{(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9),(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)\}$,

$F(\{(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3),(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3)\})$
$=\{(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3),(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9)\}$.

Consequently, C_1 and C_2 are parallel.

In view of Example 11, it is clear that the essence of parallelism is the one-to-one correspondence between the elements in the bipolar neutrosophic set. In this instance, we can see the correspondence between the following elements in *X*:

$(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3)\xrightarrow{F}(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9)$,

$(v_2,0.4,0.8,0.3,-0.7,-0.4,-0.9)\xrightarrow{F}(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)$,

$(v_3,0.1,0.6,0.8,-0.8,-0.4,-0.3)\xrightarrow{F}(v_1,0.3,0.4,0.2,-0.8,-0,5,-0.3)$.

Next, it is presented that any class of bipolar neutrosophic sets can be denoted by a general BNG in view of operation

$\wedge$ between bipolar neutrosophic set classes. Let $S_1,\cdots,S_n$ be bipolar neutrosophic sets, $C=\{S_1,\cdots,S_n\}$ be the class of these BNSs and *G* be a BNG corresponding to *C*. BNG *G* is constructed as follows: each vertex in *G* corresponds to a bipolar neutrosophic set among $S_1,\cdots,S_n$, and there are $|S_i \wedge S_j|$ edges between vertices S_i and S_j. Example 12 is presented to illustrate such kind of bipolar neutrosophic graph.

Example 12. Consider the classes of BNSs

$$C_1=\{S_1=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9)\}, S_2=\{(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2)\},$$
$$S_3=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2)\}\},$$
$$C_2=\{S_4=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2),(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8)\},$$
$$S_5=\{(v_4,0.1,0.6,0.9,-0.8,-0.3.-0.2),(v_5,0.9,0.3,0.2,-0.1,-0.8,-0.9)\}, S_6=\{(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8),(v_4,0.1,0.6,0.9,-0.8,-0.3.-0.2)\}\},$$
$$C_3=\{S_7=\{(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8),(v_6,0.3,0.6,0.9,-0.4,-0.3,-0.1)\}, S_8=\{(v_7,0.7,0.4,0.1,-0.5,-0.7,-0.9),(v_8,0.2,0.5,0.6,-0.7,-0.5,-0.3),$$
$$(v_9,0.3,0.7,0.7,-0.5.-0.4,-0.3)\}, S_9=\{(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8),(v_9,0.3,0.7,0.7,-0.5.-0.4,-0.3),(v_{10},0.6,0.5,0.5,-0.5,-0.6,-0.7)\}\}.$$

to represent the several bipolar neutrosophic graphs as determined in Fig 5.

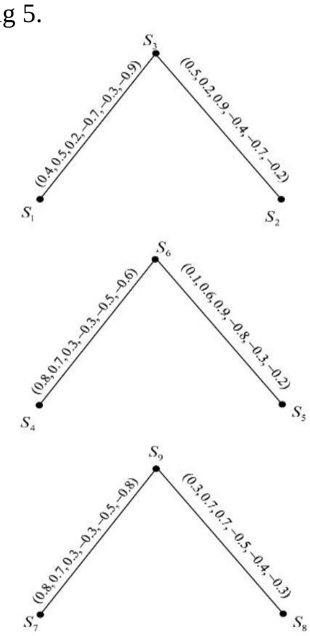

Fig 5. A bipolar neutrosophic graph represents C_1, C_2 and C_3.

Definition 8. Let $C=\{C_i : i\in I\}$ be a collection of all classes of a BNS *X*, and thus C_i can be represented by the BNG *G* ($\{|\vee C_i| : i \in I\}$ which is stated as the graph number of a BNG *G*).

Example 13. Discuss C_1, C_2 and C_3 as given in Example 12, we yield

$$\vee C_1=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2)\},$$
$$\vee C_2=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2),(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8),(v_4,0.1,0.6,0.9,-0.8,-0.3.-0.2),(v_5,0.9,0.3,0.2,-0.1,-0.8,-0.9)\},$$
$$\vee C_3=\{(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8),(v_6,0.3,0.6,0.9,-0.4,-0.3,-0.1),(v_7,0.7,0.4,0.1,-0.5,-0.7,-0.9),(v_8,0.2,0.5,0.6,-0.7,-0.5,-0.3),(v_9,0.3,0.7,0.7,-0.5.-0.4,-0.3),(v_{10},0.6,0.5,0.5,-0.5,-0.6,-0.7)\}.$$

The numbers of a bipolar neutrosophic graph $|\vee C_1|$, $|\vee C_2|$ and $|\vee C_3|$ are 2, 5 and 6 respectively.

Next, we show that any BNG *G* can be expressed by a class of BNSs, and $\wedge$ is re-formulated to an operator *N* for vertices of BNGs.

Definition 9. Let *G* be a BNG and v_i, v_j be two vertices of *G*. Suppose that v_i and v_j correspond to bipolar neutrosphic sets S_i and S_j respectively, then $N(v_i,v_j)=|S_i \wedge S_j|$. Note that $N(v_i,X)=|S_i|$ (resp. $N(v_j,X)=|S_j|$) if S_i (resp. S_j) is a bipolar neutrosophic subset of *X*.

Example 14. If $S_i=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9)\}$ and $S_j=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2)\}$, then $S_i \wedge S_j=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9)\}$ and $N(S_i,S_j)=|S_i \wedge S_j|=1$.

Theorem 3. If G_1 and G_2 are two bipolar neutrosophic graphs corresponding to two parallel classes C_1 and C_2, then $G_1 \cong G_2$.

Similar to Theorem 1 and Theorem 2, the converse of the above theorem may not hold in general since each BNG can express many classes, and the Example 15 is used to explain it.

Example 15. Consider the classes of BNSs

$$C_1=\{S_1=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9)\},$$
$$S_2=\{(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2)\},$$
$$S_3=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),$$
$$(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2)\}\},$$
$$C_2=\{S_4=\{(v_1,0.4,0.5,0.2,-0.7,-0.3,-0.9),$$
$$(v_2,0.5,0.2,0.9,-0.4,-0.7,-0.2),$$
$$(v_3,0.8,0.7,0.3,-0.3,-0.5,-0.8)\},$$
$$S_5=\{(v_4,0.1,0.6,0.9,-0.8,-0.3.-0.2),$$
$$(v_5,0.9,0.3,0.2,-0.1,-0.8,-0.9)\},$$

$$S_6 = \{(v_3, 0.8, 0.7, 0.3, -0.3, -0.5, -0.8), (v_4, 0.1, 0.6, 0.9, -0.8, -0.3. -0.2)\}\}.$$

Therefore, bipolar neutrosophic graphs corresponding to C_1 and C_2 are isomorphic to each other as manifested in Fig 6, but C_1 and C_2 are not parallel.

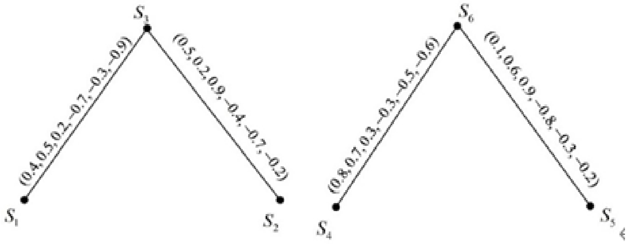

Fig 6. A bipolar neutrosophic graph represents C_1 and C_2.

D. Parallel classes in interval-valued fuzzy setting

Definition 10. Let V be a US, $IVF(V)$ be the class of all IVFSs of V, and $C_1, C_2 \subseteq IVF(V)$. We say C_1 is parallel to C_2 (denoted by $X \sim Y$, where X, Y are IVFSs in $IVF(V)$, each element in C_1 is an interval-valued partial fuzzy subset (IVPFS) of X, and each element in C_2 is an interval-valued partial fuzzy subset of Y), if there exists a bijective interval-valued mapping (BIVM) $F: X \to Y$ such that for any $c_1 \in C_1$, we have $F(c_1) = c_2$ and $c_2 \in C_2$. Let $X = Y$. Then C_1 is parallel to C_2 if there exists a BIVM $F: X \to X$ such that $F(C_1) = C_2$.

Example 16. Let $V = \{v_1, v_2, v_3\}$ be a US, $X = \{(v_1,[0.3, 0.7]), (v_2,[0.4, 0.8]), (v_3,[0.1, 0.5])\}$ be an IVFS on V. Set

$$C_1 = \{\{(v_3,[0.1,0.5])\}, \{(v_1,[0.3,0.7]), (v_2,[0.4,0.8])\}, \{(v_3,[0.1,0.5]), (v_1,[0.3,0.7])\}\},$$

$$C_2 = \{\{(v_1,[0.3,0.7])\}, \{(v_2,[0.4,0.8]), (v_3,[0.1,0.5])\}, \{(v_1,[0.3,0.7]), (v_2,[0.4,0.8])\}\}.$$

Obviously, all the elements in C_1 and C_2 are the IVPFS of X. There is a BIVM $F: X \to X$ such that

$$F(\{(v_3,[0.1,0.5])\}) = \{(v_1,[0.3,0.7])\},$$

$$F(\{(v_1,[0.3,0.7]), (v_2,[0.4,0.8])\}) = \{(v_2,[0.4,0.8]), (v_3,[0.1,0.5])\},$$

$$F(\{(v_3,[0.1,0.5]), (v_1,[0.3,0.7])\}) = \{(v_1,[0.3,0.7]), (v_2,[0.4,0.8])\}.$$

Hence, C_1 and C_2 are parallel.

By means of Example 16, it is obvious that the essence of parallelism is the one-to-one correspondence between the elements in the IVFS. In this instance, we can see the correspondence between the following elements in X:

$$(v_1,[0.3,0.7]) \xrightarrow{F} (v_2,[0.4,0.8]),$$

$$(v_2,[0.4,0.8]) \xrightarrow{F} (v_3,[0.1,0.5]),$$

$$(v_3,[0.1,0.5]) \xrightarrow{F} (v_1,[0.3,0.7]).$$

Next, it is presented that any class of IVFSs can be denoted by a general IVFG in view of operation $\wedge$ between IVFS classes. Let $S_1, \cdots, S_n$ be IVFSs, $C = \{S_1, \cdots, S_n\}$ be a class of these IVFSs and G be an IVFG corresponding to C. IVFG G is constructed as follows: each vertex in G corresponds to an interval-valued fuzzy set among $S_1, \cdots, S_n$, and there are $|S_i \wedge S_j|$ edges between vertices S_i and S_j. The following instance is applied to illustrate such a kind of IVFG.

Example 17. Consider the classes of IVFSs

$$C_1 = \{S_1 = \{(v_1,[0.6,0.8])\}, S_2 = \{(v_2,[0.3,0.7])\}, S_3 = \{(v_1,[0.6,0.8]), (v_2,[0.3,0.7])\}\},$$

$$C_2 = \{S_4 = \{(v_1,[0.6,0.8]), (v_2,[0.3,0.7]), (v_3,[0.4,0.5])\}, S_5 = \{(v_4,[0.2,0.6]), (v_5,[0.6,0.8])\}, S_6 = \{(v_3,[0.4,0.5]), (v_4,[0.2,0.6])\}\},$$

$$C_3 = \{S_7 = \{(v_3,[0.4,0.5]), (v_6,[0.3,0.6])\}, S_8 = \{(v_7,[0.1,0.7]), (v_8,[0.2,0.9]), (v_9,[0.6,0.7])\}, S_9 = \{(v_3,[0.4,0.5]), (v_9,[0.6,0.7]), (v_{10},[0.6,0.8])\}\}.$$

to represent the same IVFGs as determined in Fig 7.

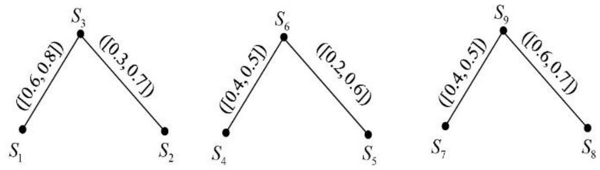

Fig 7. An IVFG represents C_1, C_2 and C_3.

Definition 11. Let $C = \{C_i : i \in I\}$ be a collection of all classes of an interval-valued fuzzy set X, and thus C_i can be represented by the same interval-valued fuzzy graph G ($\{|\vee C_i| : i \in I\}$ which is labeled as the graph number of an interval-valued fuzzy graph G).

Example 18. Focusing on C_1, C_2 and C_3 as given in Example 17, we yield

$$\vee C_1 = \{(v_1,[0.6,0.8]), (v_2,[0.3,0.7])\},$$

$$\vee C_2 = \{(v_1,[0.6,0.8]), (v_2,[0.3,0.7]), (v_3,[0.4,0.5]), (v_4,[0.2,0.6]), (v_5,[0.6,0.8])\},$$

$$\vee C_3 = \{(v_3,[0.4,0.5]), (v_6,[0.3,0.6]), (v_7,[0.1,0.7]), (v_8,[0.2,0.9]), (v_9,[0.6,0.7]), (v_{10},[0.6,0.8])\}.$$

The number of an interval-valued fuzzy graph $|\vee C_1|$, $|\vee C_2|$ and $|\vee C_3|$ are 2, 5 and 6 respectively.

Next, we show that any interval-valued fuzzy graph G can be expressed by a class of interval-valued fuzzy sets, and $\wedge$ is formualted as an operator N for vertices of interval-valued fuzzy graphs.

Definition 12. Let G be an interval-valued fuzzy graph and v_i, v_j be two vertices of G. Suppose that v_i and v_j

correspond to interval-valued fuzzy sets S_i and S_j respectively, then $N(v_i, v_j) = |S_i \wedge S_j|$. Note that $N(v_i, X) = |S_i|$ (resp. $N(v_j, X) = |S_j|$) if S_i (resp. S_j) is an interval-valued fuzzy subset of *X*.

Example 19. If $S_i = \{(v_1,[0.6,0.8])\}$ and $S_j = \{(v_1,[0.6,0.8]), (v_2,[0.3,0.7])\}$, then $S_i \wedge S_j = \{(v_1,[0.6,0.8])\}$ and $N(S_i, S_j) = |S_i \wedge S_j| = 1$.

Theorem 4. If G_1 and G_2 are two interval-valued fuzzy graphs corresponding to two parallel classes C_1 and C_2, then $G_1 \cong G_2$.

The converse of Theorem 4 may not hold in general since each interval-valued fuzzy graph can express many classes, and the Example 20 is used to explain it.

Example 20. Consider the classes of interval-valued fuzzy sets

$C_1 = \{S_1 = \{(v_1,[0.6,0.8])\}, S_2 = \{(v_2,[0.3,0.7])\}, S_3 = \{(v_1,[0.6,0.8]),(v_2,[0.3,0.7])\}\}$,

$C_2 = \{S_4 = \{(v_1,[0.6,0.8]),(v_2,[0.3,0.7]),(v_3,[0.4,0.5])\}, S_5 = \{(v_4,[0.2,0.6]),(v_5,[0.6,0.8])\}, S_6 = \{(v_3,[0.4,0.5]),(v_4,[0.2,0.6])\}\}$.

Therefore, interval-valued fuzzy graphs corresponding to C_1 and C_2 are isomorphic to each other as manifested in Fig 8, but C_1 and C_2 are not parallel.

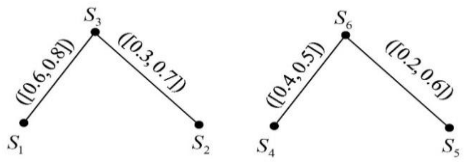

Fig 8. An IVFG represents C_1 and C_2.

E. Parallel classes in bipolar interval-valued fuzzy setting

Definition 13. Let *V* be a US, $BIVF(V)$ be the class of all BIVFSs of *V*, and $C_1, C_2 \subseteq IVF(V)$. We say C_1 is parallel to C_2 (denoted by $X \sim Y$, where X, Y are bipolar interval-valued fuzzy sets in $IVF(V)$, each element in C_1 is a bipolar interval-valued partial fuzzy subset (BIVPFS) of *X*, and each element in C_2 is an IVPFS of *Y*), if there exists a bijective bipolar interval-valued mapping (BBIVM) $F: X \to Y$ such that for any $c_1 \in C_1$, we have $F(c_1) = c_2$ and $c_2 \in C_2$. Let $X = Y$. Then C_1 is parallel to C_2 if there exists a BBIVM $F: X \to X$ such that $F(C_1) = C_2$.

Example 21. Let $V = \{v_1, v_2, v_3\}$ be a US, and $X = \{(v_1,[0.3,0.7],[-0.9,-0.4]),(v_2,[0.4,0.8],[-0.8,-0.6]), (v_3,[0.1,0.5],[-0.5,-0.2])\}$ be a bipolar interval-valued fuzzy set on V . Set

$C_1 = \{\{(v_3,[0.1,0.5],[-0.5,-0.2])\},\{(v_1,[0.3,0.7],[-0.9,-0.4]),(v_2,[0.4,0.8],[-0.8,-0.6])\}, \{(v_3,[0.1,0.5],[-0.5,-0.2]),(v_1,[0.3,0.7],[-0.9,-0.4])\}\}$,

$C_2 = \{\{(v_1,[0.3,0.7],[-0.9,-0.4])\},\{(v_2,[0.4,0.8],[-0.8,-0.6]),(v_3,[0.1,0.5],[-0.5,-0.2])\}, \{(v_1,[0.3,0.7],[-0.9,-0.4]),(v_2,[0.4,0.8],[-0.8,-0.6])\}\}$.

Obviously, all the elements in C_1 and C_2 are the IVPFSs of X . There is a BVM $F: X \to X$ such that

$F(\{(v_3,[0.1,0.5],[-0.5,-0.2])\}) = \{(v_1,[0.3,0.7],[-0.9,-0.4])\}$,

$F(\{(v_1,[0.3,0.7],[-0.9,-0.4]),(v_2,[0.4,0.8],[-0.8,-0.6])\}) = \{(v_2,[0.4,0.8],[-0.8,-0.6]),(v_3,[0.1,0.5],[-0.5,-0.2])\}$,

$F(\{(v_3,[0.1,0.5],[-0.5,-0.2]),(v_1,[0.3,0.7],[-0.9,-0.4])\}) = \{(v_1,[0.3,0.7],[-0.9,-0.4]),(v_2,[0.4,0.8],[-0.8,-0.6])\}$.

Thus, C_1 and C_2 are parallel.

In light of Example 21, it is clear that the essence of parallelism is the one-to-one correspondence between the elements in the BIVFS. In this instance, we can see the correspondence between the following elements in *X*:

$(v_1,[0.3,0.7],[-0.9,-0.4]) \xrightarrow{F} (v_2,[0.4,0.8],[-0.8,-0.6])$,

$(v_2,[0.4,0.8],[-0.8,-0.6]) \xrightarrow{F} (v_3,[0.1,0.5],[-0.5,-0.2])$,

$(v_3,[0.1,0.5],[-0.5,-0.2]) \xrightarrow{F} (v_1,[0.3,0.7],[-0.9,-0.4])$.

Next, it is presented that any class of bipolar interval-valued fuzzy sets can be denoted by a general bipolar interval-valued fuzzy graph in view of operation $\wedge$ between bipolar interval-valued fuzzy set classes. Let $S_1, \cdots, S_n$ be bipolar interval-valued fuzzy sets, $C = \{S_1, \cdots, S_n\}$ be a class of these bipolar interval-valued fuzzy sets and *G* be a BIVFG corresponding to *C*. BIVFG *G* is constructed as follows: each vertex in *G* corresponds to a BIVFS among $S_1, \cdots, S_n$, and there are $|S_i \wedge S_j|$ edges between vertices S_i and S_j. The following instance is applied to illustrate such a kind of bipolar interval-valued fuzzy graph.

Example 22. Consider the classes of bipolar interval-valued fuzzy sets

$C_1 = \{S_1 = \{(v_1,[0.6,0.8],[-0.6,-0.4])\}, S_2 = \{(v_2,[0.3,0.7],[-0.4,-0.1])\}, S_3 = \{(v_1,[0.6,0.8],[-0.6,-0.4]),(v_2,[0.3,0.7],[-0.4,-0.1])\}\}$,

$C_2 = \{S_4 = \{(v_1,[0.6,0.8],[-0.6,-0.4]),(v_2,[0.3,0.7],[-0.4,-0.1]), (v_3,[0.4,0.5],[-0.5,-0.3])\}, S_5 = \{(v_4,[0.2,0.6],[-0.9,-0.7]),(v_5,[0.6,0.8],[-0.6,-0.2])\}, S_6 = \{(v_3,[0.4,0.5],[-0.5,-0.3]),(v_4,[0.2,0.6],[-0.9,-0.7])\}\}$,

$C_3 = \{S_7 = \{(v_3,[0.4,0.5],[-0.5,-0.3]),(v_6,[0.3,0.6],[-0.5,-0.4])\}, S_8 = \{(v_7,[0.1,0.7],[-0.9,-0.8]), (v_8,[0.2,0.9],[-0.8,-0.4]),(v_9,[0.6,0.7],[-0.6,-0.3])\}, S_9 = \{(v_3,[0.4,0.5],[-0.5,-0.3]),(v_9,[0.6,0.7],[-0.6,-0.3]),(v_{10},[0.6,0.8],[-0.7,-0.6])\}\}$.

to represent the same bipolar interval-valued fuzzy graphs as determined in Fig 9.

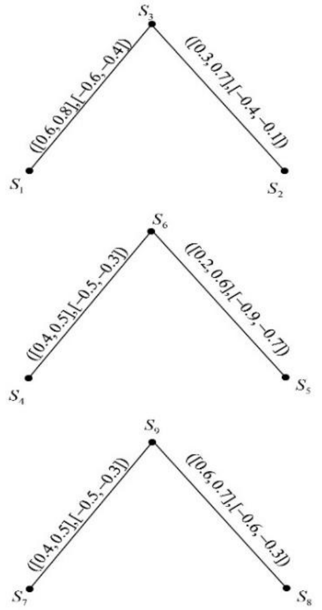

Fig 9. A bipolar interval-valued fuzzy graph represents C_1, C_2 and C_3.

Definition 14. Let $C=\{C_i : i\in I\}$ be a collection of all classes of a BIVFS *X*, and thus C_i can be formulated by the same BIVFG *G* ($\{|\vee C_i| : i\in I\}$ is formulated as the graph number of a BIVFG *G*).

Example 23. Focusing on C_1, C_2 and C_3 as given in Example 22, we yield

$$\vee C_1=\{(v_1,[0.6,0.8],[-0.6,-0.4]),(v_2,[0.3,0.7],[-0.4,-0.1])\},$$

$$\vee C_2=\{(v_1,[0.6,0.8],[-0.6,-0.4]),(v_2,[0.3,0.7],[-0.4,-0.1]),(v_3,[0.4,0.5],[-0.5,-0.3]),(v_4,[0.2,0.6],[-0.9,-0.7]),(v_5,[0.6,0.8],[-0.6,-0.2])\},$$

$$\vee C_3=\{(v_3,[0.4,0.5],[-0.5,-0.3]),(v_6,[0.3,0.6],[-0.5,-0.4]),(v_7,[0.1,0.7],[-0.9,-0.8]),(v_8,[0.2,0.9],[-0.8,-0.4]),(v_9,[0.6,0.7],[-0.6,-0.3]),(v_{10},[0.6,0.8],[-0.7,-0.6])\}.$$

The numbers of a BIVFG $|\vee C_1|$, $|\vee C_2|$ and $|\vee C_3|$ are 2, 5 and 6 respectively.

Next, we show that any BIVFG *G* can be expressed by a class of BIVFSs. Again, $\wedge$ is re-formulated to an operator *N* for vertices of BIVFGs.

Definition 15. Let *G* be a BIVFG and v_i, v_j be two vertices of *G*. Suppose that v_i and v_j correspond to bipolar interval-valued fuzzy sets S_i and S_j respectively, then $N(v_i,v_j)=|S_i\wedge S_j|$. Note that $N(v_i,X)=|S_i|$ (resp. $N(v_j,X)=|S_j|$) if S_i (resp. S_j) is a BIVFS of *X*.

Example 24. If $S_i=\{(v_1,[0.6,0.8],[-0.6,-0.4])\}$ and $S_j=\{(v_1,[0.6,0.8],[-0.6,-0.4]),\ (v_2,[0.3,0.7],\ [-0.4,-0.1])\}$, th $S_i\wedge S_j=\{(v_1,[0.6,0.8],[-0.6,-0.4])\}$ and $N(S_i,S_j)=|S_i\wedge S_j|=1$.

Theorem 5. If G_1 and G_2 are BIVFGs corresponding to two parallel classes C_1 and C_2, then $G_1\cong G_2$.

The converse of Theorem5 may not hold in general since each bipolar interval-valued fuzzy graph can express many classes, and the Example 25 is used to explain it.

Example 25. Consider the classes of bipolar interval-valued fuzzy sets

$$C_1=\{S_1=\{(v_1,[0.6,0.8],[-0.6,-0.4])\},\ S_2=\{(v_2,[0.3,0.7],[-0.4,-0.1])\},\ S_3=\{(v_1,[0.6,0.8],[-0.6,-0.4]),(v_2,[0.3,0.7],[-0.4,-0.1])\}\},$$

$$C_2=\{S_4=\{(v_1,[0.6,0.8],[-0.6,-0.4]),(v_2,[0.3,0.7],[-0.4,-0.1]),(v_3,[0.4,0.5],[-0.5,-0.3])\},$$
$$S_5=\{(v_4,[0.2,0.6],[-0.9,-0.7]),(v_5,[0.6,0.8],[-0.6,-0.2])\},$$
$$S_6=\{(v_3,[0.4,0.5],[-0.5,-0.3]),(v_4,[0.2,0.6],[-0.9,-0.7])\}\}.$$

Therefore, bipolar interval-valued fuzzy graphs corresponding to C_1 and C_2 are isomorphic to each other as manifested in Fig 10, but C_1 and C_2 are not parallel.

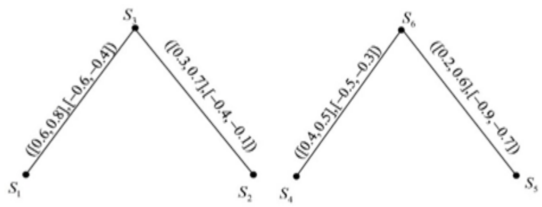

Fig 10. A bipolar interval-valued fuzzy graph represents C_1 and C_2.

IV. Fuzzy Topological Graphs and Algebraic Operations in Distinct Settings

We generate fuzzy topological spaces (FTSs) in view of FS graphs in three kinds of settings respectively. Several algebraic operations on BFTGs (resp. NTGs, BNTGs, IVFTGs and BIVFTGs) such as $\vee,\wedge,\le$ are defined on vertices by $v_{S_1}\vee v_{S_2}\vee\cdots=v_{S_1\vee S_2\cdots}$, $v_{S_1}\wedge v_{S_2}=v_{S_1\wedge S_2}$ and $v_{S_1}\le v_{S_2}$ if $S_1\le S_2$. The following results and examples are divided into five settings respectively.

A. *Algebraic operations in bipolar fuzzy setting*

Definition 16. A bipolar fuzzy topology on a bipolar fuzzy set $BS=\{(v_1,a_1,b_1),\cdots,(v_n,a_n,b_n)\}$ ($0\le a_1,\cdots,a_n\le 1$ and $-1\le b_1,\cdots,b_n\le 0$) can be established in terms of a bipolar fuzzy graph *G* such that each vertex in *G* is a class in *BS* and the edge number between two vertices is the cardinality of the intersection of corresponding two classes of *BS* and the positive degree (resp. negative degree) of edges is the positive degree (negative degree) of each vertex in its intersection. The subscripts of bipolar fuzzy pseudograph (with loops), discrete bipolar topological graph (no loop) and simple bipolar fuzzy graph (one edge between two adjacent vertices) are marked by *p*, *d* and *s* respectively.

Theorem 4. Let $|E_p(G)|$, $|E_d(G)|$ and $|E_s(G)|$ be the edge number of a bipolar fuzzy pseudograph, discrete bipolar topological graph and simple BFG on the bipolar fuzzy set $BS=\{(v_1,a_1,b_1),\cdots,(v_n,a_n,b_n)\}$, respectively. Then, $|E_p(G)|=n2^{n-2}(2^{n-1}-1)+n2^{n-1}$, $|E_d(G)|=n2^{n-2}(2^{n-1}-1)$ and $|E_s(G)|=\dfrac{2^{2n}-2^n-3^n+1}{2}$.

Example 26. Let $BS=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2)\}$, then $|E_p(G)|=30$, $|E_d(G)|=18$ and $|E_s(G)|=15$.

A BFTG deduced in terms of a bipolar fuzzy graph can be stated by the following result.

Theorem 6. Let *G* be a BFG that meets the conditions:

- *G* has a unique isolated vertex which is represented by $\varnothing$;
- There is a vertex *v* adjacent to other vertices in $G-\{\varnothing\}$, and $\mu_R^+(v_i,v)\le\mu_R^+(v_i,BS)\le\mu_R^+(v,BS)$ and $\mu_R^-(v_i,v)\ge\mu_R^-(v_i,BS)\ge\mu_R^-(v,BS)$ for any $v_i\in V(G)-\{\varnothing\}$;
- Let v_1 and v_2 be any two distinct vertices. We have $v_1\wedge v_2, v_1\vee v_2\in V(G)$.

Then, the class τ of vertices is a BFTG.

The next theorem is used to compute the size of *G* by means of fuzzy topological graph τ.

Theorem 7. The edge number of a BFTG is expressed by a bipolar fuzzy topology $\tau=\{\varnothing,\{(v_1,a_1,b_1)\},\{(v_1,a_1,b_1),(v_2,a_2,b_2)\},\cdots,X=\{(v_1,a_1,b_1),\cdots,(v_n,a_n,b_n)\}\}$.

The tricks of proof Theorem 6 and Theorem 7 are similar to what's described in Atef et al. [30] and we skip it here.

Example 27. Let $BS=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2)\}$ with a bipolar fuzzy topological space $\tau=\{\varnothing,S_1=\{(v_1,0.6,-0.5)\},S_2=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6)\},X=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2)\}\}$. See Fig 11 for the BFTG, and the positive and negative degrees of edges are 2.1 and -2.1 respectively.

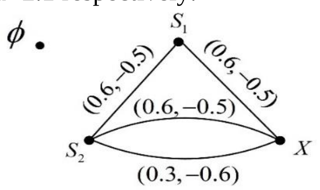

Fig 11. A bipolar topological graph in Example 27.

Each BFTG can be expressed by a BFG, but the reverse may not be true. We present the following two examples to explain it in detail.

Example 28. Let *G* be a BFG drawn in Fig 12, and a BFTG be constructed in terms of the following schemes: the only isolated vertex is formulated by $\varnothing$; the vertex has maximum degree four represented by $X=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2)\}$. Since $|S_2\wedge X|=2$, the vertex represents S_2 is $\{(v_1,0.6,-0.5),(v_2,0.3,-0.6)\}$. Similarly, $|S_3\wedge X|=2$ and the corresponding vertex is denoted by $S_3=\{(v_1,0.6,-0.5),(v_3,0.1,-0.2)\}$. $|S_2\wedge S_3|=1$, and the MF value of edge connect S_2 and S_3 is $(0.6,-0.5)$. $|S_1\wedge X|=|S_1\wedge S_2|=|S_1\wedge S_3|=1$ and hence S_1 is denoted by $\{(v_1,0.6,-0.5)\}$. Therefore, $\tau=\{\varnothing,S_1=\{(v_1,0.6,-0.5)\},S_2=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6)\},S_3=\{(v_1,0.6,-0.5),(v_3,0.1,-0.2)\},X=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2)\}$ is a bipolar fuzzy topology and this graph is called a BFTG.

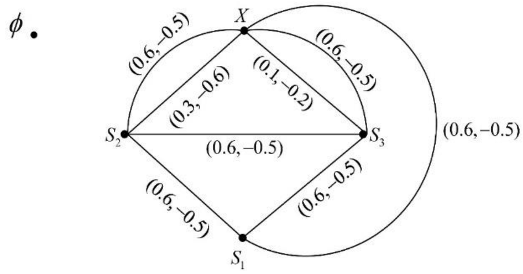

Fig 12. A bipolar fuzzy graph which is a bipolar fuzzy topological graph.

Example 29. The graph *G* in Fig 13 is not a BFTG, where

$$X=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2),(v_4,0.5,-0.5)\},$$
$$S_1=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6)\},$$
$$S_2=\{(v_1,0.6,-0.5),(v_3,0.1,-0.2)\},$$
$$|S_1\wedge X|=|S_2\wedge X|=2,\ |S_1\wedge S_2|=1,$$
$$S_1\wedge X=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6)\},$$
$$S_2\wedge X=\{(v_1,0.6,-0.5),(v_3,0.1,-0.2)\},$$
$$S_1\wedge S_2=\{(v_1,0.6,-0.5)\}.$$

However, $\tau=\{\varnothing,S_1=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6)\},S_2=\{(v_1,0.6,-0.5),(v_3,0.1,-0.2)\},X=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2),(v_4,0.5,-0.5)\}\}$ is not a bipolar fuzzy topology because $S_1\cup S_2=\{(v_1,0.6,-0.5),(v_2,0.3,-0.6),(v_3,0.1,-0.2)\}\notin\tau$.

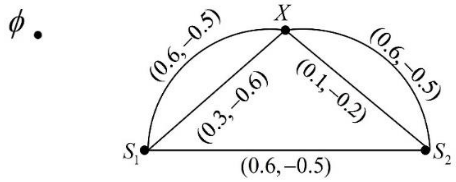

Fig 13. A BFG which is not a BFTG.

B. Algebraic operations in neutrosophic setting

Definition 17. A neutrosophic topology on a neutrosophic set $NS=\{(v_1,a_1,b_1,c_1),\cdots,(v_n,a_n,b_n,c_n)\}$ (here $0\le a_1,\cdots,a_n\le 1$ are membership values of truthness, $0\le b_1,\cdots,b_n\le 1$ are membership values of indeterminacy, and $0\le c_1,\cdots,c_n\le 1$ are membership values of falsity) can be established in terms of a NG *G* such that each vertex in *G* is a class in *NS* and edge number between two vertices is the cardinality of intersection of corresponding to two classes of *NS* and the truthness degree (resp. indeterminacy degree and falsity degree) of edges is the truthness degree (resp. indeterminacy degree and falsity degree) of each vertex in its intersection. The neutrosophic pseudograph (with loops), discrete neutrosophic topological graph (no loop) and simple neutrosophic graph (one edge between two adjacent vertices) mark their subscripts by *NP*, *ND* and *NS* respectively.

Theorem 8. Let $|E_{NP}(G)|$, $|E_{ND}(G)|$ and $|E_{NS}(G)|$ be the edge number of a neutrosophic pseudograph, discrete neutrosophic topological graph and simple neutrosophic graph on the bipolar fuzzy set $NS=\{(v_1,a_1,b_1,c_1),\cdots,(v_n,a_n,b_n,c_n)\}$, respectively. Then, $|E_{NP}(G)|=n2^{n-2}(2^{n-1}-1)+n2^{n-1}$, $|E_{ND}(G)|=n2^{n-2}(2^{n-1}-1)$ and $|E_{NS}(G)|=\frac{2^{2n}-2^n-3^n+1}{2}$.

Example 30. Let $NS=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8)\}$. Then $|E_{NP}(G)|=30$, $|E_{ND}(G)|=18$ and $|E_{NS}(G)|=15$.

A NTG deduced in terms of a neutrosophic graph can be stated by the following result.

Theorem 9. Let *G* be a NG that meets the conditions:

- *G* has a unique isolated vertex which is represented by $\varnothing$;
- There is a vertex *v* adjacent to other vertices in $G-\{\varnothing\}$, and $\mu_{\mathrm{T}}(v_i,v)\le\mu_{\mathrm{T}}(v_i,NS)\le\mu_{\mathrm{T}}(v,NS)$, $\mu_{\mathrm{N}}(v_i,v)\ge\mu_{\mathrm{N}}(v_i,NS)\ge\mu_{\mathrm{N}}(v,NS)$ and $\mu_{\mathrm{F}}(v_i,v)\ge\mu_{\mathrm{F}}(v_i,NS)\ge\mu_{\mathrm{F}}(v,NS)$ for any $v_i\in V(G)-\{\varnothing\}$;
- Let v_1 and v_2 be any two distinct vertices. We have $v_1\wedge v_2, v_1\vee v_2\in V(G)$.

Then, the class τ of vertices is a NTG.

The next theorem is used to compute the size of *G* by means of neutrosophic topological graph τ.

Theorem 9. The edge number of a NTG is expressed by a neutrosophic topology $\tau=\{\varnothing,\{(v_1,a_1,b_1,c_1)\},\{(v_1,a_1,b_1,c_1),(v_2,a_2,b_2,c_2)\},\cdots,X=\{(v_1,a_1,b_1,c_1),\cdots,(v_n,a_n,b_n,c_n)\}\}$.

Example 31. Let $NS=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8)\}$ with a neutrosophic topological space $\tau=\{\varnothing,S_1=\{(v_1,0.6,0.5,0.4)\},S_2=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5)\},X=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8)\}\}$. See Fig 14 for the neutrosophic topological graph, and the truthness degree, indeterminacy degree and falsity degree of edges are 2.1, 2.1 and 1.7 respectively.

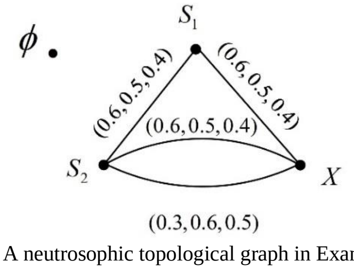

Fig 14. A neutrosophic topological graph in Example 31.

Each neutrosophic topological graph can be expressed by a neutrosophic graph, but the reverse may not be true. We present the following two examples to explain it in detail.

Example 32. Let *G* be a neutrosophic graph drawn in Fig 15, and a neutrosophic topological graph be constructed in terms of the following schemes: the only isolated vertex is formulated by $\varnothing$; the vertex has the maximum degree four represented by $X=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8)\}$. Since $|S_2\wedge X|=2$, the vertex represents S_2 is $\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5)\}$. Similarly, $|S_3\wedge X|=2$ and the corresponding vertex is denoted by $S_3=\{(v_1,0.6,0.5,0.4),$. $|S_2\wedge S_3|=1$, and the MF value of the edge connecting S_2 and S_3 is (0.6, 0.5, 0.4). $|S_1\wedge X|=|S_1\wedge S_2|=|S_1\wedge S_3|=1$ and hence S_1 is denoted by $\{(v_1,0.6,0.5,0.4)\}$. Therefore, $\tau=\{\varnothing,S_1=\{(v_1,0.6,0.5,0.4)\},S_2=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5)\},S_3=\{(v_1,0.6,0.5,0.4),(v_3,0.1,0.2,0.8)\},X=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8)\}\}$ is a neutrosophic topology and this graph is called a neutrosophic topological graph.

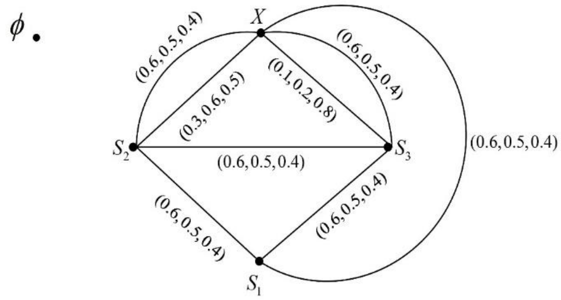

Fig 15. A NG which is a NTG.

Example 33. The graph *G* in Fig 16 is not a neutrosophic topological graph, where

$X=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8),(v_4,0.5,0.5,0.5)\}$,

$$S_1=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5)\},$$

$$S_2=\{(v_1,0.6,0.5,0.4),(v_3,0.1,0.2,0.8)\},$$

$$|S_1\wedge X|=|S_2\wedge X|=2,\ |S_1\wedge S_2|=1,$$

$$S_1\wedge X=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5)\},$$

$$S_2\wedge X=\{(v_1,0.6,0.5,0.4),(v_3,0.1,0.2,0.8)\},$$

$$S_1\wedge S_2=\{(v_1,0.6,0.5,0.4)\}.$$

However, $\tau=\{\varnothing,S_1=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5)\},S_2=\{(v_1,0.6,0.5,0.4),(v_3,0.1,0.2,0.8)\},X=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8),(v_4,0.5,0.5,0.5)\}\}$ is not a neutrosophic topology because $S_1\cup S_2=\{(v_1,0.6,0.5,0.4),(v_2,0.3,0.6,0.5),(v_3,0.1,0.2,0.8)\}\notin\tau$.

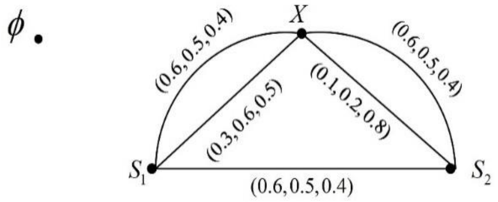

Fig 16. A neutrosophic graph which is not a neutrosophic topological graph.

C. Algebraic operations in bipolar neutrosophic setting

Definition 18. A bipolar neutrosophic topology on a bipolar neutrosophic set $BNS=\{(v_1,a_1,b_1,c_1,a_1^{'},b_1^{'},c_1^{'}),\cdots,(v_n,a_n,b_n,c_n,a_n^{'},b_n^{'},c_n^{'})\}$ (here $0\le a_1,\cdots,a_n\le 1$ are positive membership values of truthness, $0\le b_1,\cdots,b_n\le 1$ are positive membership values of indeterminacy, $0\le c_1,\cdots,c_n\le 1$ are positive membership values of falsity, $-1\le a_1^{'},\cdots,a_n^{'}\le 0$ are negative membership values of truthness, $-1\le b_1^{'},\cdots,b_n^{'}\le 0$ are negative membership values of indeterminacy, $-1\le c_1^{'},\cdots,c_n^{'}\le 0$ are negative membership values of falsity) can be established in terms of a BNG G such that each vertex in G is a class in *BNS* and edge number between two vertices is the cardinality of intersection of corresponding two classes of *BNS*, the positive truthness degree (resp. positive indeterminacy degree and positive falsity degree) of edges is the positive truthness degree (resp. positive indeterminacy degree and positive falsity degree) of each vertex in its intersection, and the negative truthness degree (resp. negative indeterminacy degree and negative falsity degree) of edges is the negative truthness degree (resp. negative indeterminacy degree and negative falsity degree) of each vertex in its intersection. The bipolar neutrosophic pseudograph (with loops), bipolar discrete neutrosophic topological graph (no loops) and bipolar simple neutrosophic graph (one edge between two adjacent vertices) mark their subscripts by *BNP*, *BND* and *BNS* respectively.

Theorem 10. Let $|E_{BNP}(G)|$, $|E_{BND}(G)|$ and $|E_{BNS}(G)|$ be the edge number of a bipolar neutrosophic pseudograph, bipolar discrete neutrosophic topological graph and bipolar simple neutrosophic graph on the bipolar fuzzy set $BNS=\{(v_1,a_1,b_1,c_1,a_1^{'},b_1^{'},c_1^{'}),\cdots,(v_n,a_n,b_n,c_n,a_n^{'},b_n^{'},c_n^{'})\}$, respectively. Then, $|E_{BNP}(G)|=n2^{n-2}(2^{n-1}-1)+n2^{n-1}$, $|E_{BND}(G)|=n2^{n-2}(2^{n-1}-1)$ and $|E_{BNS}(G)|=\dfrac{2^{2n}-2^n-3^n+1}{2}$.

Example 34. Let $BNS=\{(v_1,0.6,0.5,0.4,-0.5,-0.6,-0.8),(v_2,0.3,0.6,0.5,-0.6,-0.2,-0.4),(v_3,0.1,0.2,0.8,-0.9,-0.7,-0.2)\}$. Then $|E_{BNP}(G)|=30$, $|E_{BND}(G)|=18$ and $|E_{BNS}(G)|=15$.

A bipolar neutrosophic topological graph deduced in terms of a bipolar neutrosophic graph can be stated by the following result.

Theorem 11. Let G be a BNG that meets the conditions:

- G has a unique isolated vertex which is represented by $\varnothing$;
- There is a vertex v adjacent to other vertices in $G-\{\varnothing\}$, and $\mu_{\mathrm{T}}^P(v_i,v)\le\mu_{\mathrm{T}}^P(v_i,BNS)\le\mu_{\mathrm{T}}^P(v,BNS)$, $\mu_{\mathrm{N}}^P(v_i,v)\ge\mu_{\mathrm{N}}^P(v_i,BNS)\ge\mu_{\mathrm{N}}^P(v,BNS)$, $\mu_{\mathrm{F}}^P(v_i,v)\ge\mu_{\mathrm{F}}^P(v_i,BNS)\ge\mu_{\mathrm{F}}^P(v,BNS)$, $\mu_{\mathrm{T}}^N(v_i,v)\ge\mu_{\mathrm{T}}^N(v_i,BNS)\ge\mu_{\mathrm{T}}^N(v,BNS)$, $\mu_{\mathrm{N}}^N(v_i,v)\le\mu_{\mathrm{N}}^N(v_i,BNS)\le\mu_{\mathrm{N}}^N(v,BNS)$, $\mu_{\mathrm{F}}^N(v_i,v)\le\mu_{\mathrm{F}}^N(v_i,BNS)\le\mu_{\mathrm{F}}^N(v,BNS)$ for any $v_i\in V(G)-\{\varnothing\}$;
- Let v_1 and v_2 be any two distinct vertices. We have $v_1\wedge v_2, v_1\vee v_2\in V(G)$.

Then, the class τ of vertices is a BNTG.

The next theorem is used to compute the size of G by means of bipolar neutrosophic topological graph τ.

Theorem 12. The edge number of a BNTG is expressed by a bipolar neutrosophic topology $\tau=\{\varnothing,\{(v_1,a_1,b_1,c_1,a_1^{'},b_1^{'},c_1^{'})\},\{(v_1,a_1,b_1,c_1,a_1^{'},b_1^{'},c_1^{'}),(v_2,a_2,b_2,c_2,a_2^{'},b_2^{'},c_2^{'})\},\cdots,X=\{(v_1,a_1,b_1,c_1,a_1^{'},b_1^{'},c_1^{'}),\cdots,(v_n,a_n,b_n,c_n,a_n^{'},b_n^{'},c_n^{'})\}\}$.

Example 35. Let $BNS=\{(v_1,0.6,0.5,0.4,-0.5,-0.6,-0.8),(v_2,0.3,0.6,0.5,-0.6,-0.2,-0.4),(v_3,0.1,0.2,0.8,-0.9,-0.7,-0.2)\}$ with a bipolar neutrosophic topological space $\tau=\{\varnothing, S_1=\{(v_1,0.6,0.5,0.4,-0.5,-0.6,-0.8)\},S_2=\{(v_1,0.6,0.5,-0.6,-0.8),(v_2,0.3,0.6,0.5,-0.6,-0.2,-0.4)\},X=\{(v_1,0.6,0.5,0.4,-0.5,-0.6,-0.8),(v_2,0.3,0.6,0.5,-0.6,-0.2,-0.4),(v_3,0.1,0.2,0.8,-0.9,-0.7,-0.2)\}\}$. See Fig 17 for the bipolar neutrosophic topological graph, and the positive truthness degree, positive indeterminacy degree, positive falsity degree, negative truthness degree, negative indeterminacy degree, and negative falsity degree of edges are 2.1, 2.1, 1.7, -2.1, -2, -2.8 respectively.

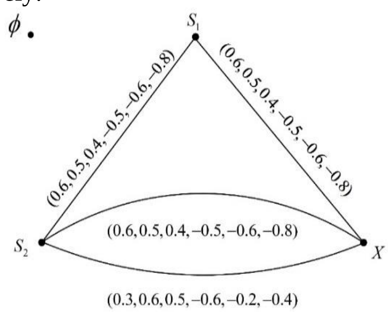

Fig 17. A bipolar neutrosophic topological graph in Example 35.

Each BNTG can be expressed by a BFG, but the reverse may not be true. We present the following two examples to explain it in detail.

Example 36. Let G be a BNG drawn in Fig 15, and a bipolar neutrosophic topological graph be constructed in terms of the following schemes: the only isolated vertex is formulated by $\varnothing$; the vertex has the maximum degree four represented by $X=\{(v_1,0.6,0.5,0.4,-0.5,-0.6,-0.8),$

$(v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4)$, $(v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\}$. Since $|S_2 \wedge X| = 2$, the vertex representing S_2 is $\{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4)\}$. Similarly, $|S_3 \wedge X| = 2$ and the corresponding vertex are denoted by $S_3 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\}$. $|S_2 \wedge S_3| = 1$, and the MF value of edge connect S_2 and S_3 is $(0.6, 0.5, 0.4, -0.5, -0.6, -0.8)$. $|S_1 \wedge X| = |S_1 \wedge S_2| = |S_1 \wedge S_3| = 1$ and hence S_1 is denoted by $\{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8)\}$. Therefore, $\tau = \{\varnothing, S_1 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8)\}, S_2 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4)\}, S_3 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\}, X = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\}\}$ is a bipolar neutrosophic topology and this graph is called a BNGT.

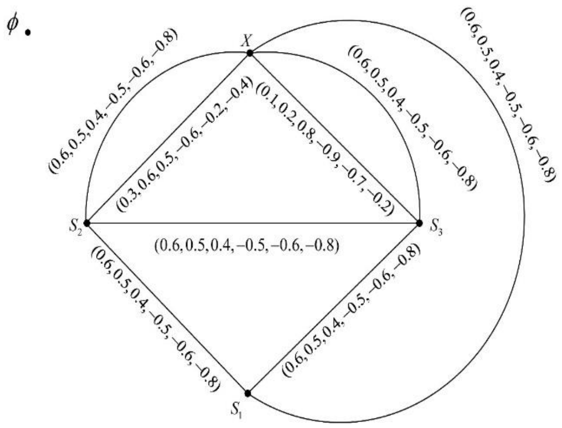

Fig 18. A BNG which is a BNTG.

Example 37. The graph G in Fig 19 is not a bipolar neutrosophic topological graph where $X = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2), (v_4, 0.5, 0.5, 0.5, -0.5, -0.5, -0.5)\}$,

$$S_1 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4)\},$$

$$S_2 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\},$$

$$|S_1 \wedge X| = |S_2 \wedge X| = 2, \ |S_1 \wedge S_2| = 1,$$

$$S_1 \wedge X = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4)\},$$

$$S_2 \wedge X = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\},$$

$$S_1 \wedge S_2 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8)\}.$$

However, $\tau = \{\varnothing, S_1 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4)\}, S_2 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\}, X = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2), (v_4, 0.5, 0.5, 0.5, -0.5, -0.5, -0.5)\}\}$ is not a bipolar neutrosophic topology because $S_1 \cup S_2 = \{(v_1, 0.6, 0.5, 0.4, -0.5, -0.6, -0.8), (v_2, 0.3, 0.6, 0.5, -0.6, -0.2, -0.4), (v_3, 0.1, 0.2, 0.8, -0.9, -0.7, -0.2)\} \notin \tau$.

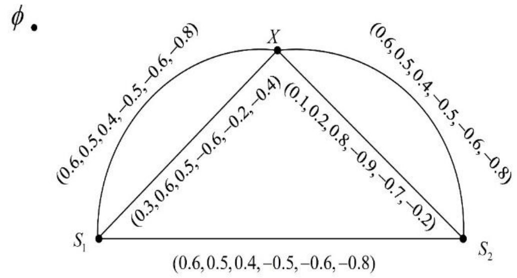

Fig 19.. ABNG which is not a BNTG

D. Algebraic operations in interval-valued fuzzy setting

Definition 19. An IVFT on an IVFS $IVFS = \{(v_1, [a_1, b_1]), \cdots, (v_n, [a_n, b_n])\}$ (here $0 \le a_i \le b_i \le 1$ for $i \in \{1, \cdots, n\}$) can be established in terms of an IVFG G such that each vertex in G is a class in *IVFS* and edge number between two vertices is the cardinality of intersection of corresponding two classes of *IVFS*. The interval-valued fuzzy pseudograph (with loops), discrete IVFTG (no loops) and simple IVFG (one edge between two adjacent vertices) mark their subscripts by *IVFP*, *IVFD* and *IVFS* respectively.

Theorem 13. Let $|E_{IVFP}(G)|$, $|E_{IVFD}(G)|$ and $|E_{IVFS}(G)|$ be the edge number of an interval-valued fuzzy pseudograph, discrete IVFTG and simple IVFG on the BFS $IVFS = \{(v_1, [a_1, b_1]), \cdots, (v_n, [a_n, b_n])\}$, respectively. Then, $|E_{IVFP}(G)| = n2^{n-2}(2^{n-1} - 1) + n2^{n-1}$, $|E_{IVFD}(G)| = n2^{n-2}(2^{n-1} - 1)$ and $|E_{IVFS}(G)| = \dfrac{2^{2n} - 2^n - 3^n + 1}{2}$.

Example 38. Let $IVFS = \{(v_1, [0.3, 0.6]), (v_2, [0.5, 0.7]), (v_3, [0.1, 0.4])\}$. Then $|E_{IVFP}(G)| = 30$, $|E_{IVFD}(G)| = 18$ and $|E_{IVFS}(G)| = 15$.

An IVFTG deduced in terms of an interval-valued fuzzy graph can be stated by the following result.

Theorem 15. Let G be an interval-valued fuzzy graph that meets the conditions:

- G has a unique isolated vertex which is represented by $\varnothing$;
- There is a vertex v adjacent to other vertices in $G - \{\varnothing\}$, and $\mu^{+l}(v_i, v) \le \mu^{+l}(v_i, IVFS) \le \mu^{+l}(v, IVFS)$, and

$\mu^{+u}(v_i, v) \le \mu^{+u}(v_i, IVS) \le \mu^{+u}(v, IVS)$ for any $v_i \in V(G) - \{\varnothing\}$;

- Let v_1 and v_2 be any two distinct vertices. We have $v_1 \wedge v_2, v_1 \vee v_2 \in V(G)$.

Then, the class τ of vertices is an IVFTG.

The next theorem is used to compute the edge number of G by means of IVFTG τ.

Theorem 16. The edge number of an IVFTG is expressed by an interval-valued fuzzy topology $\tau = \{\varnothing, \{(v_1,[a_1,b_1])\}, \{(v_1,[a_1,b_1]),(v_2,[a_2,b_2])\}, \cdots, X = \{(v_1,[a_1,b_1]), \cdots, (v_n,[a_n,b_n])\}\}$.

Example 39. Let $IVFS = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7]),(v_3,[0.1,0.4])\}$ with an interval-valued fuzzy topological space $\tau = \{\varnothing, S_1 = \{(v_1,[0.3,0.6])\}, S_2 = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7])\}, X = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7]),(v_3,[0.1,0.4])\}\}$. See Fig 20 for the interval-valued fuzzy topological graph.

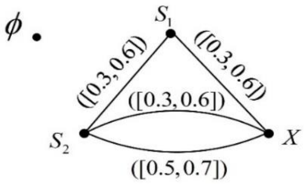

Fig 20. An IVFTG in Example 39.

Each IVFTG can be expressed by an IVFG, but the reverse may not be true. We present the following two examples to explain it in detail.

Example 40. Let G be an IVFG drawn in Fig 21, and an IVFTG be constructed in terms of the following schemes: the only isolated vertex is formulated by $\varnothing$; the vertex has maximum degree four represented by $X = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7]),(v_3,[0.1,0.4])\}$ .Since $|S_2 \wedge X| = 2$, the vertex representing S_2 is $\{(v_1,[0.3,0.6]),(v_2,[0.5,0.7])\}$. Similarly, $|S_3 \wedge X| = 2$ and the correspond vertex is denoted by $S_3 = \{(v_1,[0.3,0.6]),(v_3,[0.1,0.4])\}$. $|S_2 \wedge S_3| = 1$, and the MF value of the edge connecting S_2 and S_3 is $([0.3,0.6])$. $|S_1 \wedge X| = |S_1 \wedge S_2| = |S_1 \wedge S_3| = 1$ and hence S_1 is denoted by $\{(v_1,[0.3,0.6])\}$. Therefore, $\tau = \{\varnothing, S_1 = \{(v_1,[0.3,0.6])\}, S_2 = \{(v_1,[0.3,\{(v_1,[0.3,0.6]),(v_2,[0.5,0.7])\}, S_3 = \{(v_1,[0.3,0.6]),(v_3,[0.1,0.4])\}, X = \{(v_1,[0.3,0.6]), (v_2,[0.5,0.7]),(v_3,[0.1,0.4])\}\}$ is an IVFT and this graph is called an IVFTG.

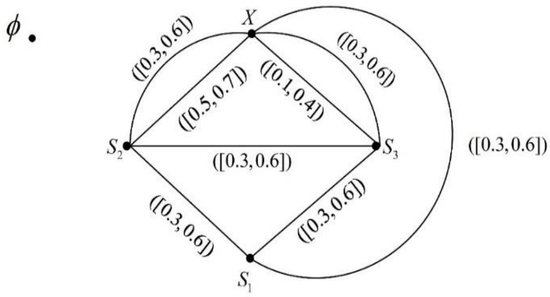

Fig 21. An IVFG which is an IVFTG.

Example 41. The graph G in Fig 22 is not an IVFTG, where

$$X = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7]),(v_3,[0.1,0.4]),(v_4,[0.6,0.8])\},$$

$$S_1 = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7])\},$$

$$S_2 = \{(v_1,[0.3,0.6]),(v_3,[0.1,0.4])\},$$

$$|S_1 \wedge X| = |S_2 \wedge X| = 2, \ |S_1 \wedge S_2| = 1,$$

$$S_1 \wedge X = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7])\},$$

$$S_2 \wedge X = \{(v_1,[0.3,0.6]),(v_3,[0.1,0.4])\},$$

$$S_1 \wedge S_2 = \{(v_1,[0.3,0.6])\}.$$

However, $\tau = \{\varnothing, S_1 = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7])\}, S_2 = \{(v_1,[0.3,0.6]),(v_3,[0.1,0.4])\}, X = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7]),(v_3,[0.1,0.4]),(v_4,[0.6,0.8])\}\}$ is not an IVFT because $S_1 \cup S_2 = \{(v_1,[0.3,0.6]),(v_2,[0.5,0.7]),(v_3,[0.1,0.4])\} \notin \tau$

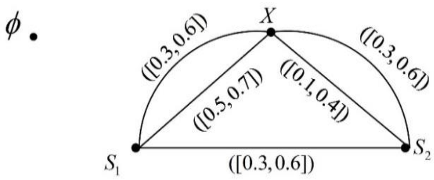

Fig 22. An IVFG which is not an IVFTG.

E. Algebraic operations in bipolar interval-valued fuzzy setting

Definition 20. A bipolar interval-valued fuzzy topology on a bipolar interval-valued fuzzy set $BIVFS = \{(v_1,[a_1,b_1],[c_1,d_1]), \cdots, (v_n,[a_n,b_n],[c_n,d_n])\}$ (here $0 \le a_i \le b_i \le 1$ and $-1 \le c_i \le d_i \le 0$ for $i \in \{1,\cdots,n\}$) can be established in terms of a BIVFG G such that each vertex in G is a class in *BIVFS* and edge number between two vertices is the cardinality of intersection of corresponding to two classes of *BIVFS*. The bipolar interval-valued fuzzy pseudograph (BIVFP) (with loops), bipolar discrete interval-valued fuzzy topology graph (BDIVFTG) (no loops) and bipolar simple interval-valued fuzzy graph (BSIVFG) (one edge between two adjacent vertices) mark their subscripts by *BIVFP*, *BIVFD* and *BIVFS* respectively.

Theorem 17. Let $|E_{BIVFP}(G)|$, $|E_{BIVFD}(G)|$ and $|E_{BIVFS}(G)|$ be the edge number of a BIVFP, BDIVFTG and BSIVFG on the

BFS $BIVFS=\{(v_1,[a_1,b_1],[c_1,d_1]),\cdots,(v_n,[a_n,b_n],[c_n,d_n])\}$, respectively. Then, $|E_{BIVFP}(G)|=n2^{n-2}(2^{n-1}-1)+n2^{n-1}$, $|E_{BIVFD}(G)|=\ n2^{n-2}(2^{n-1}-1)$ and $|E_{BIVFS}(G)|=\frac{2^{2n}-2^n-3^n+1}{2}$

Example 42. Let $BIVFS=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5]),(v_3,[0.1,0.4],[-0.4,-0.2])\}$ .Then $|E_{BIVFP}(G)|=30$, $|E_{BIVFD}(G)|=18$ and $|E_{BIVFS}(G)|=15$.

A BIVFTG deduced in terms of a bipolar interval-valued fuzzy graph can be stated by the following result.

Theorem 18. Let *G* be a BIVFG that meets the conditions:

- *G* has a unique isolated vertex which is represented by $\varnothing$;
- There is a vertex *v* adjacent to other vertices in $G-\{\varnothing\}$, and $\mu^{+l}(v_i,v)\le\mu^{+l}(v_i,\ BIVFS)\le\mu^{+l}(v,BIVFS)$, $\mu^{+u}(v_i,v)\le\mu^{+u}(v_i,BIVS)\le\mu^{+u}(v,BIVS)$, $\mu^{-l}(v_i,v)\ge\mu^{-l}(v_i,BIVFS)\ge\mu^{-l}(v,BIVFS)$, $\mu^{-u}(v_i,v)\ge\mu^{-u}(v_i,BIVS)\ge\mu^{-u}(v,BIVS)$ for any $v_i\in V(G)-\{\varnothing\}$;
- Let v_1 and v_2 be any two distinct vertices. We have $v_1\wedge v_2, v_1\vee v_2\in V(G)$.

Then, the class τ of vertices is a bipolar interval-valued fuzzy topological graph.

The next theorem is used to compute the edge number of *G* by means of a bipolar interval-valued fuzzy topological graph τ.

Theorem 19. The edge number of a bipolar interval-valued fuzzy topological graph is expressed by a bipolar interval-valued fuzzy topology $\tau=\{\varnothing,\{(v_1,[a_1,b_1],[c_1,d_1])\},\{(v_1,[a_1,b_1],[c_1,d_1]),(v_2,[a_2,b_2],[c_2,d_2])\},\ \cdots,X=\{(v_1,[a_1,b_1],[c_1,d_1]),\cdots,(v_n,[a_n,b_n],[c_n,d_n])\}\}$.

Example 43. Let $NS=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5]),\ (v_3,[0.1,0.4],\ [-0.4,-0.2])\}$ with a bipolar interval-valued fuzzy topological space $\tau=\{\varnothing,S_1=\{(v_1,[0.3,0.6],[-0.9,-0.3])\},S_2=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5])\},\quad X=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5]),(v_3,[0.1,0.4],[-0.4,-0.2])\}\}$ See Fig 23 for the IVFTG.

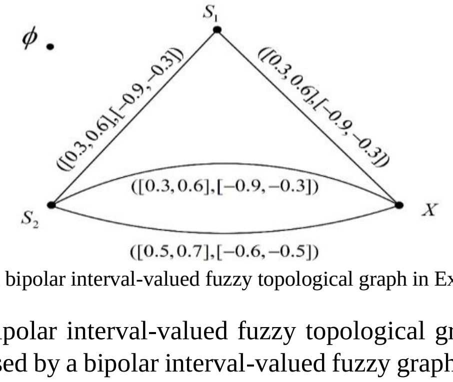

Fig 23. A bipolar interval-valued fuzzy topological graph in Example 43.

Each bipolar interval-valued fuzzy topological graph can be expressed by a bipolar interval-valued fuzzy graph, but the reverse may not be true. We present the following two examples to explain it in detail.

Example 40. Let *G* be a BIVFG drawn in Fig 24, and a BIVFTG be constructed in terms of the following schemes: the only isolated vertex is formulated by $\varnothing$; the vertex has maximum degree four represented by $X=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5]),\ (v_3,[0.1,0.4],[-0.4,-0.2])\}$. Since $|S_2\wedge X|=2$, the vertex representing S_2 is $\{(v_1,[0.3,0.6],[-0.9,-0.3]),\ (v_2,[0.5,0.7],\ [-0.6,-0.5])\}$. Similarly, $|S_3\wedge X|=2$ and the corresponding vertex is denoted by $S_3=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_3,[0.1,0.4],[-0.4,-0.2])\}$. $|S_2\wedge S_3|=1$, and the membership value of edge connect S_2 and S_3 is $([0.3,0.6],[-0.9,-0.3])$. $|S_1\wedge X|=|S_1\wedge S_2|=|S_1\wedge S_3|=1$ and hence S_1 denoted by $\{(v_1,[0.3,0.6],[-0.9,-0.3])\}$. Therefore, $\tau=\{\varnothing,S_1=\{(v_1,[0.3,0.6],\ [-0.9,-0.3])\},\ S_2=\{(v_1,[0.3,0.6],[-0.9,-0.3])(v_2,[0.5,0.7],[-0.6,-0.5])\},S_3=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_3,[0.1,0.4],[-0.4,-0.2])\},X=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5]),(v_3,[0.1,0.4],[-0.4,-0.2])\}\}$ is a bipolar interval-valued fuzzy topology and this graph is called a bipolar interval-valued fuzzy topological graph.

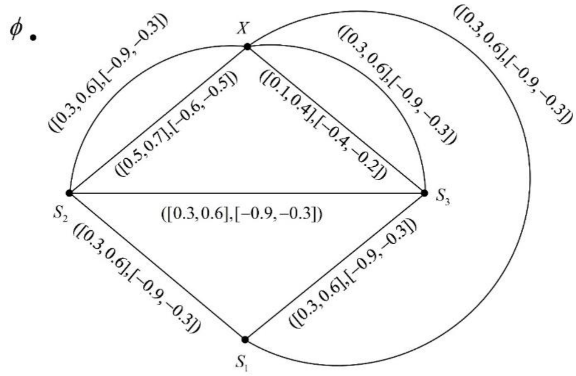

Fig 24. A BIVFG which is a BIVFTG.

Example 41. The graph *G* in Fig 25 is not a BIVFTG where

$X=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5]),(v_3,[0.1,0.4],[-0.4,-0.2]),(v_4,[0.6,0.8],[-0.7,-0.1])\}$,

$S_1=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5])\}$,

$S_2=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_3,[0.1,0.4],[-0.4,-0.2])\}$, $|S_1\wedge X|=|S_2\wedge X|=2$, $|S_1\wedge S_2|=1$,

$S_1\wedge X=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7],[-0.6,-0.5])\}$,

$S_2\wedge X=\{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_3,[0.1,0.4],[-0.4,-0.2])\}$,

$$S_1 \wedge S_2 = \{(v_1,[0.3,0.6],[-0.9,-0.3])\}.$$

However, $\tau = \{\varnothing, S_1 = \{(v_1,[0.3,0.6],[-0.9,-0.3]), (v_2,[0.5,0.7],[-0.6,-0.5])\}, S_2 = \{(v_1, [0.3,0.6], [-0.9,-0.3]),(v_3,[0.1,0.4],[-0.4,-0.2])\}, X = \{(v_1,[0.3, 0.6],[-0.9,-0.3]),(v_2, [0.5,0.7],[-0.6,-0.5]),(v_3, [0.1,0.4],[-0.4,-0.2]), (v_4,[0.6,0.8],[-0.7,-0.1])\}\}$ is not a bipolar interval-valued fuzzy topology because $S_1 \cup S_2 = \{(v_1,[0.3,0.6],[-0.9,-0.3]),(v_2,[0.5,0.7], [-0.6,-0.5]),(v_3,[0.1,0.4],[-0.4,-0.2])\} \notin \tau$.

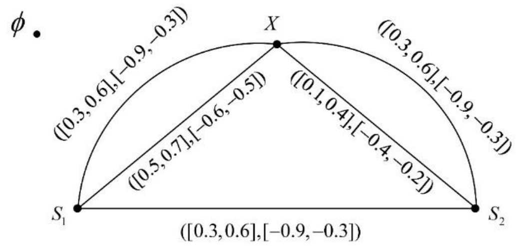

Fig 25. A BIVFG which is not a BIVFTG.

V. Conclusion

Graphs are a common model functioned to reveal the relationship between things, and the relationship between fuzzy sets can also be presented by graph structures. In this article, we characterize the fuzzy topology from the perspective of the fuzzy graph. Five settings are discussed respectively: BIF setting, NS setting, BNS setting, IVFS setting and BIVFS setting. Parallel classes and topological spaces in these settings are presented, and several examples are depicted to show the expression of theorems and concepts. Due to the wide applications of BFS, NS and IVFS, the results derived in this paper have potential application prospects, especially in the circumstances that there are two different angles, positive and negative, to describe the uncertain features of issues.

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