# Bicomplex r-Parameter Mittag-Leffler Function and Associated Properties

Anil Kumar Yadav, Rupakshi Mishra Pandey, Vishnu Narayan Mishra

***Abstract*—The extension of the exponential function is the Mittag-Leffler function. In physical implications, the Mittag-Leffler function is becoming more and more crucial, as a result a number of researchers are investigating different generalizations and extensions associated with the Mittag-Leffler function. In this present article, we define the bicomplex r-parameter Mittag-Leffler function which is an extension of bicomplex one parameter Mittag-Leffler function. In addition, various properties of this function like integral representation, differential relation, duplication formula, bicomplex C-R equation, analyticity, region of convergence, order and type are established. Additionally, the bicomplex r-parameter Mittag-Leffler function's Laplace transform and Caputo fractional derivative are produced.**

***Index Terms*—mittag-Leffler function, bicomplex numbers, gamma function, exponential function.**

## I. Introduction

BICOMPLEX numbers have grabbed a considerable amount of interest and have undergone notable research. A lot of studies have been conducted on bicomplex numbers over a long period of time. The bicomplex numbers and its numerous properties have been highlighted by Corrado Segre [25] in his work that was published in 1892. Researchers have been exploring the distinct algebraic and geometric characteristics of bicomplex numbers along with its implications over the past few years. For a deeper analysis readers may follow [16], [22], [23]. Recent apporaches have focused on extending holomorphic and meromorphic functions [6], [7], the integral trasforms [1], [2] as well as a number of other functions, including the Hurwitz Zeta function [11], [4], Polygamma function [10], bicomplex analysis and Hilbert space [14], [15], Riemann Zeta function [21], Gamma and Beta functions [12] in the bicomplex variable from their complex variable.

Numerous investigators have lately defined several generalizations and extensions of the Mittag-Leffler functions and these are beneficial for studying fractional calculus, see [5], [8], [9], [13]. Mittag-Leffler function is observed while examining the fractional form of several differential equations. For investigating the bicomplex form of such types of fractional differential and integral equations in a bicomplex field, a bicomplex variant of the Mittag-Leffler function would be needed. Agarwal et al. [3], [26] have defined the bicomplex Mittag-Leffler function for one, two parameters recently.

According to the analysis of the aformentioned cited literature, there are substantial research gap relating to the generalised version of the bicomplex Mittag- Lefler function. This paper aims is to fulfill that gap. An endeavor has been made to define the r-parameter bicomplex Mittag-Leffler function which is the generalization of one and two parameter bicomplex Mittag-Leffler function. Moreover, its different properties such as integral relations, differential relations, duplication formulae, C-R equation, Laplace and Caputo fractional derivative have been derived.

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Anil Kumar Yadav is a Research Scholar in Department of Mathematics, Amity Institute of Applied Sciences, Amity University Uttar Pradesh, India. E-mail: anilisha9192@gmail.com.

Rupakshi Mishra Pandey is an Associate Professor in Department of Mathematics, Amity Institute of Applied Sciences, Amity University Uttar Pradesh, India. E-mail: rmpandey@amity.edu.

Vishnu Narayan Mishra is a Professor in Department of Mathematics, Indira Gandhi National Tribal University, Amarkantak, Madhya Pradesh, India. E-mail: vnm@igntu.ac.in.

## II. Preliminaries

**Definition 2.1** (Bicomplex number). A bicomplex number $\phi$ is defined [25] as

$$\phi = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1 i_2\delta_3,\ (\delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}), \quad (1)$$

where $i_1^2 = i_2^2 = -1$. Also it can be written as

$$\phi = (\delta_0 + i_1\delta_1) + i_2(\delta_2 + i_1\delta_3),\ (\delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R})$$

or

$$\phi = \mu_1 + i_2\mu_2, \qquad (\mu_1, \mu_2 \in \mathbb{C}) \quad (2)$$

where $\mu_1 = \delta_0 + i_1\delta_1$, $\mu_2 = \delta_2 + i_1\delta_3$, whereas $\mathbb{C}$ denotes the collection of complex numbers.

Suppose $\overline{\mathbb{C}}$ represents the collection of bicomplex numbers.

$$\overline{\mathbb{C}} = \{\phi : \phi = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1 i_2\delta_3 \ \mid \delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}\} \quad (3)$$

or

$$\overline{\mathbb{C}} = \{\phi : \phi = \mu_1 + i_2\mu_2 \ \mid \mu_1, \mu_2 \in \mathbb{C}\}.$$

We will also make use of the notations $\delta_0$ = Re($\phi$), $\delta_1$ = $\text{Im}_{i_1}(\phi)$, $\delta_2$ = $\text{Im}_{i_2}(\phi)$, $\delta_3$=$\text{Im}_j(\phi)$.
where $i_1 i_2 = i_2 i_1 = j$, $\iota_1 j = j\iota_1 = -\iota_2$, $i_2 j = j i_2 = -i_1$, $j^2 = 1$.

Let $\phi = \mu_1 + i_2\mu_2 \in \overline{\mathbb{C}}$, then $\phi$ is non invertible if and only if,

$$\mu_1^2 + \mu_2^2 = 0 \implies \mu_1 + i_1\mu_2 = 0 \ or \ \mu_1 - i_1\mu_2 = 0. \quad (4)$$

Null cone ($\mathbb{NC}$) [24] is the collection of all non-invertible element, and it is described in the following manner

$$\mathbb{NC} = \mathbb{O}_2 = \{\phi : \phi = \mu_1 + i_2\mu_2 \ \mid \ \mu_1^2 + \mu_2^2 = 0\}. \quad (5)$$

The two non trivial idempotent zero divisors $e_1$ and $e_2$ in $\overline{\mathbb{C}}$ are defined [18] in such a way

$$e_1 = \frac{1 + i_1 i_2}{2} = \frac{1 + j}{2}, \quad e_2 = \frac{1 - i_1 i_2}{2} = \frac{1 - j}{2}$$

such that

$$e_1 + e_2 = 1, \quad e_1.e_2 = 0, \quad e_1^2 = e_1, \quad e_2^2 = e_2.$$

**Definition 2.2** (Idempotent Representations). Every bicomplex number $\phi = \mu_1 + i_2\mu_2 \in \overline{\mathbb{C}}$ has the following unique idempotent representation in terms of $e_1$ and $e_2$ is given as

$$\phi = \mu_1 + i_2\mu_2 = (\mu_1 - i_1\mu_2)e_1 + (\mu_1 + i_1\mu_2)e_2 = \phi_1 e_1 + \phi_2 e_2, \tag{6}$$

where $\phi_1 = \mu_1 - i_1\mu_2$ and $\phi_2 = \mu_1 + i_1\mu_2$.

Projection mappings $P_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}_1 \subseteq \mathbb{C}$, $P_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}_2 \subseteq \mathbb{C}$ for a bicomplex number $\phi = \mu_1 + i_2\mu_2$ are defined [22] as

$$\begin{aligned} P_1(\phi) = P_1(\mu_1 + i_2\mu_2) &= P_1[(\mu_1 - i_1\mu_2)e_1 + (\mu_1 + i_1\mu_2)e_2] \\ &= (\mu_1 - i_1\mu_2) \in \overline{\mathbb{C}}_1 \end{aligned}$$

and

$$\begin{aligned} P_2(\phi) = P_2(\mu_1 + i_2\mu_2) &= P_1[(\mu_1 - i_1\mu_2)e_1 + (\mu_1 + i_1\mu_2)e_2] \\ &= (\mu_1 + i_1\mu_2) \in \overline{\mathbb{C}}_2, \end{aligned}$$

where
$\overline{\mathbb{C}}_1 = \{\phi_1 : \phi_1 = \mu_1 - i_1\mu_2 \ |\mu_1, \mu_2 \in \mathbb{C}\}$ and $\overline{\mathbb{C}}_2 = \{\phi_2 : \phi_2 = \mu_1 + i_1\mu_2 \ |\mu_1, \mu_2 \in \mathbb{C}\}$.

**Definition 2.3** (Modulus). For any element,
$\phi = \mu_1 + i_2\mu_2 \in \overline{\mathbb{C}}$.
The real modulus of $\phi$ is expressed [22] as

$$|\phi| = \sqrt{|\mu_1|^2 + |\mu_2|^2}.$$

The $i_1$ -modulus of $\phi$ is expressed as

$$|\phi|_{i_1} = \sqrt{\mu_1^2 + \mu_2^2}.$$

The $i_2$ -modulus of $\phi$ is expressed as

$$|\phi|_{i_2} = \sqrt{|\mu_1|^2 - |\mu_2|^2 + 2Re(\mu_1\bar{\mu_1})i_2}.$$

The $j$ -modulus of $\phi$ is expressed as

$$|\phi|_j = |\mu_1 - i_1\mu_2|e_1 + |\mu_1 + i_1\mu_2|e_2.$$

**Definition 2.4** (Norm). For any element, $\phi = \mu_1 + i_2\mu_2 = \phi_1 e_1 + \phi_2 e_2 = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1 i_2\delta_3 \in \overline{\mathbb{C}}$, the norm of $\phi$ is given [22] by

$$\begin{aligned} ||\phi|| = \sqrt{|\mu_1|^2 + |\mu_2|^2} &= \frac{1}{\sqrt{2}}\sqrt{|\phi_1|^2 + |\phi_2|^2} \\ &= \sqrt{\delta_0^2 + \delta_1^2 + \delta_2^2 + \delta_3^2}. \end{aligned}$$

**Definition 2.5** (Absolute value). For any element $\phi = \mu_1 + i_2\mu_2 = \phi_1 e_1 + \phi_2 e_2 = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1 i_2\delta_3 \in \overline{\mathbb{C}}$, the absolute value of $\phi$ is given [22] by

$$\begin{aligned} |\phi|_{abs} = \sqrt{|\mu_1^2 + \mu_2^2|} &= \sqrt{|(\mu_1 - i_1\mu_2)(\mu_1 + i_1\mu_2)|} \\ &= \sqrt{|\phi_1\phi_2|} = \sqrt{|\phi_1||\phi_2|}. \end{aligned}$$

**Definition 2.6** (Argument). For any element $\phi = \mu_1 + i_2\mu_2 = \phi_1 e_1 + \phi_2 e_2 = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1 i_2\delta_3 \in \overline{\mathbb{C}}$, the hyperbolic argument of $\phi$ is given [22] by

$$arg_j(\phi) = arg(\phi_1)e_1 + arg(\phi_2)e_2.$$

**Definition 2.7** (see [22]).) Suppose $h : U \subseteq \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, here $U$ indicates an open set. Also assume that $h(\mu_1 + \iota_2\mu_2) = h_1(\mu_1, \mu_2) + i_2 h_2(\mu_1, \mu_2)$. Then the fuction $h$ is $\overline{\mathbb{C}}$-holomorphic on U iff $h_1$ and $h_2$ are holomorphic in $U$, and satisfy

$$\frac{\partial h_1}{\partial \mu_1} = \frac{\partial h_2}{\partial \mu_2} \quad \text{and} \quad \frac{\partial h_1}{\partial \mu_2} = -\frac{\partial h_2}{\partial \mu_1} \quad \text{on } U.$$

above equations are said to be complexified Cauchy-Riemann equations.

**Theorem 2.8** (see [19]). Assume that $\sum_{u=0}^{\infty} a_u\phi^u$ is a power series having the component series $\sum_{u=0}^{\infty} b_u\phi_1^u$ and $\sum_{u=0}^{\infty} c_u\phi_2^u$, where $a_u = b_u e_1 + c_u e_2$. Then the series converges, diverges accordingly $N(\phi) < R$, $N(\phi) > R$ correspondingly, where $R$ is the same radius of convergence of both component series. Also $N(\phi)$ is given by

$$N(\phi) = \sqrt{||\phi||^2 + \sqrt{||\phi||^4 - |\phi|_{abs}^4}} = \max(|\phi_1|, |\phi_2|), \tag{7}$$

such that
$||\phi|| = \frac{1}{\sqrt{2}}\sqrt{|\phi_1|^2 + |\phi_2|^2}$ and $|\phi|_{abs} = \sqrt{|\phi_1||\phi_2|}$.

**Theorem 2.9** (Decomposition theorem see [20]). Suppose $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ be the component regions of $\overline{\mathbb{C}}$ in the $\phi_1$ and $\phi_2$ plane respectively. If the function $h(\phi)$ is analytic in a region $\overline{\mathbb{C}}$ then there exists a unique pair of complex-valued analytic functions $h_1(\phi_1)$ and $h_2(\phi_2)$ defined in the regions $\overline{\mathbb{C}}_1$ and $\overline{\mathbb{C}}_2$ respectively, in this manner

$$h(\phi) = h_1(\phi_1)e_1 + h_2(\phi_2)e_2 \tag{8}$$

for all $\phi$ in $\overline{\mathbb{C}}$.

**Definition 2.10**. The bicomplex gamma function (see [12]) is defined by

$$\frac{1}{\Gamma(\phi)} = \phi e^{\gamma\phi}\prod_{s=1}^{\infty}\Big(1 + \frac{\phi}{s}e^{(-\frac{\phi}{s})}\Big), \quad \phi \in \overline{\mathbb{C}} \tag{9}$$

provided that $\mu_1 \neq \frac{-(a+b)}{2}$ and $\mu_2 \neq \iota_1\frac{(b-a)}{2}$, where $a, b \in \mathbb{N}\cup\{0\}$.
Also, $\Omega$ ($0 \leq \Omega \leq 1$) the Euler constant is expressed as

$$\Omega = \lim_{m\to\infty}(H_m - \log\text{ m}), \ H_m = \sum_{k=1}^{m}\frac{1}{k}.$$

**Definition 2.11**. The Mittag-Leffler function defined by Mittag [17] in one parameter is expressed as

$$\mathbb{E}_\gamma(\mu) = \sum_{u=0}^{\infty}\frac{\mu^n}{\Gamma(\gamma u + 1)} \ , \ Re(\gamma) > 0, \ \mu \in \mathbb{C} \tag{10}$$

with the help of gamma function property and Taylor cofficients of Cauchy inequality (see [9] p.18) there exist a number $x \geqslant 0$ and $a(x)$ a positive number in this manner

$$M_{\mathbb{E}_\gamma(a)} = \max_{|z|=a}|\ \mathbb{E}_\gamma(\mu)| < e^{a^x}, \ \ \forall\ a > a(x), \tag{11}$$

the above equation shows that the function $\mathbb{E}_\gamma(\mu)$ is entire. Also for each $Re(\gamma) > 0$, type $\varrho$ and order $\tau$ of Mittag-Leffler function (10) is provided by

$$\varrho = \frac{1}{\tau\sigma} \lim\sup_{x\to\infty} (x|a_x|^{\frac{\tau}{x}}) = 1. \quad (12)$$

and

$$\tau = \lim\sup_{x\to\infty} \frac{x\ logx}{log\frac{1}{|a_x|}} = \frac{1}{Re(\gamma)}. \quad (13)$$

## III. Bicomplex $r$-parameter Mittag-Leffler function

Since the bicomplex Mittag-Leffler function for one parameter is described [3] as

$$\mathbb{E}_\nu(\chi) = \sum_{u=0}^{\infty} \frac{\chi^u}{\Gamma(\nu u + 1)}, \quad (14)$$

where $\nu, \chi, \in \overline{\mathbb{C}}$, $\chi = \mu_1 + i_2\mu_2$, $(\mu_1, \mu_2 \in \mathbb{C})$ and $|Im_j(\nu)| < Re(\nu)$.

On the behalf of equation (14), we introduce the bicomplex Mittag-Leffler function for the r-parameter which is defined as

$$E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) = E_{\omega,\upsilon}^{(p,q)_s}(\phi) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u + \upsilon))^s} \frac{\phi^u}{u!} \quad (15)$$

with the conditions $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$; where $\omega, \upsilon, \phi \in \overline{\mathbb{C}}$, $p_\iota, q_\iota \in \mathbb{C}$, $\iota = 1, 2, \cdots, s$ and $r = s+2$.

Also $(p_1)_{q_1u}$ represents the generalized Pochhammer symbol which is given by

$$(p_1)_{q_1u} = \frac{\Gamma(p_1 + q_1u)}{\Gamma(p_1)}$$

In particular $(p_1)_{q_1u} = (q_1)^{q_1u} \prod_{s=1}^{q_1} \left(\frac{p_1+s-1}{q_1}\right)_u$, whenever $q_1 \in \mathbb{N}$.

The subsequent theorem gives a strong justification of the definition of bicomplex r-parameter Mittag-Leffler function (15).

**Theorem III.1.** *Let $p_\iota, q_\iota \in \mathbb{C}; \omega, \upsilon, \phi \in \overline{\mathbb{C}}$ where $\phi = \mu_1 + i_2\mu_2 = \phi_1e_1 + \phi_2e_2$ and $\omega = \omega_1e_1 + \omega_2e_2 = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1i_2\delta_3, \upsilon = \upsilon_1e_1 + \upsilon_2e_2 = \varepsilon_0 + i_1\varepsilon_1 + i_2\varepsilon_2 + i_1i_2\varepsilon_3$ with $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$. Then*

$$E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u + \upsilon))^s} \frac{\phi^u}{u!}.$$

*Proof:* Consider the function

$$E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u + \upsilon))^s} \frac{\phi^u}{u!} \quad (16)$$

By using the idempotent representation,

$$E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_1 u + \upsilon_1))^s} \frac{\phi_1^u}{u!} e_1 + \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_2 u + \upsilon_2))^s} \frac{\phi_2^u}{u!} e_2$$

$$E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) = E_{\omega_1,\upsilon_1}^{(p,q)_s}(\phi_1)e_1 + E_{\omega_2,\upsilon_2}^{(p,q)_s}(\phi_2)e_2$$

where $\phi = \phi_1e_1 + \phi_2e_2$ and $\omega = \omega_1e_1 + \omega_2e_2$, $\upsilon = \upsilon_1e_1 + \upsilon_2e_2$.

Now,

$$E_{\omega_1,\upsilon_1}^{(p,q)_s}(\phi_1) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_1 u + \upsilon_1))^s} \frac{\phi_1^u}{u!}$$

above equation represent the complex Mittag-Leffler function that is convergent for $Re(\omega_1) > 0, Re(\upsilon_1) > 0$.

Similarly,

$$E_{\omega_2,\upsilon_2}^{(p,q)_s}(\phi_2) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_2 n + \upsilon_2))^s} \frac{\phi_2^u}{u!}$$

is also complex Mittag-Leffler function convergent for $Re(\omega_2), Re(\upsilon_2) > 0$.

Since $E_{\omega_1,\upsilon_1}^{(p,q)_s}(\phi_1)$ and $E_{\omega_2,\upsilon_2}^{(p,q)_s}(\phi_2)$ are convergent, so by Ringleb decomposition theorem, the function $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ is also convergent in the disk $\overline{\mathbb{C}}$.

Further, let $\omega = \delta_0 + i_1\delta_1 + i_2\delta_2 + i_1i_2\delta_3 = \omega_1e_1 + \omega_2e_2$ where $\omega_1 = (\delta_0 + \delta_3) + i_1(\delta_1 - \delta_2)$ and $\omega_2 = (\delta_0 - \delta_3) + i_1(\delta_1 + \delta_2)$.

Now since $Re(\omega_1) > 0$ and $Re(\omega_2) > 0$,
$\implies \delta_0 + \delta_3 > 0$ and $\delta_0 - \delta_3 > 0$
$\implies |\delta_3| < \delta_0 = |Im_j(\omega)| < Re(\omega)$.

Similarly, we can proof that $|Im_j(\upsilon)| < Re(\upsilon)$.
This complete the proof. ■

## IV. C-R Equation, duplication formula and Recurrence Relations.

**Theorem IV.1.** *The function $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ satisfies the bicomplex Cauchy-Riemann equations.*

*Proof:* Consider the r-parameter bicomplex Mittag-Leffler function

$$E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) = \sum_{u=0}^{\infty} \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u + \upsilon))^s} \frac{\phi^u}{u!}$$

Now by using the idempotent property we get,

$$\begin{aligned} E_{\omega,\upsilon}^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}(\phi) &= E_{\omega_1,\upsilon_1}^{(p,q)_s}(\phi_1)e_1 + E_{\omega_2,\upsilon_2}^{(p,q)_s}(\phi_2)e_2 \\ &= E_{\omega_1,\upsilon_1}^{(p,q)_s}(\mu_1 - \iota_1\mu_2)e_1 + E_{\omega_2,\upsilon_2}^{(p,q)_s}(\mu_1 + \iota_2\mu_2)e_2 \\ &= E_{\omega_1,\upsilon_1}^{(p,q)_s}(\mu_1 - i_1\mu_2)\Big(\frac{1+i_1i_2}{2}\Big) + E_{\omega_2,\upsilon_2}^{(p,q)_s}(\mu_1 + i_1\mu_2)\Big(\frac{1-i_1\iota_2}{2}\Big) \\ &= \frac{1}{2}\Big(E_{\omega_1,\upsilon_1}^{(p,q)_s}(\mu_1 - i_1\mu_2) + E_{\omega_2,\upsilon_2}^{(p,q)_s}(\mu_1 + i_1\mu_2)\Big) \\ &\quad + \iota_2\Big(\frac{\iota_1}{2}\big(E_{\omega_1,\upsilon_1}^{(p,q)_s}(\mu_1 - i_1\mu_2) - E_{\omega_2,\upsilon_2}^{(p,q)_s}(\mu_1 + i_1\mu_2)\big)\Big) \\ &= f_1(\mu_1, \mu_2) + i_2f_2(\mu_1, \mu_2), \end{aligned}$$

where
$f_1(\mu_1, \mu_2) = \frac{1}{2}\Big(E_{\omega_1,\upsilon_1}^{(p,q)_s}(\mu_1 - i_1\mu_2) + E_{\omega_2,\upsilon_2}^{(p,q)_s}(\mu_1 + i_1\mu_2)\Big)$,
$f_2(\mu_1, \mu_2) = \frac{i_1}{2}\Big(E_{\omega_1,\upsilon_1}^{(p,q)_s}(\mu_1 - i_1\mu_2) - E_{\omega_2,\upsilon_2}^{(p,q)_s}(\mu_1 + i_1\mu_2)\Big)$.

Now,

$$\frac{\partial f_1}{\partial \mu_1}=\frac{1}{2}\big(E'^{(p,q)_s}_{\omega_1,\upsilon_1}(\mu_1-i_1\mu_2)+E'^{(p,q)_s}_{\omega_2,\upsilon_2}(\mu_1+i_1\mu_2)\big),$$
$$\frac{\partial f_1}{\partial \mu_2}=\frac{1}{2}\big(-i_1E'^{(p,q)_s}_{\omega_1,\upsilon_1}(\mu_1-i_1\mu_2)+i_1E'^{(p,q)_s}_{\omega_2,\upsilon_2}(\mu_1+i_1\mu_2)\big),$$
$$\frac{\partial f_2}{\partial \mu_1}=\frac{i_1}{2}\big((E'^{(p,q)_s}_{\omega_1,\upsilon_1}(\mu_1-i_1\mu_2)-E'^{(p,q)_s}_{\omega_2,\upsilon_2}(\mu_1+i_1\mu_2)\big),$$
$$\frac{\partial f_2}{\partial \mu_2}=\frac{i_1}{2}\big(-i_1E'^{(p,q)_s}_{\omega_1,\upsilon_1}(\mu_1-i_1\mu_2)-i_1E'^{(p,q)_s}_{\omega_2,\upsilon_2}(\mu_1+i_1\mu_2)\big).$$

from these four equations we observe that,

$$\frac{\partial f_1}{\partial \mu_1}=\frac{\partial f_2}{\partial \mu_2}\text{ and }\frac{\partial f_1}{\partial \mu_2}=-\frac{\partial f_2}{\partial \mu_1}.$$

this shows that the bicomplex Cauchy-Riemann equations are satisfied by the function $E^{(p,q)_s}_{\omega,\upsilon}(\phi)$. ■

**Theorem IV.2.** *(Duplication formula). Let $p_\iota, q_\iota \in \mathbb{C};\ \omega,\upsilon,\phi\in\overline{\mathbb{C}}$ where $\phi=\mu_1+i_2\mu_2=\phi_1e_1+\phi_2e_2$ and $\omega=\omega_1e_1+\omega_2e_2=\delta_0+i_1\delta_1+i_2\delta_2+i_1i_2\delta_3, \upsilon=\upsilon_1e_1+\upsilon_2e_2=\varepsilon_0+i_1\varepsilon_1+i_2\varepsilon_2+i_1i_2\varepsilon_3$ with $|Im_j(\omega)|<Re(\omega)$, $|Im_j(\upsilon)|<Re(\upsilon)$. Then*

(a) $E^{(p,2q)_s}_{2\omega,\upsilon}(\phi^2)=\frac{1}{2}\big(E^{(p,q)_s}_{\omega,\upsilon}(\phi)+E^{(p,q)_s}_{\omega,\upsilon}(-\phi)\big)$.

(b) $\phi E^{(p,q)_s}_{2\omega,\omega+\upsilon}(\phi^2)=\frac{1}{2}\big(E^{(p,q)_s}_{\omega,\upsilon}(\phi)-E^{(p,q)_s}_{\omega,\upsilon}(-\phi)\big)$.

*Proof:* (a) Suppose $T$ represents the R.H.S of the above equation,
i.e,

$$\begin{aligned}T&=\frac{1}{2}\Big(\sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^u}{\omega!}+\\&\qquad\sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{(-\phi)^u}{u!}\Big)\\&=\frac{1}{2}\Big(\sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^u+(-\phi)^u}{u!}\Big)\\&=\frac{1}{2}\Big(2\frac{1}{(\Gamma(\upsilon))^s}+2\frac{(p_1)_{q_12}(p_2)_{q_22}\cdots(p_s)_{q_s2}}{(\Gamma(2\omega+\upsilon))^s}\frac{\phi^2}{2!}+\\&\qquad 2\frac{(p_1)_{q_14}(p_2)_{q_24}\cdots(p_s)_{q_s4}}{(\Gamma(4\omega+\upsilon))^s}\frac{\phi^4}{4!}+\cdots\Big)\\&=\sum_{u=0}^{\infty}\frac{(p_1)_{q_12u}(p_2)_{q_22u}\cdots(p_s)_{q_s2u}}{(\Gamma(2\omega u+\upsilon))^s}\frac{\phi^{2u}}{2u!}\\&=\sum_{m=0}^{\infty}\frac{(p_1)_{2q_1u}(p_2)_{2q_2u}\cdots(p_s)_{2q_su}}{(\Gamma(2\omega u+\upsilon))^s}\frac{(\phi^2)^u}{2u!}\\&=E^{(p,2q)_s}_{2\omega,\upsilon}(\phi^2).\end{aligned}$$

(b) Suppose $T$ represents the R.H.S of the above equation,
i.e,

$$\begin{aligned}T&=\frac{1}{2}\Big(\sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^u}{u!}-\\&\qquad\sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{(-\phi)^u}{u!}\Big)\\&=\frac{1}{2}\Big(\sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^u-(-\phi)^u}{u!}\Big)\\&=\frac{1}{2}\Big(2\frac{(p_1)_{q_11}(p_2)_{q_21}\cdots(p_s)_{q_s1}}{(\Gamma(\omega+\upsilon))^s}\frac{\phi}{1!}+\\&\qquad 2\frac{(p_1)_{q_13}(p_2)_{q_23}\cdots(p_s)_{q_s3}}{(\Gamma(3\omega+\upsilon))^s}\frac{\phi^3}{3!}+\\&\qquad 2\frac{(p_1)_{q_15}(p_2)_{q_25}\cdots(p_s)_{q_s5}}{(\Gamma(5\omega+\upsilon))^s}\frac{\phi^5}{5!}+\cdots\Big)\\&=\frac{1}{2}\Big(2\frac{(p_1)_{q_11}(p_2)_{q_21}\cdots(p_s)_{q_s1}}{(\Gamma(0+\omega+\upsilon))^s}\frac{\phi}{1!}+\\&\qquad 2\frac{(p_1)_{q_13}(p_2)_{q_23}\cdots(p_s)_{q_s3}}{(\Gamma(2\omega+\omega+\upsilon))^s}\frac{\phi^3}{3!}+\\&\qquad 2\frac{(p_1)_{q_15}(p_2)_{q_25}\cdots(p_s)_{q_s5}}{(\Gamma(2(2\omega)+\omega+\upsilon))^s}\frac{\phi^5}{5!}+\cdots\Big)\\&=\sum_{u=0}^{\infty}\frac{(p_1)_{q_1(2u+1)}(p_2)_{q_2(2u+1)}\cdots(p_s)_{q_s(2u+1)}}{(\Gamma(2\omega u+\omega+\upsilon))^s}\\&\qquad\times\frac{\phi^{2u+1}}{2u+1!}\\&=\phi\sum_{u=0}^{\infty}\frac{(p_1)_{q_1(2u+1)}(p_2)_{q_2(2u+1)}\cdots(p_s)_{q_s(2u+1)}}{(\Gamma(2\omega u+\omega+\upsilon))^s}\\&\qquad\times\frac{(\phi^2)^u}{2u+1!}\\&=\phi E^{(p,q)_s}_{2\omega,\omega+\upsilon}(\phi^2).\end{aligned}$$

■

**Theorem IV.3.** *(Recurrence Relations for the bicomplex r-parameter Mittag-Leffler function). Let $p_\iota, q_\iota\in\mathbb{C};\omega,\upsilon,\phi\in\overline{\mathbb{C}}$ where $\phi=\mu_1+i_2\mu_2=\phi_1e_1+\phi_2e_2$ and $\omega=\alpha_1e_1+\alpha_2e_2=a_0+i_1a_1+i_2a_2+i_1i_2a_3, \upsilon=\beta_1e_1+\beta_2e_2=b_0+i_1b_1+i_2b_2+i_1i_2b_3$ with $|Im_j(\omega)|<Re(\omega)$, $|Im_j(\upsilon)|<Re(\upsilon)$. Then*

(a) $E^{(p,q)_s}_{\omega,\upsilon}(\phi)=\frac{1}{\Gamma\upsilon}+\phi\mathbb{E}^{(p,q)_s}_{\omega,\upsilon+\omega}(\phi)$

(b) $\mathbb{E}^{(p,q)_s}_{\omega,\upsilon}(\phi)=\upsilon\mathbb{E}^{(p,q)_s}_{\omega,\upsilon+1}(\phi)+\omega\phi\frac{d}{d\phi}\mathbb{E}^{(p,q)_s}_{\omega,\upsilon+1}(\phi)$

(c) $$\begin{aligned}\mathbb{E}^{(p,q)_s}_{\omega,r+1}(\phi)&=r(r+2)\mathbb{E}^{(p,q)_s}_{\omega,r+3}(\phi)\\&\quad+\omega\phi(\omega+2(r+1))\mathbb{E}'^{(p,q)_s}_{\omega,r+3}(\phi)\\&\quad+\phi^2\mathbb{E}''^{(p,q)_s}_{\omega,r+3}(\phi)+\mathbb{E}^{(p,q)_s}_{\omega,r+2}(\phi)\end{aligned}$$

*Proof:* (a) By using the idempotent property, we have

$$\begin{aligned}\mathbb{E}^{(p,q)_s}_{\omega,\upsilon}(\phi)&=\mathbb{E}^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)e_1+\mathbb{E}^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)e_2\\&=\Big(\frac{1}{\Gamma\upsilon_1}+\phi_1\mathbb{E}^{(p,q)_s}_{\omega_1,\upsilon_1+\omega_1}(\phi_1)\Big)e_1\\&\quad+\Big(\frac{1}{\Gamma\upsilon_2}+\phi_2\mathbb{E}^{(p,q)_s}_{\omega_2,\upsilon_2+\omega_2}(\phi_2)\Big)e_2\\&=\frac{1}{\Gamma\upsilon}+\phi\mathbb{E}^{(p,q)_s}_{\omega,\upsilon+\omega}(\phi)\end{aligned}$$

(b) By utilizing the idempotent property, we have

$$\begin{aligned}\mathbb{E}^{(p,q)_s}_{\omega,\upsilon}(\phi) &= \mathbb{E}^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)e_1 + \mathbb{E}^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)e_2\\ &= \Big(\upsilon_1\mathbb{E}^{(p,q)_s}_{\omega_1,\upsilon_1+1}(\phi_1)\\ &\qquad + \omega_1\phi_1\frac{d}{d\phi_1}\mathbb{E}^{(p,q)_s}_{\omega_1,\upsilon_1+1}(\phi_1)\Big)e_1\\ &\quad+\Big(\upsilon_2\mathbb{E}^{(p,q)_s}_{\omega_2,\upsilon_2+1}(\phi_2)\\ &\qquad + \omega_2\phi_2\frac{d}{d\phi_2}\mathbb{E}^{(p,q)_s}_{\omega_2,\upsilon_2+1}(\phi_2)\Big)e_2\\ &= \upsilon\mathbb{E}^{(p,q)_s}_{\omega,\upsilon+1}(\phi) + \omega\phi\frac{d}{d\phi}\mathbb{E}^{(p,q)_s}_{\omega,\upsilon+1}(\phi)\end{aligned}$$

(c) By using the idempotent property, we have

$$\begin{aligned}\mathbb{E}^{(p,q)_s}_{\omega,r+1}(\phi) &= \mathbb{E}^{(p,q)_s}_{\omega,r+1}(\phi_1)e_1 + \mathbb{E}^{(p,q)_s}_{\omega,r+1}(\phi_2)e_2\\ &= \Big(r(r+2)\mathbb{E}^{(p,q)_s}_{\omega,r+3}(\phi_1)\\ &\quad+\omega\phi_1(\omega+2(r+1))\mathbb{E}'^{(p,q)_s}_{\omega,r+3}(\phi_1)\\ &\quad+\phi_1^2\mathbb{E}''^{(p,q)_s}_{\omega,r+3}(\phi_1)+\mathbb{E}^{(p,q)_s}_{\omega,r+2}(\xi_1)\Big)e_1\\ &\quad+\Big(r(r+2)\mathbb{E}^{(p,q)_s}_{\omega,r+3}(\phi_2)\\ &\quad+\omega\phi_2(\omega+2(r+1))\mathbb{E}'^{(p,q)_s}_{\omega,r+3}(\phi_2)\\ &\quad+\phi_2^2\mathbb{E}''^{(p,q)_s}_{\omega,r+3}(\phi_2)+\mathbb{E}^{(p,q)_s}_{\omega,r+2}(\phi_2)\Big)e_2\\ &= r(r+2)\mathbb{E}^{(p,q)_s}_{\omega,r+3}(\phi)\\ &\quad+a\phi(\omega+2(r+1))\mathbb{E}'^{(p,q)_s}_{\omega,r+3}(\phi)\\ &\quad+\phi^2\mathbb{E}''^{(p,q)_s}_{\omega,r+3}(\phi)+\mathbb{E}^{(p,q)_s}_{\omega,r+2}(\phi)\end{aligned}$$

■

## V. Analiticity, order and type of $r$-parameter bicomplex function.

**Theorem V.1.** *In bicomplex domain, the function $E^{(p,q)_s}_{\omega,\upsilon}(\phi)$ with $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$ is an entire function.*

*Proof:* Assume that $\sum\limits_{u=0}^{\infty} a_u\phi^u$ shows a bicomplex series, s.t $a_u, \phi \in \overline{\mathbb{C}}$; $a_u = b_ue_1 + c_ue_2$, $\phi = \phi_1e_1 + \phi_2e_2$. So by the Ringleb decomposition theorem,

$$\sum_{u=0}^{\infty} a_u\phi^u = \Big(\sum_{u=0}^{\infty} b_u\phi_1^u\Big)e_1 + \Big(\sum_{u=0}^{\infty} c_u\phi_2^u\Big)e_2$$

the above series converges in $\in \overline{\mathbb{C}}$ whenever both series $\sum\limits_{u=0}^{\infty} b_u\phi_1^u$ and $\sum\limits_{u=0}^{\infty} c_u\phi_2^u$ converges in $\mathbb{C}$.

Also by using idempotent property the function can be expressed as-

$$E^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}_{\omega,\upsilon}(\phi) = E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)e_1 + E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)e_2.$$

Since

$$E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1) = \sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_1u+\upsilon_1))^s}\frac{\phi_1^u}{u!},\quad for\ Re(\omega_1)>0,\ Re(\upsilon_1)>0$$

and

$$E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2) = \sum_{u=0}^{\infty}\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_2u+\upsilon_2))^s}\frac{\phi_2^u}{u!},\quad for\ Re(\omega_2)>0, Re(\upsilon_2)>0$$

above two equations represents the complex Mittag-Leffler function of infinite radius of convergence (Let R), so $|\phi_1| < R$, $|\phi_2| < R$.

From equation (7),

$$\begin{aligned}N(\phi) &= \sqrt{||\phi||^4 - \sqrt{||\phi||^4 - |\phi|^4_{abs}}}\\ &= \max(|\phi_1|,|\phi_2|) < R.\end{aligned}$$

So the function $E^{(p,q)_s}_{\omega,\upsilon}(\phi)$ is also converges with infinite radius of convergence in the bicomplex domain. As in $\mathbb{C}$, the functions $E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)$ and $E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)$ are entire therefore in $\overline{\mathbb{C}}$ the function $E^{(p,q)_s}_{\omega,\upsilon}(\phi)$ is entire also. ■

**Theorem V.2.** *The bicomplex Mittag-Leffler function $E^{(p,q)_s}_{\omega,\upsilon}(\phi)$ is an entire function of finite order $\tau = \frac{\delta_0-\delta_3 j}{(\delta_0^2-\delta_3^2)}$ and type $\varrho = 1$.*

*Proof:* By using idempotent property, we can write

$$E^{p_1,q_1;p_2,q_2;\ldots;p_s,q_s}_{\omega,\upsilon}(\phi) = E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)e_1 + E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)e_2,$$

here the functions $E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)$ for $Re(\omega_1) > 0$, $Re(\upsilon_1) > 0$ and $E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)$ for $Re(\omega_2) > 0$, $Re(\upsilon_2) > 0$ are the complex Mittag-Leffler function.

Also the functions $E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)$ and $E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)$ are entire functions, so there exist the numbers $k_1, k_2 > 0$ and positive numbers $r_1(k_1), r_2(k_2)$, such that

$$M_{E^{(p,q)_s}_{\omega_1,\upsilon_1}(r_1)} = \max_{|\phi_1|=r_1}|E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)| < e^{r_1^{k_1}}, \forall r_1 > r_1(k_1),$$

and

$$M_{E^{(p,q)_s}_{\omega_2,\upsilon_2}(r_2)} = \max_{|\phi_2|=r_2}|E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)| < e^{r_2^{k_2}}, \forall r_2 > r_2(k_2).$$

Let us consider $r = \max(r_1, r_2)$ and $k = \max(k_1, k_2)$, then we get

$$M_{E^{(p,q)_s}_{\omega_1,\upsilon_1}(r_1)} = \max_{|\phi_1|=r}|E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)| < e^{r_1^{k_1}} < e^{r^k},$$

and

$$M_{E^{(p,q)_s}_{\omega_2,\upsilon_2}(r_2)} = \max_{|\phi_2|=r}|E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)| < e^{r_2^{k_2}} < e^{r^k}.$$

So we have,

$$M_{E^{(p,q)_s}_{\omega,\upsilon}(r)} = \max|E^{(p,q)_s}_{\omega,\upsilon}(\phi)|_j$$

[using j-modulus of bicomplex number]

$$\begin{aligned}M_{E^{(p,q)_s}_{\omega,\upsilon}(r)} &= \max|E^{(p,q)_s}_{\omega_1,\upsilon_1}(\phi_1)|e_1\\ &\quad+\max|E^{(p,q)_s}_{\omega_2,\upsilon_2}(\phi_2)|e_2\\ M_{E^{(p,q)_s}_{\omega,\upsilon}(r)} &< e^{r^k}e_1 + e^{r^k}e_2\\ M_{E^{(p,q)_s}_{\omega,\upsilon}(r)} &= e^{r^k}\end{aligned}$$

this shows that $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ is an entire function.

Also, the order $\tau$ for the function $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ with $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$ is given by

$$\begin{aligned}\tau &= \limsup_{k\to\infty} \frac{k\ logk}{log\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{1}{u!}} \\ &= \Big(\limsup_{k\to\infty} \frac{k\ logk}{log\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_1u+\upsilon_1))^s}\frac{1}{u!}}\Big)e_1 + \\ &\quad \Big(\limsup_{k\to\infty} \frac{k\ logk}{log\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_2u+\upsilon_2))^s}\frac{1}{u!}}\Big)e_2 \\ &= \Big(\frac{1}{Re(\omega_1)}\Big)e_1 + \Big(\frac{1}{Re(\omega_2)}\Big)e_2 = \\ &\quad \Big(\frac{1}{\delta_0+\delta_3}\Big)e_1 + \Big(\frac{1}{\delta_0-\delta_3}\Big)e_2 = \frac{\delta_0-\delta_3 j}{(\delta_0^2-\delta_3^2)},\end{aligned}$$

where $\delta_0 > |\delta_3| \implies \delta_0^2 - \delta_3^2 \neq 0$.

Also the type $\varrho$ of bicomplex r-parameter Mittag-Leffler $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ is given by

$$\begin{aligned}\varrho &= \frac{i}{\varrho\tau}\ (\limsup_{k\to\infty} k|a_k|^{\frac{\tau}{k}}_j) \\ &= \frac{i}{\varrho\tau}\ \Big(\limsup_{k\to\infty} k\frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{1}{u!}|^{\frac{\tau}{k}}_j\Big) \\ &= \frac{i}{\varrho\tau}\ \Big(\limsup_{k\to\infty} k\frac{(p_1)_{q_1u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_1 n+\upsilon_1))^s}\frac{1}{u!}|^{\frac{\tau}{k}}_j\Big)\ e_1 + \\ &\quad \frac{i}{\varrho\tau}\ \Big(\limsup_{k\to\infty} k\frac{(p_1)_{q_1u}\cdots(p_s)_{q_su}}{(\Gamma(\omega_2 u+\upsilon_2))^s}\frac{1}{u!}|^{\frac{\tau}{k}}_j\Big)\ e_2 \\ &= 1\cdot e_1 + 1\cdot e_2 \\ &= 1.\end{aligned}$$

■

## VI. Integral and Differential relations.

**Theorem VI.1.** *(Integral Relations): The function $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ with $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$ satisfies the following integral relation.*

(a) $\frac{1}{\Gamma(\alpha u+1)}\int_0^\infty e^{-z}E_{\omega,\upsilon}^{(p,q)_s}(z^\alpha\phi) = E_{\omega,\upsilon}^{(p,q)_s}(\phi)$.

(b) $\int_0^\infty e^{-z}E_{\omega,\upsilon}^{(p,q)_s}(z\phi)dt = \frac{1}{1-\phi}$.

*Proof:* (a) Suppose $T$ represents the L.H.S of the above equation.
i.e,

$$\begin{aligned}T &= \frac{1}{\Gamma(\alpha u+1)}\int_0^\infty e^{-z}E_{\omega,\upsilon}^{(p,q)_s}(z^\alpha\phi)dz \\ &= \frac{1}{\Gamma(\alpha u+1)}\int_0^\infty e^{-z}\times \\ &\quad \Big(\sum_{u=0}^\infty \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{(z^\alpha\phi)^u}{u!}\Big)dz \\ &= \frac{1}{\Gamma(\alpha u+1)}\int_0^\infty e^{-z}\times \\ &\quad z^{\alpha u}\Big(\sum_{u=0}^\infty \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^u}{u!}\Big)dz \\ &= \frac{1}{\Gamma(\alpha u+1)}\int_0^\infty e^{-z}\times \\ &\quad z^{(\alpha u+1)-1}\Big(\sum_{u=0}^\infty \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^u}{u!}\Big)dz \\ &= \frac{1}{\Gamma(\alpha u+1)}\int_0^\infty e^{-z}z^{(\alpha u+1)-1}E_{\omega,\upsilon}^{(p,q)_s}(\phi)dz \\ &= E_{\omega,\upsilon}^{(p,q)_s}(\phi).\end{aligned}$$

(b) Suppose $T$ represents the L.H.S of the above equation, i.e

$$\begin{aligned}T &= \int_0^\infty e^{-z}E_{\omega,\upsilon}^{(p,q)_s}(z\phi)dz \\ &= \int_0^\infty e^{-z}\Big(\sum_{u=0}^\infty \frac{(p_1)_{q_1u}(p_2)_{q_2u}\cdots(p_s)_{q_su}}{(\Gamma(\omega u+\upsilon))^s} \\ &\qquad \times\frac{(z\phi)^u}{u!}\Big)dz \\ &= \int_0^\infty e^{-z}z^u\Big\{\frac{(p_1)_{q_10}(p_2)_{q_20}\cdots(p_s)_{q_s0}}{(\Gamma(\upsilon))^s} + \\ &\qquad \frac{(p_1)_{q_11}(p_2)_{q_21}\cdots(p_s)_{q_s1}}{(\Gamma(\omega+\upsilon))^s}\frac{\phi^u}{u!} + \cdots\cdots\Big\}dz\end{aligned}$$

on taking the values of $p_\iota = q_\iota = 1$, $\forall\ \iota = 1, 2, \cdots, s$ and $\omega = \upsilon = 1$.

$$\begin{aligned} &= \int_0^\infty e^{-z}z^{(u+1)-1}\Big\{\frac{(1)_{1\cdot0}(1)_{1\cdot0}\cdots(1)_{1\cdot0}}{(\Gamma(1))^s} + \\ &\qquad \frac{(1)_{1\cdot1}(1)_{1\cdot1}\cdots(1)_{1\cdot1}}{(\Gamma(2))^s}\frac{\phi}{1!} + \cdots\Big\}dz \\ &= \int_0^\infty e^{-z}z^{(u+1)-1}\Big\{\frac{\{(1)_{1\cdot0}\}^s}{(\Gamma(1))^s} + \frac{\{(1)_{1\cdot1}\}^s}{(\Gamma(2))^s}\frac{\phi}{1!} + \\ &\qquad \frac{\{(1)_{1\cdot2}\}^s}{(\Gamma(3))^s}\frac{\phi^2}{2!} + \cdots\Big\}dz \\ &= \int_0^\infty e^{-z}z^{(u+1)-1}\Big\{1 + \frac{\phi}{1!} + \frac{\phi^2}{2!} + \cdots\Big\}dz \\ &= \int_0^\infty e^{-z}z^{(u+1)-1}\Big(\sum_{k=0}^\infty \frac{\phi^k}{k!}\Big)dz \\ &= \frac{1}{1-\phi}.\end{aligned}$$

■

**Theorem VI.2.** *(Integral Relations). Let $\phi$, $\lambda \in \overline{\mathbb{C}}$, $\lambda \notin \mathbb{NC}$ where $\phi = \mu_1 + i_2\mu_2 = \phi_1e_1 + \phi_2e_2$, $\upsilon > 0$, Then*

(a) $\phi^\upsilon E_{1,\upsilon+1}^{(p,q)_s}(\lambda\phi) = \frac{1}{\Gamma\upsilon}\int_0^\phi (\phi-t)^{\upsilon-1}e^{\lambda t}dt$,

(b) $\phi^\upsilon E_{2,\upsilon+1}^{(p,q)_s}(\lambda\phi^2) = \frac{1}{\Gamma\upsilon}\int_0^\phi (\phi-t)^{\upsilon-1}cosh\sqrt{\lambda t}\ dt$,

(c) $\phi^{\upsilon+1} E_{2,\upsilon+1}^{(p,q)_s}(\lambda\phi^2) = \frac{1}{\Gamma\upsilon}\int_0^\phi (\phi-t)^{\upsilon-1}\frac{sinh\sqrt{\lambda t}}{\sqrt{\lambda}}dt$,

*Proof:* (a) Suppose $S$ represents the L.H.S of the above equation, i.e,

$$S = \phi^{\upsilon} E_{1,\upsilon+1}^{(p,q)_s}(\lambda\phi)$$

using the idempotent property, we have

$$= \big(\phi_1^{\upsilon} E_{1,\upsilon+1}^{(p,q)_s}(\lambda_1\phi_1)\big)e_1 + \big(\phi_2^{\upsilon} E_{1,\upsilon+1}^{(p,q)_s}(\lambda_2\phi_2)\big)e_2$$
$$= \left(\frac{1}{\Gamma\upsilon}\int_0^{\phi_1}(\phi_1 - t)^{\upsilon-1}e^{\lambda_1 t}dt\right)e_1 + \left(\frac{1}{\Gamma\upsilon}\int_0^{\phi_2}(\phi_2 - t)^{\upsilon-1}e^{\lambda_2 t}dt\right)e_2$$
$$= \frac{1}{\Gamma\upsilon}\int_0^{\phi}(\phi - t)^{\upsilon-1}e^{\lambda t}dt.$$

(b) Suppose $S$ represents the L.H.S of the above equation, i.e,

$$S = \phi^{\upsilon} E_{2,\upsilon+1}^{(p,q)_s}(\lambda\phi^2)$$

By using the idempotent property, we have

$$= \big(\phi_1^{\upsilon} E_{2,\upsilon+1}^{(p,q)_s}(\lambda_1\phi_1^2)\big)e_1 + \big(\phi_2^{\upsilon} E_{2,\upsilon+1}^{(p,q)_s}(\lambda_2\phi_2^2)\big)e_2$$
$$= \left(\frac{1}{\Gamma\upsilon}\int_0^{\phi_1}(\phi_1 - t)^{\upsilon-1}cosh\sqrt{\lambda_1}tdt\right)e_1 + \left(\frac{1}{\Gamma\upsilon}\int_0^{\phi_2}(\phi_2 - t)^{\upsilon-1}cosh\sqrt{\lambda_2}tdt\right)e_2$$
$$= \frac{1}{\Gamma\upsilon}\int_0^{\phi}(\phi - t)^{\upsilon-1}cosh\sqrt{\lambda}tdt$$

(c) Suppose $S$ represents the L.H.S of the above equation. i.e,

$$S = \phi^{\upsilon+1} E_{2,\upsilon+2}^{(p,q)_s}(\lambda\phi^2)$$

using the idempotent property, we have

$$= \big(\phi_1^{\upsilon+1} E_{2,\upsilon+2}^{(p,q)_s}(\lambda_1\phi_1^2)\big)e_1 + \big(\phi_2^{\upsilon+1} E_{2,\upsilon+2}^{(p,q)_s}(\lambda_2\phi_2^2)\big)e_2$$
$$= \left(\frac{1}{\Gamma\upsilon}\int_0^{\phi_1}(\phi_1 - t)^{\upsilon-1}\frac{sinh\sqrt{\lambda_1}t}{\sqrt{\lambda_1}}dt\right)e_1 + \left(\frac{1}{\Gamma\upsilon}\int_0^{\phi_2}(\phi_2 - t)^{\upsilon-1}\frac{sinh\sqrt{\lambda_2}t}{\sqrt{\lambda_2}}dt\right)e_2$$
$$= \frac{1}{\Gamma\upsilon}\int_0^{\phi}(\phi - t)^{\upsilon-1}\frac{sinh\sqrt{\lambda}t}{\sqrt{\lambda}}dt$$

■

**Theorem VI.3.** *(Differential Relations). For $\eta > 0$, the function $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ with $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$ satisfies the differential relation.*

(a) $\left(\frac{d}{d\phi}\right)^{\eta} E_{\omega,\upsilon}^{(p,q)_s}(\phi^{\eta}) = E_{\eta}(\phi^{\eta})$

(b) $\left(\frac{d}{d\phi}\right)^{\eta}\left(\phi^{\upsilon-1}E_{\omega,\upsilon}^{(p,q)_s}(\phi^{\omega})\right) = \phi^{\upsilon-\eta-1}E_{\omega,\upsilon-\eta}^{(p,q)_s}(\phi^{\omega})$.

*Proof:* (a) Suppose $S$ represents the L.H.S of the above equation. i.e,

$$S = \left(\frac{d}{d\phi}\right)^{\eta} E_{\omega,\upsilon}^{(p,q)_s}(\phi^{\eta})$$
$$= \left(\frac{d}{d\phi}\right)^{m}\sum_{u=0}^{\infty}\frac{(p_1)_{q_1 u}(p_2)_{q_2 u}\cdots(p_s)_{q_s u}}{(\Gamma(\omega u+\upsilon))^s}\frac{\phi^{mu}}{mu!}$$
$$= \left(\frac{d}{d\phi}\right)^{m}\Big\{\frac{(p_1)_{q_1 0}(p_2)_{q_2 0}\cdots(p_s)_{q_s 0}}{\Gamma(\upsilon)^s} + \frac{(p_1)_{q_1 1}(p_2)_{q_2 1}\cdots(p_s)_{q_s 1}}{\Gamma(\omega+\upsilon)^s}\frac{\phi^{m}}{m!} + \frac{(p_1)_{q_1 2}(p_2)_{q_2 2}\cdots(p_s)_{q_s 2}}{\Gamma(2\omega+\upsilon)^s}\frac{\phi^{2m}}{2m!} + \cdots\Big\}$$

on taking the values of $p_{\iota} = q_{\iota} = 1$, $\forall\ \iota = 1, 2, \cdots, s$ and $\omega = \upsilon = 1$.

$$= \left(\frac{d}{d\phi}\right)^{\eta}\Big\{\frac{(1)_{1\cdot 0}(1)_{1\cdot 0}\cdots(1)_{1\cdot 0}}{(\Gamma(1))^s} + \frac{(1)_{1\cdot 1}(1)_{1\cdot 1}\cdots(1)_{1\cdot 1}}{(\Gamma(2))^s}\frac{\phi^{\eta}}{\eta!} + \frac{(1)_{1\cdot 2}(1)_{1\cdot 2}\cdots(1)_{1\cdot 2}}{(\Gamma(3))^s}\frac{\phi^{2\eta}}{2\eta!} + \cdots\Big\}$$
$$= \left(\frac{d}{d\phi}\right)^{\eta}\Big\{\frac{\{(1)_{1\cdot 0}\}^s}{(\Gamma(1))^s} + \frac{\{(1)_{1\cdot 1}\}^s}{(\Gamma(2))^s}\frac{\phi^{\eta}}{\Gamma(\eta+1)} + \frac{\{(1)_{1\cdot 2}\}^s}{(\Gamma(3))^s}\frac{\phi^{2\eta}}{\Gamma(2\eta+1)} + \cdots\Big\}$$
$$= \left(\frac{d}{d\phi}\right)^{\eta}(1) + \left(\frac{d}{d\phi}\right)^{\eta}\frac{\phi^{\eta}}{\Gamma(\eta+1)} + \left(\frac{d}{d\phi}\right)^{\eta}\frac{\phi^{2\eta}}{\Gamma(2\eta+1)} + \left(\frac{d}{d\phi}\right)^{\eta}\frac{\phi^{3\eta}}{\Gamma(3\eta+1)}\cdots$$
$$= 1 + \Big\{\Big(\frac{(2\eta)(2\eta-1)\cdots(2\eta-(\eta-1))}{(2\eta)(2\eta-1)\cdots(2\eta-(\eta-1))\Gamma(2\eta-(\eta-1))}\Big) \times \phi^{(2\eta-\eta)}\Big\} + \Big\{\Big(\frac{(3\eta)(3\eta-1)\cdots(3\eta-(\eta-1))}{(3\eta)(3\eta-1)\cdots(3\eta-(\eta-1))\Gamma(3\eta-(\eta-1))}\Big) \times \phi^{(3\eta-\eta)}\Big\} + \cdots$$
$$= 1 + \frac{\phi^{(2\eta-\eta)}}{\Gamma(2\eta-(\eta-1))} + \frac{\phi^{(3\eta-\eta)}}{\Gamma(3\eta-(\eta-1))} + \cdots$$
$$= \sum_{l=1}^{\infty}\frac{\phi^{(l\eta-\eta)}}{\Gamma(l\eta-(\eta-1))}, \quad \text{taking } l \to l+1$$
$$= \sum_{l=0}^{\infty}\frac{\phi^{l\eta}}{\Gamma(l\eta+1)} = E_{\eta}(\phi^{\eta}).$$

(b) Suppose $S$ represents the L.H.S of the above equation. i.e,

$$S = \left(\frac{d}{d\phi}\right)^{\eta}\left(\phi^{\upsilon-1}E_{\omega,\upsilon}^{(p,q)_s}(\phi^{\omega})\right)$$

using the idempotent property, we have

$$= \left(\frac{d}{d\phi_1}\right)^{\eta}\left(\phi_1^{\upsilon_1-1}E_{\omega,\upsilon_1}^{(p,q)_s}(\phi_1^{\omega})\right)e_1 + \left(\frac{d}{d\phi_2}\right)^{\eta}\left(\phi_2^{\upsilon_2-1}E_{\omega,\upsilon_2}^{(p,q)_s}(\phi_2^{\omega})\right)e_2$$

$$
\begin{aligned}
&= \left(\phi_1^{\upsilon_1-\eta-1} E_{\omega,\upsilon_1-\eta}^{(p,q)_s}(\phi_1^{\omega})\right) e_1 \\
&\quad + \left(\phi_2^{\upsilon_2-\eta-1} E_{\omega,\upsilon_2-\eta}^{(p,q)_s}(\phi_2^{\omega})\right) e_2 \\
&= \phi^{\upsilon-\eta-1} E_{\omega,\upsilon-\eta}^{(p,q)_s}(\phi^{\omega}), \quad \eta \geq 1.
\end{aligned}
$$

■

## VII. Bicomplex Laplace transform and Caputo fractional derivative

Assume $f$ be a function then the $\mu^{th}$ order Caputo fractional derivative is described in such a way

$$
{}^{C}D^{\mu} f(\kappa) = \frac{1}{\Gamma(l-\mu)} \int_0^{\kappa} \frac{f^{(l)}(\varsigma)}{(\kappa-\varsigma)^{\mu+1-l}} d\varsigma, \qquad (17)
$$

where, $l-1 < \mu \leq l, \ l \in \mathbb{N}, \ \kappa > 0$.

Let $s = s_1e_1 + s_2e_2 = \delta_0 + \delta_1 i_1 + \delta_2 i_2 + \delta_3 i_1 i_2 \in \overline{\mathbb{C}}$ and $f(\kappa)$ represents a real valued function whose order is $m \in \mathbb{R}$. Then, for $l-1 < \mu \leq l, \ l \in \mathbb{N}$ the bicomplex Laplace transform of equation (17) is expressed by

$$
L({}^{C}D^{\mu} f(\kappa); s) = s^{\mu} F(s) - \sum_{m=0}^{l-1} s^{\mu-m-1} f^{m}(0), \qquad (18)
$$

The bicomplex Laplace transform of $f(t)$ is given by $F(s)$. Also for all $s \in D$, it is convergent. Where $D$ is defined below

$D = \{s : H_{\rho}(s)$ denotes the right half plane : $\delta_0 > m + |\delta_3|\}$

As for all $s \in D$, $F(s)$ is convergent therefore the bicomplex Laplace transform defined in equation (18) is also convergent.

**Theorem VII.1.** *Let $s$, $\omega$, $\upsilon$, $\lambda \in \overline{\mathbb{C}}$ where $s = s_1e_1 + s_2e_2$, then the Laplace transform of $E_{\omega,\upsilon}^{(p,q)_s}(\phi)$ is expressed by*

$$
L[\phi^{\upsilon-1} E_{\omega,\upsilon}^{(p,q)_s}(\lambda\phi^{\omega}); s] = \frac{s^{\omega-\upsilon}}{s^{\omega}-\lambda} \qquad (19)
$$

*where $|Im_j(\omega)| < Re(\omega)$, $|Im_j(\upsilon)| < Re(\upsilon)$, $|Im_j(s)| < Re(s)$, $|\lambda s^{-\omega}|_j < 1$.*

*Proof:* Suppose $S$ represents the L.H.S of the above equation.
i.e,

$$
S = L[\phi^{\upsilon-1} E_{\omega,\upsilon}^{(p,q)_s}(\lambda\phi^{\omega}); s]
$$

using the idempotent property, we have

$$
\begin{aligned}
&= L[\phi^{\upsilon_1-1} E_{\omega_1,\upsilon_1}^{(p,q)_s}(\lambda\phi_1^{\omega}); s_1] e_1 + \\
&\quad L[\phi^{\upsilon_2-1} E_{\omega_2,\upsilon_2}^{(p,q)_s}(\lambda\phi_2^{\omega}); s_2] e_2 \\
&= \left(\int_0^{\infty} e^{-s_1\phi} E_{\omega_1,\upsilon_1}^{(p,q)_s}(\lambda_1\phi_1^{\omega})\phi^{\upsilon_1-1} d\phi\right) e_1 + \\
&\quad \left(\int_0^{\infty} e^{-s_2\phi} E_{\omega_2,\upsilon_2}^{(p,q)_s}(\lambda_2\phi_2^{\omega})\phi^{\upsilon_2-1} d\phi\right) e_2 \\
&= \frac{s^{\omega_1-\upsilon_1}}{s_1^{\omega_1}-\lambda_1} e_1 + \frac{s^{\omega_2-\upsilon_2}}{s_2^{\omega_2}-\lambda_2} e_2, \\
&\qquad (|\lambda_1 s_1^{-\omega_1}| < 1, \ |\lambda_2 s_2^{-\omega_2}| < 1) \\
&= \frac{s^{\omega-\upsilon}}{s^{\omega}-\lambda}.
\end{aligned}
$$

here,

$$
\begin{aligned}
|\lambda s^{-\omega}|_j &= |\lambda_1 s_1^{-\omega_1}| e_1 + |\lambda_2 s_2^{-\omega_2}| e_2 \\
&< 1.e_1 + 1.e_2 = 1.
\end{aligned}
$$

■

## VIII. Conclusion.

In this article, the bicomplex r-parameter Mittag-Leffler function has been introduced and some of its characteristics have been discovered such as integral representation, differential representation, duplication formula, order and type and bicomplex C-R equation. Besides that Laplace transform and Caputo fractional derivative are also achieved. By using the concept of fractional calculus on this function some interesting results can be achieved in future. The importance of the Mittag-Leffler function is continually expanding in applied sciences and engineering. When working with differential and integral equations of fractional order, it is extremely useful. As a result the large class of function which are present in electromagnetism, signal theory and quantum theory can be addressed more generally using bicomplex space.

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