An Extension of Analysis of Multiple Period Bus Transit System

Yu-Lan Wang, Chiu-Tang Lin, Junling Zhang, and Xiaolin Li

***Abstract*—This paper extends the paper of feeder bus systems. In their systems, the condition to guarantee the existence of the optimal solution is not discussed. We investigate the Hessian matrix to find criterion to locate the maximum solution and derive the closed form solution for the profit function. Moreover, we derive a formulated estimation for the route spacing in order to provide readers with a detailed look into existing methods, and their characteristics and applicability to analyze the closed form profit function. In our note, we have found: (a) To find the criterion to insure the profit function has critical points and show the condition to guarantee that the small one is uniquely maximum point; (b) To solve the maximum profit in a closed form and find the condition to insure the maximum profit is positive such that operation of the bus system is profitable; (c) To construct a better formulated approximation and compare numerical examples from their paper.**

***Index Terms*—Analytical approach, Headway of bus, Public transportation, Bus service area**

I. Introduction

OPTIMIZING various decision variables, such as service headway, route length, and route spacing, is crucial in the design of transit systems. Over the past decades, analytical optimization models have been developed to simultaneously address these factors when optimizing bus systems. However, it is important to acknowledge that certain decision variables, like stop spacing, route spacing, and the length of local service zones, may exhibit spatial variation while remaining practically unchangeable over time in real transit systems. This dynamic nature necessitates a comprehensive understanding of the interplay between these variables to achieve efficient and effective transit system design. In this paper, we delve into the complexities associated with jointly optimizing these factors and explore potential solutions to improve the design and functionality of transit systems. In the realm of optimizing public transportation systems, various decision variables play a crucial role. While some factors, such as service headways, exhibit variation across different periods, they typically remain consistent along specific routes. Extensive literature exists on the analytical approaches used to optimize these systems, with many studies assuming a fixed demand and minimizing a total cost function comprising operator and user costs (e.g., Vuchic and Newell [1], Tsao and Schonfeld [2], Kuah and Perl [3]). However, these approaches may have limited applicability due to their assumption of zero demand elasticity. To address this limitation and incorporate additional factors, such as demand elasticity, financial constraints, and the impact of fleet size on congestion, for an urban bus service, Oldfield and Bly [4] developed an analytic model to determine the optimal vehicle size. Notably, those papers discussed above did not account for time-dependent demand.

By minimizing user wait time, with a fixed fleet size, in contrast, Newell [5] treated demand as a smooth function of time, optimizing the schedule for a transit route, and incorporated one or more demand peaks. This paper aims to expand on existing research by considering the influence of fleet size, financial constraints, and demand elasticity on congestion in the optimization of public transportation systems. Furthermore, we take into account the time-dependent nature of demand to refine the scheduling process and enhance user experience. By addressing these important factors, we strive to develop a comprehensive and effective model for optimizing transit systems that better aligns with real-world scenarios. Without demand elasticity to obtain analytic solutions, in analyzing a commuter bus system, Clarens and Hurdle [6] have considered time-dependent demand. In the realm of optimizing public transportation systems, the development of multiple period models has been instrumental. Chang and Schonfeld [7] have introduced these models, which enable certain system characteristics, such as route structure, to be fixed based on the best compromise over different time periods. Simultaneously, other characteristics, within each specific period, to optimize service headways. These models have been formulated and compared under four distinct conditions: equilibrium demand scenarios, steady equilibrium demand, cyclical fixed demand, and steady fixed demand. When considering fixed demand, the primary optimization objective is to minimize the total system cost, encompassing both user and operator costs. Conversely, for equilibrium demand scenarios, the objective functions aim to maximize operator profit and social welfare. The outcomes of these models yield closed-form solutions for optimized bus services, including route spacing, headway, fare, and cost, accounting for different demand conditions. This paper builds upon the pioneering work of Chang and Schonfeld [7]

Manuscript received May 17, 2023; revised September 13, 2023.

This work was supported in part by the Weifang University of Science and Technology, with the unified social credit code: 52370700MJE3971020.

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by examining and analyzing the implications of their multiple period models. By considering various demand scenarios and incorporating different optimization objectives, we aim to deepen our understanding of the optimal design and operation of public transportation systems. The insights gained from this research will contribute to the development of more effective and efficient bus services, providing valuable guidance for transit planners and policymakers.

This paper adopts a proof of Hessian Matrix in order to provide readers with a detailed look into existing methods, and their characteristics and applicability to analysis of the close form profit function. Also, we believe it comprehensive result furnishes decision makers and researchers much information and more confidence for both comparisons and applications of analytic approaches. In our note, we have found that the following improvements: (a) To find the criterion to insure the profit function has critical points for fare and show the condition to guarantee that small one is uniquely maximum point between the bigger and small one; (b) To solve the maximum profit in a closed form and find the condition to insure the maximum profit is positive such that to run the bus system is profitable; (c) To construct a better formulated approximation for routing space and compare numerical example results from Chang and Schonfeld [7] by our solving procedure.

II. Assumptions and Notation

To be compatible with Chang and Schonfeld [7], we will adopt the same assumptions and notation as theirs.

z: ratio of wait time/headway.

y: express ratio = express speed/local speed.

W: width of bus service area.

V: local bus speed.

T: duration time (service hours).

r: route spacing.

q: potential or fixed demand density.

P: total system profit.

M: passenger average trip time, $\mathrm{M} = \frac{\mathrm{L}}{2\mathrm{V}} + \frac{\mathrm{W}}{2\mathrm{bV}} + \frac{\mathrm{J}}{\mathrm{yV}}$.

L: length of bus service area.

k: invariant component of the demand function, $\mathrm{k} = 1 - \frac{\mathrm{d}}{4\mathrm{g}}\mathrm{e_x} - \mathrm{Me_v}$.

J: express distance.

h: headway. g: access speed.

f: fare.

e_w: demand elasticity parameter for waiting time.

e_x: demand elasticity parameter for access time.

e_v: demand elasticity parameter for in-vehicle time.

e_p: demand elasticity parameter for fare.

D: bus average round trip time, $\mathrm{D} = \frac{2\mathrm{L}}{\mathrm{V}} + \frac{\mathrm{W}}{\mathrm{bV}} + \frac{2\mathrm{J}}{\mathrm{yV}}$.

d: bus stop spacing.

B: bus operating cost.

b: non-stop ratio = non-stop speed / local speed.

III. Review of Chang and Schonfeld [7]

According to the Chang and Schonfeld [7], it states that profit is total revenue minus operator cost. Firstly, we study the profit, $P = P(f,h,r)$, of bus transit system under steady equilibrium demand, Based on Chang and Schonfeld [7], since they have derived that

$$P = P(f,h,r) = -\frac{BDTW}{hr} + fqTLW\left(k - e_p f - \frac{e_x r}{4g} - e_w hz\right). \quad (3.1)$$

Secondly, we review the results of Chang and Schonfeld [7]. Owing to analyze the relation between (a) number of zones and (b) headway, they derived the system of first partial derivative equations

$$\frac{\partial P}{\partial r} = -\frac{e_x}{4g} fqTLW + \frac{BDTW}{hr^2}, \quad (3.2)$$

$$\frac{\partial P}{\partial h} = -e_w fqzTLW + \frac{BDTW}{h^2 r}, \quad (3.3)$$

and

$$\frac{\partial P}{\partial f} = -e_p fqTLW, + qTLW\left(k - e_p f - \frac{e_x r}{4g} - e_w hz\right). \quad (3.4)$$

Chang and Schonfeld [7] tried to solve the zeros of the first partial derivative system such that they will solve

$$\frac{e_x}{4g} fqTLW = \frac{BDTW}{hr^2}, \quad (3.5)$$

$$e_w fqzTLW = \frac{BDTW}{h^2 r}, \quad (3.6)$$

and

$$qTLW\left(k - e_p f - \frac{e_x r}{4g} - e_w hz\right) = e_p fqTLW. \quad (3.7)$$

From Equations (3.5) and (3.6), they implied

$$h = \frac{e_x}{4gze_w} r. \quad (3.8)$$

Plugging Equation (3.8) into Equation (3.7), they got

$$f = \frac{1}{2e_p}\left(k - \frac{e_x}{2g} r\right). \quad (3.9)$$

Substituting Equations (3.8) and (3.9) into Equation (3.6), they had

$$-\frac{e_x}{2gk} r^4 + r^3 - \frac{32g^2 zBDe_p e_w}{kqLe_x^2} = 0. \quad (3.10)$$

They claimed that if route spacing is less than two miles, then the first term in Equation (3.10) is relatively small to be neglected. Therefore, the remaining equation implies the next approximated solution for route spacing:

$$r \approx \left(\frac{32g^2 zBDe_p e_w}{kqLe_x^2}\right)^{\frac{1}{3}}. \quad (3.11)$$

IV. Our improved results

To derive the condition for the existence and uniqueness of positive solutions for route spacing.

First, we will show that Equation (3.10) in general does not have only one solution. Motivated by Equation (3.10) and for later use, first we consider the following function

$$f(r) = -a_4 r^4 + r^3 - a_0, \tag{4.1}$$

with $a_4 > 0$ and $a_0 > 0$, where

$$a_4 = \frac{e_x}{2gk} r^4, \tag{4.2}$$

and

$$a_0 = \frac{32g^2 zBDe_p e_w}{kqLe_x^2}. \tag{4.3}$$

Based on Equation (4.1), we derive that

$$f'(r) = r^2(3 - 4a_4 r), \tag{4.4}$$

and

$$f''(r) = 6r(1 - 2a_4 r). \tag{4.5}$$

According to Equations (4.4) and (4.5), we solve the zeros for them such that we know that $f(r)$ increases for $r \in \left(-\infty, \frac{3}{4a_4}\right)$ and decreases for $r \in \left(\frac{3}{4a_4}, \infty\right)$; moreover, $f(r)$ is concave up for $r \in \left(0, \frac{1}{2a_4}\right)$ and concave down for $r \in (-\infty, 0)$, and $r \in \left(\frac{1}{2a_4}, \infty\right)$. Therefore, $f(r)$ has its maximum value at $r = \frac{3}{4a_4}$, and

$$f\left(\frac{3}{4a_4}\right) = \frac{27}{256a_4^3} - a_0. \tag{4.6}$$

Now from our earlier discussion, we derive the Lemma 1 and Lemma 2 which are described as follows:

Lemma 1. $f(r) = 0$ has positive solution if and only if $f\left(\frac{3}{4a_4}\right) \geq 0$ if and only if $27 \geq 256a_0 a_4^3$.

Lemma 2. The profit $P(f, h, r)$ has critical points if and only if $\frac{27}{1024} \geq \frac{zBDe_p e_w e_x}{gk^4 qL}$.

In the numerical example, we will show the condition that from the practical point of view $\frac{27}{1024} > \frac{zBDe_p e_w e_x}{gk^4 qL}$ is hold. Therefore, Equation (3.10) has two positive roots, say r_1 and r_2 with $r_1 < \frac{3}{4a_4} < r_2$.

From this point onward, we shall also prove that r_2 is not a local maximum point and find the condition to insure that r_1 is a local maximum point, then to prove that r_1 is the global maximum point. Hence, we consider the Hessian Matrix of $P(f, h, r)$, then

$$H = \lfloor h_{ij} \rfloor_{3\times3},$$

$$= -wT \begin{bmatrix} \frac{2BD}{hr^3} & \frac{BD}{h^2 r^2} & \frac{qLe_x}{4g} \\ \frac{BD}{h^2 r^2} & \frac{2BD}{h^3 r} & qzLe_w \\ \frac{qLe_x}{4g} & qzLe_w & 2qLe_p \end{bmatrix}. \tag{4.7}$$

By Equations (3.8) and (3.10),we have the three principal determinants as

$$\det[h_{ij}]_{1\times1} = -wT\frac{2BD}{hr^3} < 0, \tag{4.8}$$

$$\det[\mathrm{h}_{ij}]_{2\times2} = (-\mathrm{wT})^2 \frac{3\mathrm{B}^2\mathrm{D}^2}{\mathrm{h}^4\mathrm{r}^4} > 0, \tag{4.9}$$

and

$$\det\lfloor h_{ij} \rfloor_{3\times3} = -\frac{w^3 T^3}{h^4 r^4},$$

$$+\frac{q^2 BDL^2 e_x{}^2 k}{4g^2 ze_w}\left(\frac{56g^2 zBDe_p e_w}{kqLe_x^2} - r^3\right). \tag{4.10}$$

We know the followings are equivalent: (i) r_2 is a local maximum point; (ii) The three principal determinants are alternative between positive and negative each other; (iii) $\det\lfloor h_{ij} \rfloor_{3\times3} < 0$; (iv) $\frac{7}{4}a_0 > r_2^3$.

By $f\left(\frac{3}{4a_4}\right) \geq 0$ then $27 \geq 256a_0 a_4^3$ and

$$\frac{7}{4}a_0 < \left(\frac{3}{4a_4}\right)^3 < r_2^3. \tag{4.11}$$

Hence r_2 is not a local maximum point. Here, we begin to obtain the criterion to insure that r_1 is a local maximum point. The followings are equivalent: (a) r_1 is a local maximum point; (b) The three principal determinants are alternative between positive and negative each other; (c) $\det\lfloor h_{ij} \rfloor_{3\times3} < 0$; (d) $\frac{7}{4}a_0 > r_1{}^3$.

By Equation (4.11), it shows

$$\left(\frac{7}{4}a_0\right)^{\frac{1}{3}} < \left(\frac{3}{4a_4}\right). \tag{4.12}$$

We imply that $r_1 < \left(\frac{7}{4}a_0\right)^{\frac{1}{3}} < \left(\frac{3}{4a_4}\right)$ if and only if $f\left(\left(\frac{7}{4}a_0\right)^{\frac{1}{3}}\right) > 0$ that is $108 > 2401a_0 a_4^3$.

We summarize the results in Lemmas 3 and 4 as follows.

Lemma 3. If $\frac{27}{1024} > \frac{zBDe_p e_w e_x}{gk^4 qL}$, then r_2 is not a local maximum point.

Lemma 4. If $\frac{27}{2401} > \frac{zBDe_p e_w e_x}{gk^4 qL}$, then r_1 is the global maximum point.

V. Criteria to derive a positive optimal profit

In this Section, we will obtain a formulated solution for maximum profit and show the condition for the maximum value is positive

When P has critical solutions ($27 \ge 256 a_0 a_4^3$), at the critical point, we have

$$\frac{e_x r}{4g} = e_w hz, \tag{5.1}$$

$$k - e_p f - \frac{e_x r}{4g} - e_w hz = e_p f, \tag{5.2}$$

$$k - \frac{e_x r}{2g} = 2e_p f, \tag{5.3}$$

and Equation (3.10), then the profit function is

$$P = \frac{qTLW}{8e_p}\left(\frac{e_x r}{g} - k\right)\left(\frac{e_x r}{g} - 2k\right). \tag{5.4}$$

We recall our findings that

$$\frac{3}{4a_4} = \frac{3gk}{2e_x}, \tag{5.5}$$

such that we imply that

$$\frac{e_x}{g} r_1 < \frac{3k}{2} < \frac{e_x}{g} r_2. \tag{5.6}$$

From Equation (3.10), we know that

$$\frac{-r^3}{2k}\left(\frac{e_x}{g} r - 2k\right) = \frac{32 g^2 zBDe_p e_w}{kqLe_x^2}, \tag{5.7}$$

then we derive that

$$\frac{e_x}{g} r_1 - 2k < 0, \tag{5.8}$$

and

$$\frac{e_x}{g} r_2 - 2k < 0. \tag{5.9}$$

We will find the condition for $\frac{e_x}{g} r_1 - k < 0$. We know that $\frac{e_x}{g} r_1 - k < 0$ if and only if $f\left(\frac{gk}{e_x}\right) > 0$.

We compute that

$$f\left(\frac{gk}{e_x}\right) = \frac{g^3 k^3}{2e_x^3} - \frac{32 g^2 zBDe_p e_w}{kqLe_x^2},$$

$$= \frac{1}{16 a_4^3} - a_0, \tag{5.10}$$

so we derive the following Lemma 5.

Lemma 5. The condition to guarantee the maximum value of *P* being positive is $1 > 16 a_0 a_4^3$.

VI. Our new approximated route spacing

In this Section, we will construct an improved formulated approximation for route spacing.

Chang and Schonfeld [7] used $a_0^{1/3}$ as the formulated approximation for r_1, whereas we will locate a better formulated approximated solution, say $a_0^{1/3} + \varepsilon$, for r_1. Thus, we would have

$$f\left(a_0^{1/3} + \varepsilon\right) = -a_0^{4/3} a_4 + \left(3a_0^{2/3} - 4a_0 a_4\right)\varepsilon,$$
$$+\left(3a_0^{1/3} - 6a_0^{2/3} a_4\right)\varepsilon^2 + \left(1 - 4a_0^{1/3} a_4\right)\varepsilon^3 - a_4 \varepsilon^4. \tag{6.1}$$

In other words, when we compute $f\left(a_0^{1/3} + \varepsilon\right) \approx 0$ by neglecting the ε^2 and the higher terms, we can obtain that

$$\varepsilon = \frac{a_0^{\frac{2}{3}} a_4}{3 - 4a_0^{\frac{1}{3}} a_4}, \tag{6.2}$$

By the way in which ε was defined, we recall that if $f\left(\frac{3}{4a_4}\right) \ge 0$ then

$$27 \ge 256 a_0 a_4^3. \tag{6.3}$$

Based on Equation (6.3), we know that

$$3 - 4a_0^{\frac{1}{3}} a_4 > 0, \tag{6.4}$$

and

$$\varepsilon > 0. \tag{6.5}$$

VII. Numerical comparison of R_1 and R_2

We consider the same numerical example as Chang and Schonfeld [7]. They had the following parameters: $b = 2$, $d = 0.25$, $g = 2.5$, $q = 67.8$, $y = 2$, $z = 0.5$, $B = 32.5$, $J = 4$, $L = 3$, $V = 15$, $W = 2$, $e_p = 0.07$, $e_v = 0.35$, $e_w = 0.7$, $e_x = 0.7$, $D = \frac{2L}{V} + \frac{2J}{yV} + \frac{W}{bV}$, $M = \frac{L}{2V} + \frac{J}{yV} + \frac{W}{2bV}$, and $k = 1 - \frac{de_x}{4g} - Me_v$.

Since $a_0 a_4^3 = 0.005144$, $27/256 = 0.105$ and $108/2401 = 0.045$, we obtain that Equation (3.10) has

Table 1. Comparison among solutions.

	exact solution	$r_1 \approx a_0^{1/3}$	$r_1 \approx a_0^{1/3} + \varepsilon$	Chang and Schonfeld [7]
route spacing	$r = 1.174$	$r = 1.096$	$r = 1.178$	$r = 1.17$
headway	$h = 0.235$	$h = 0.219$	$h = 0.236$	$h = 0.234$
fare	$f = 5.178$	$f = 5.255$	$f = 5.173$	$f = 4.98$
total profit	$P = 5903.295$	$P = 5880.124$	$P = 5903.219$	$P = 5034$

two solutions. From the help of a computer capability, we know that $r_1 = 1.174$, and $r_2 = 6.318$. Since the restriction in Lemma 4 is satisfied, we claim that the derived r_1 is the global maximum point.

Moreover, we compute the profit and list them in the Table 1. For easy comparison with the results of Chang and Schonfeld [7], we quote their results from Page 474, Table 3, column 4, then list them in Table 1.

It is shown that our formulated approximation,

$$r_1 \approx a_0^{1/3} + \varepsilon , \tag{7.1}$$

is comparatively accurate and can represent the exact maximum solution.

Based on Table 1, our formulated approximated route spacing provides a very accurate estimation for the exact route spacing.

VIII. A Related Inventory model

In this section, we study inventory systems with a temporary purchasing discount. We recall the inventory model proposed by Aucamp and Kuzdrall [8] with the following objective function,

$$Z = S + C_D Q + B\exp\left(\frac{-r_0(Q_0+Q)}{D}\right)$$

$$+\frac{Cr_1}{r_0}\left(Q_0 + Q - \frac{D}{r_0}\left(1-\exp\left(\frac{-r_0(Q_0+Q)}{D}\right)\right)\right), \tag{8.1}$$

where

$$B = \frac{(S + CQ_E + A)}{H}, \tag{8.2}$$

with

$$A = C\frac{r_1}{r_0}\left(Q_E - \frac{D}{r_0}H\right), \tag{8.3}$$

and

$$H = 1 - \exp(-r_0 Q_E / D), \tag{8.4}$$

are three abbreviations to simplify the expressions in Equation (8.1).

Aucamp and Kuzdrall [8] derived that the solution for $\frac{dZ}{dQ} = 0$ to obtain that

$$Q^* = \frac{D}{r_0}\ln\frac{r_1CD + (G/H)}{r_1CD + r_0C_DD} - Q_0 . \tag{8.5}$$

with

$$G = r_0^2(S + CQ_E + A), \tag{8.6}$$

to simplify the expressions in Equation (8.5).

Aucamp and Kuzdrall [8] mentioned that it must check that whether $Q^* > 0$ or not. Moreover, $Z(Q^*)$ needs to compare with $Z(0) - S$ to decide whether to accept the special reduction or not. They only used a numerical example to check those considerations. They did not consider the convexity property from the second derivative. We will provide more detailed explanation.

IX. Our Improvements

We rewrite the objective function as

$$Z = S + C_D\left(\frac{D}{r_0}x - Q_0\right)$$

$$+\frac{CDr_1}{r_0^2}\left(e^{-x} - 1 + x\right) + Be^{-x}, \tag{9.1}$$

where x is a new variable with

$$x = \frac{r_0}{D}(Q_0 + Q) \tag{9.2}$$

under the restriction

$$x \geq \frac{r_0 Q_0}{D}. \tag{9.3}$$

We know that

$$\frac{dZ}{dx} = \frac{C_D D}{r_0} - \frac{Cr_1D}{r_0^2}\left(e^{-x} - 1\right) - Be^{-x}, \tag{9.4}$$

and

$$\frac{d^2Z}{dx^2} = \left(\frac{Cr_1D}{r_0^2} + B\right)e^{-x} > 0 . \tag{9.5}$$

The solution for $\frac{dZ}{dx} = 0$ is denoted by $x^{\#}$, where

$$x^{\#} = \ln\left(\frac{Br_0^2 + CDr_1}{C_DDr_0 + CDr_1}\right). \tag{9.6}$$

Hence, we prove the main contribution of our study that the result, $x^{\#} = \ln\left(\frac{Br_0^2 + CDr_1}{C_DDr_0 + CDr_1}\right)$ of Equation (9.6) is the best solution without considering the restriction of Equation (9.3).

If $x^{\#} > \frac{r_0 Q_0}{D}$, then $x^{\#}$ is the minimum solution under the restriction of Equation (9.3). If $x^{\#} \le \frac{r_0 Q_0}{D}$, form the convexity property in Equation (9.5), then $\frac{r_0 Q_0}{D}$ is the minimum solution.

X. Direction for Future Research

The direction for the future research is twofold. First, we will find the condition to show that $x^{\#} > \frac{r_0 Q_0}{D}$ is satisfied. Second, we will find the criterion to insure that
$Z(x=0) - S > Z(x^{\#})$.

On the other hand, we had published several papers with respect to pattern recognition to find some interesting problems that deserve to be investigated further: (i) How to apply our discrete results to continuous setting? (ii) Testing my theoretical derivations to an application situation. For the past many years, we had published several articles related to bus service model with a rectangular service area. There are some interesting problems that deserve further investigation: (a) How to avoid using the route width as a continuous variable that will result in unreasonable partition for the area width? (b) If we adopt the partition number of the area width as a discrete variable, then how can we find the optimal solution? (c) Comparison between continuous results with discrete results to check whether or not previous theoretical findings based on continuous variable acceptable?

Based on our another research project with similarity measures, we had published two papers to point out that this research direction is a hot spot for practitioners. We found a related field to partition a network into communities. Zhang et al. [9] aroused our attention. We may apply our previous results to show that within a community, the similarity measure value is greater than that of outside the community. We find several unsolved problems: (1) How to find clicks for a network? (2) Based on clicks, how to generate a group? (3) How to compute modularity values? (4) How to convert a graph data of adjacency matrix to an intuitionistic fuzzy set environment?

There are several recently published papers that are important to shed the light for the hot spot in the further study such that we list them in the following. Tang et al. [10] study a supermarket during the Chinese new year period by customer analysis. Yang et al. [11] developed a new information system according to reciprocal accumulation generation operation and vector continued fractions. Assis and Coelho [12] studied a distant learning and teaching project by temperature control as an education implement. According to machine learning procedure, Zhang et al. [13] developed for super-resolution image for morphological sparse areas. Wan et al. [14] found the optimal solution for a retailer warehouse by allocation arrangement. Tobar et al. [15] studied segmentation problem with label enhancement and base representation. Adhitya et al. [16] considered loads and concrete structures under earthquake. Zhu et al. [17] examined the optimal solution for train schedule with carbon emission consideration. With the breaking wave effect, Unyapoti and Pochai [18] constructed a binary arrangement of a wave crest model and a shoreline evolution model. Purwani et al. [19] applied the Newton-Raphson algorithm with Aitken extrapolation method to approximate stock volatility. Alomari and Massoun [20] used the Caputo fractional derivative to locate a numerical solution. Mane and Lodhi [21] considered singularly perturbed equations with numerical solution by cubic approach. Based on above discussion, we provide several possible directions for researchers for future studies.

XI. A Related Problem of Car Sensor

We study a related problem in the paper of Hua et al. [22] to examine their arrangement of locations for car sensor under a network consideration.

It is supposed that there are D_1, $D_2, \ldots, D_t$ column vectors, where the weight of D_j is denoted as w_j for $j = 1, \ldots, t$. We predict that the assumption should be added as non-zero column vectors. In Hua et al. [22], they assumed that let Ψ be the collection of all independent subsets of $\{D_j : j = 1, 2, \ldots, t\}$, and then their goal is to find

$$\max\left\{\sum_{w_i \in G} w_i : G \subset \Psi\right\}. \tag{11.1}$$

The proposed method in Hua et al. [22] is to rearrange the order of $\{D_j : j = 1, 2, \ldots, t\}$ depending on their weights such that

$$w_{g(1)} \ge w_{g(2)} \ge \cdots \ge w_{g(n)}. \tag{11.2}$$

Hua et al. [22] considered that (i) $\{D_{g(j)} : j = 1\}$, (ii) $\{D_{g(j)} : j = 1, 2\}$, ..., (t) $\{D_{g(j)} : j = 1, 2, \ldots t\}$ in this order to select independent subset to derive a subset as $\{D_{p(k)} : k = 1, \ldots, m\}$ then the total weight for this independent sub-family is derived as

$$\sum_{k=1}^{m} w_{p(k)}. \tag{11.3}$$

The goal in Hua et al. [22] is to prove that

$$\sum_{k=1}^{m} w_{p(k)} = \max\left\{\sum_{w_i \in G} w_i : G \subset \Psi\right\}. \tag{11.4}$$

To save the precious space of this journal, we will not cite the detailed proof of Hua et al. [22] in our paper. Those interested readers please directly refer to Hua et al. [22] for their solution procedure.

The inherent problem of Equation (11.2) proposed by Hua et al. [22] is the expression of Equation (11.2) is not unique.

Consequently, the proof in Hua et al. [22] is questionable, because they did not prove that for two different expressions of Equation (11.2), then they will derive the same maximum value.

XII. Our Solution Procedure

We can rearrange $\{w_j : j = 1,\ldots,t\}$ in a finite communities such that in each community, w_j has the same value that is, if there are s communities to partition $\{w_j : j = 1,\ldots,t\}$ as

$$\{w_j : j = 1,\ldots,t\} = \bigcup_{k=1}^{s} \Theta_k \,, \tag{12.1}$$

such that if w_a and w_b in the same community, then $w_a = w_b$, if $w_c \in \Theta_i$ and $w_d \in \Theta_j$ then $w_c > w_d$ if and only if $i < j$ such that we express $\{w_j : j = 1,\ldots,t\}$ as

$$\begin{aligned} w_\alpha &= w_\beta = \cdots (\in \Theta_1) > \\ w_\chi &= w_\delta = \cdots (\in \Theta_2) > \cdots \\ > w_\psi &= w_\zeta = \cdots (\in \Theta_s). \end{aligned} \tag{12.2}$$

Hence, we define the corresponding D_j as follows,

$$\{D_j : j = 1,2,\ldots,t\} = \bigcup_{k=1}^{s} \mathrm{M}_k \,, \tag{12.3}$$

where for $k = 1,2,\ldots,s$,

$$\mathrm{M_k} = \{\mathrm{D_i} : \mathrm{w_i} \in \Theta_\mathrm{k}\}. \tag{12.4}$$

We will prove by the Principle of Finite Induction on the rank of $\{D_j : j = 1,2,\ldots,t\}$.

We assume the rank of $\{D_j : j = 1,2,\ldots,t\}$ is z .

We assume that $z = 1$ then any D_j is a base for $\{D_j : j = 1,2,\ldots,t\}$. By the approach of Hua et al. [22], $D_{g(1)} \in M_1$ and then

$$w_{g(1)} = \max \{w_k : k = 1,\ldots,t\} \geq w_j \,, \tag{12.5}$$

for any D_j being a base for $\{D_j : j = 1,2,\ldots,t\}$ to prove that the approach proposed by Hua et al. [22] can attain the maximum.

However, Hua et al. [22] did not consider that Equation (11.2) can have different ordering such that their solution procedure is incomplete.

We assume that our approach is valid for $z = 1,2,\ldots,u$ and then we assume that the rank of $\{D_j : j = 1,2,\ldots,t\}$ is $u+1$.

For the subspace generated by $\{D_\alpha, D_\beta, \cdots\} = M_1$, we assume that the rank of M_1 is z_1 with $z_1 \leq u+1$.

By our approach, there is a subset consisting of z_1 independent vector that is a base for the subspace generated by M_1. This base will be denoted as M_2.

For any other selection of base for $\{D_j : j = 1,2,\ldots,t\}$, we denote this base as $\mathrm{M_3}$.

We define M_2 as follows,

$$\mathrm{M_2} = \{\mathrm{D_j} : \mathrm{D_j} \in \mathrm{M_3}, \mathrm{w_j} \in \Theta_1\}. \tag{12.6}$$

We denote the rank of $\mathrm{M_3}$ as $\#(M_3)$. Owing to $\#(M_3)$ must less than or equal to the rank of M_1 to imply the $\#(M_3) \leq z_1$. We will divide into two cases: Case (a) $\#(M_3) < z_1$, and Case (b) $\#(M_3) = z_1$.

To simplify the expression, we will define that

$$\Theta = \{w_i : i = 1,\ldots,t\}. \tag{12.7}$$

For Case (a) $\#(M_3) < z_1$, the subspace generates by $\overline{\Phi}_2$ cannot contain all elements in $\overline{\Phi}_1$, because the dimension relation. Hence, we can select an element, denoted as C_α where C_α is in the subspace generates by $\overline{\Phi}_1$, we denoted it as

$$\mathrm{C}_\alpha \in \langle \overline{\Phi_1} \rangle. \tag{12.8}$$

On the other hand, C_α is not in the subspace generates by M_2, and then we can apply the replacement theorem of bases in linear algebra to replace one element in the base $\mathrm{M_3}$, we denote it as D_β.

Owing to the independent relation, we know that D_β is not in M_2. Hence, we imply that w_β is not in Θ_1, and then

$$\mathrm{w}_\beta < \mathrm{w}_\alpha. \tag{12.9}$$

Based on Equation (12.9), we obtain that

$$\textstyle\sum_{\mathrm{D_i} \in \mathrm{M_3}} \mathrm{w_i} < \sum_{\mathrm{D_j} \in \mathrm{M_4}} \mathrm{w_j}, \tag{12.10}$$

where

$$\mathrm{M_4} = \{\mathrm{D}_\alpha\} \cup \mathrm{M_3} - \{\mathrm{D}_\beta\}, \tag{12.11}$$

is a new base that add D_α and delete D from $\mathrm{M_3}$.

Base on our above discussion, we know that the base $\mathrm{M_3}$ will not attain the maximum value. Consequently, we know that Case (b) is valid. We conclude our derivations in the following theorem.

Theorem 1. We prove that for any base which attains the maximum value of the optimal problem, then all of them have the cardinal number of selected sub-base form $\{\mathrm{D_i} : \mathrm{w_i} \in \Theta_1\}$.

After we verify our theorem 1, we consider the remaining subspace $\{\mathrm{D_j} : \mathrm{w_j} \in \Theta_\mathrm{k},\ \mathrm{k} = 2,3,\ldots,\mathrm{s}\}$. Owing to the rank of $\{\mathrm{D_j} : \mathrm{w_j} \in \Theta_\mathrm{k},\ \mathrm{k} = 2,3,\ldots,\mathrm{s}\}$ is small or equal to u. We apply the Principle of Finite Induction to show that our approach can derive a base that attain the maximum value problem.

Based on our above discussion, we present a simplified version to locate a base to attain the maximum value of car sensor allocation problem in a traffic design network.

XIII. Conclusion

This study attempts to provide a better solving procedure for route spacing and profit function from the point of view of Chang and Schonfeld [7]. The above results show the condition for existence of route spacing and demonstrate the uniqueness maximum point of profit function. Moreover, results obtained show that the formulated approximation $r_1 \approx a_0^{1/3}$ of Chang and Schonfeld [7] is not good enough in their own paper because of their lack of consideration for this approximation. On the other hand, our formulated approximation $r_1 \approx a_0^{1/3} + \varepsilon$ is very accurate and can represent the exact maximum solution. Moreover, we discuss Aucamp and Kuzdrall [8] to point out their questionable findings and then provide our improvements. We studied a car sensor problem proposed by Hus et al. [22] to present a proof to show that their algorithm can derive the optimal solution. We also provide several possible directions for practitioners to help them to locate future research trend.

REFERENCES

[1] V. Vuchic, G. Newell, "Rapid transit interstation spacing for minimum travel time," *Transportation Science*, vol. 2, 1968, pp. 359-374.

[2] S.M. Tsao, P.M. Schonfeld, "Branched transit services: an analysis," *Journal of Transportation Engineering*, vol. 110, 1984, pp. 112-128.

[3] G.K. Kuah, J. Perl, "Optimization of feeder bus routes and bus-stop spacing," *Journal of Transportation Engineering*, vol. 114, 1988, pp. 341-354.

[4] R.H. Oldfield, P.H. Bly, "An analytic investigation of optimal bus size," *Transportation Research Part B-Methodological*, vol. 22B, 1988, pp. 319-337.

[5] G.F. Newell, "Dispatching policies for a transportation routes," *Transportation Science*, vol. 5, 1971, pp. 91-105.

[6] C.C. Clarens, V.F. Hurdle, "An operating strategy for a commuter bus system," *Transportation Science*, vol. 9, 1975, pp. 1-20.

[7] S.K. Chang, P.M. Schonfeld, "Multiple period optimization of bus transit systems," *Transportation Research Part B-Methodological*, vol. 25, no. 6, 1991, pp. 453-478.

[8] D.C. Aucamp, P.J. Kuzdrall, “Order Quantities With Temporary Price Reductions,” *The Journal of Operational Research Society*, vol. 40, no. 10, 1989, pp. 937-940.

[9] Z. Zhang, X. Jiang, L. Ma, S. Tang, Z. Zheng, "Detecting communities in clustered networks based on group action on set," *Physica A*, vol. 390, 2011, pp. 1171–1181.

[10] Y. T. Tang, H. H. Wu, H. W. Yang, M. M. Lo, "Customers' Behavior Analysis Before and During the Chinese New Year: An Empirical Study from A Supermarket in Taiwan," *IAENG International Journal of Applied Mathematics*, vol. 53, no.2, 2023, pp. 684-689.

[11] X. Yang, D. Ding, X. Zhang, "New Information Model MGRM (1, n) Based on Vector Continued Fractions and Reciprocal Accumulation Generation Operation," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, 2023, pp. 678-683.

[12] W. O. Assis, A. D. Coelho, "Use of Educational Tool for Modeling Temperature Control Systems as a Remote Teaching and Learning Strategy in Engineering," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, 2023, pp. 630-640.

[13] J. Zhang, J. Tang, X. Feng, "Multi Morphological Sparse Regularized Image Super-Resolution Reconstruction Based on Machine Learning Algorithm," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, 2023, pp. 622-629.

[14] Y. Wan, S. Wang, Y. Hu, Y. Xie, "Multiobjective Optimization of the Storage Location Allocation of a Retail E-commerce Picking Zone in a Picker-to-parts Warehouse," *Engineering Letters*, vol. 31, no. 2, 2023, pp. 481-493.

[15] A. D. Tobar, J. C. Aguirre, D. A. Cardenas-Pena, A. M. Alvarez-Meza, C.G. Castellanos-Dominguez, "Hippocampus Segmentation using Patch-based Representation and ROC Label Enhancement," *Engineering Letters*, vol. 31, no. 2, 2023, pp. 504-510.

[16] B. B. Adhitya, A. Costa, K. Verawati, Wadirin, "Study on the Failure Performance of Reinforced Concrete and Composite Concrete Structures due to Non-Linear Time History Earthquake Loads," *Engineering Letters*, vol. 31, no. 2, 2023, pp. 544-553.

[17] C. Zhu, X. Yang, Z. Wang, J. Fang, J. Wang, L. Cheng, "Optimization for the Train Plan with Flexible Train Composition Considering Carbon Emission," *Engineering Letters*, vol. 31, no. 2, 2023, pp. 562-573.

[18] P. Unyapoti, N. Pochai, "A Combination of A Shoreline Evolution Model and A Wave Crest Model on T-Head Groin Structures With the Breaking Wave Effect," *IAENG International Journal of Computer Science*, vol. 50, no. 2, 2023, pp. 347-358.

[19] S. Purwani, A. F. Ridwan, R. A. Hidayana, S. Sukono, "Secant Method with Aitken Extrapolation Outperform Newton-Raphson Method in Estimating Stock Implied Volatility," *IAENG International Journal of Computer Science*, vol. 50, no. 2, 2023, pp. 368-374.

[20] A. K. Alomari, Y. Massoun, "Numerical Solution of Time Fractional Coupled Korteweg-de Vries Equation with a Caputo Fractional Derivative in Two Parameters," *IAENG International Journal of Computer Science*, vol. 50, no. 2, 2023, pp. 388-393.

[21] S. Mane, R. K. Lodhi, "Cubic B-spline Technique for Numerical Solution of Singularly Perturbed Convection-Diffusion Equations with Discontinuous Source Term," *IAENG International Journal of Computer Science*, vol. 50, no. 2, 2023, pp. 402-407.

[22] S. R. Hua, S. Peeta, C. H. Chu, "Identification of vehicle sensor locations for link-based network traffic applications," *Transportation Research Part B*, vol. 43, 2009, pp. 873–894.

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