Stability of Generalized Radical Functional Equation on Non-Archimedean Normed Space

Koushika Dhevi Sankar and Sangeetha Sampath

***Abstract*—In this paper, we discuss the generalized Hyers-Ulam-Rassias stability of the radical functional equation**

$$g(\sqrt[n]{ax^n+by^n})+g(\sqrt[n]{ax^n-by^n})=2ag(x)$$

in the non-Archimedean normed space. Also we proved some results for the same.

***Index Terms*—Hyers-Ulam-Rassias stability, radical functional equation, non-Archimedean normed space.**

I. Introduction

In 1940, S.M. Ulam raised the problem on functional equation."Let (G_1, *) be a group and let (G_2,$\diamond$,d) be a metric group with the metric d(.,.). Given $\epsilon > 0$ does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfy the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all x, y $\in G_1$ then there is a homomorphism H:$G_1 \to G_2$ with

$$d(h(x),H(x))<\epsilon$$

for all $x \in G_1$?" [16]. In 1941, Hyers provided responses using Banach spaces instead of group homomorphism [12].

The stability theory of functional equation arises when we substitute the functional equation with an inequality that pertubates to the equation. Thus, the stability concern for a functional equation is how the solution of the relevant inequality differs from the solution of the provided functional equation.[18], [15]

In 2012, Khodaei et al. discussed the approximation of radical functional equations related to quadratic and quartic mappings [13]. In 2016, Ghazanfari and Alizadehz addressed the stability of radical cubic functional equation in quasi β-Banach spaces [2].

In 2017, Sintunavarat and Aiemsomboon gave a new type of stability of a radical quadratic functional equation using Brzdek's fixed point theorem[1]. Further, Iz-iddine EL-Fassi studied a new kind of hyperstability for radical cubic functional equation in non-Archimedean metric spaces [6]. In 2018, Iz-iddine EL-Fassi discussed new stability results for the radical sextic functional equation related to quadratic mappings in (2, β) Banach spaces [7].

Youssef Aribou and Samir Kabbaj studied a new functional inequality in non-Archimedean Banach spaces related to radical cubic functional equation [3]. In 2019, Iz-iddine EL-Fassi studied solution and approximation of radical quintic mapping in quasi-β Banach spaces [9].

In 2016, Iqbal M. Batina et al. discussed the common fixed point theorem in Non-Archimedean Menger PM-spaces using CLR property with applications to functional equations [5]. In 2018, Iz-iddine EL-Fassi studied a new type of approximation for the radical quintic functional equation in non-Archimedean (2,β) Banach spaces [8]. In 2020, Kandhasamy and Emanuel studied the stability of radical septic functional equation [11]. In 2021, Iz-iddine EL-Fassi et al. gave the fixed point approach to stability of kth radical functional equation in non-Archimedean (n, β) Banach spaces [10] .

In our study, we discuss the generalized Hyers-Ulam-Rassias stability of the generalize radical functional equation

$$g(\sqrt[n]{ax^n+by^n})+g(\sqrt[n]{ax^n-by^n})=2ag(x) \qquad (1)$$

where $n, a, b \in \mathbb{Z}^+$ and $n > 1$ in the non-Archimedean normed space.

Let us define the following notation,

$$G(x,y)=g(\sqrt[n]{ax^n+by^n})+g(\sqrt[n]{ax^n-by^n})-2ag(x). \quad (2)$$

Overall our consideration, X be an additive group and Y be a complete non-Archimedean normed space.

II. Preliminaries

Definition 2.1. [14] A functional equation is an equation in which both sides contain a finite number of functions, some are known and some are unknown.

Example 2.1. f(x+y)=f(x)+f(y) is the Cauchy Additive Functional Equation

Definition 2.2. [14] A solution of a functional equation is a function which satisfies the equation.

Example 2.2. (i) f(x)=kx is a solution of the Cauchy functional equation f(x+y)=f(x)+f(y)
(ii) f(x)=cx +a is the solution of the Jensen functional equation f($\frac{x+y}{2}$)=$\frac{f(x)+f(y)}{2}$

Definition 2.3. [14] A functional equation F is stable if any

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function f satisfying the equation F approximately is near to exact solution of F.

Definition 2.4. [4], [17]. If $\mathbb{F}$ is any field then a valuation (of rank 1) is a map $|.| : \mathbb{F} \to \mathbb{R}$, satisfying the following axioms:

$$(i)|x| \geq 0$$
$$(ii)|x| = 0, \quad when \quad x = 0$$
$$(iii)|xy| = |x||y|$$
$$(iv)|x + y| \leq |x| + |y|$$

for all x,y $\in \mathbb{F}$.
The valuation is said to be non-Archimedean, if the following stronger form of inequality (iv) holds, namely

$$|x + y| \leq max\{|x|, |y|\}.$$

Definition 2.5. [4] Let p be a positive prime number. For every non-zero rational number x there exists a unique integer α such that

$$x = p^{\alpha}.\frac{a}{b}$$

with some integer a and b not divisible by p. we define

$|x|_p$=$\frac{1}{p^{\alpha}}$ when x $\neq 0$, $|0|_p = 0$ when $x = 0$.

So called p-adic valuation.

Example 2.3. Take $x = \frac{162}{13}$. Suppose we want to find its 3-adic absolute value (hence p=3). Expressed in the p-adic form, we obtain

$$x = 81.\frac{2}{13} = 3^4.\frac{2}{13}$$

which mean $|x|_3 = \frac{1}{3^4}$.
13-adic absolute value for x. It will simply be $|x|_{13} = 13$ because

$$x = 13^{-1}.162$$
$$|x|_{13} = \frac{1}{13^{-1}} = 13$$

Definition 2.6. [17] A sequence $\{x_n\}$ in $\mathbb{K}$ is called a Cauchy sequence with respect to a non-Archimedean valuation $|.|$, if and only if

$$|x_{n+1} - x_n| \to 0, \quad as \quad n \to \infty.$$

Definition 2.7 [4] If every Cauchy sequence of $\mathbb{K}$ has a limit in $\mathbb{K}$, then $\mathbb{K}$ is said to be Complete.

Example 2.4 The field $\mathbb{Q}_p$ of p-adic number is the completion of $\mathbb{Q}$ with respect to $|.|_p$[17]

Definition 2.8 [17] A complete normed linear space is called a Banach Space.

Definition 2.9 [4], [17] Let X be a vector space over a field $\mathbb{K}$ with a non-trivial non-Archimedean valuation $|.|$. Then, a function $\|.\| : X \to \mathbb{R}$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff x=0 for all x$\in$X
(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all x $\in$ X and $\alpha \in \mathbb{K}$
(iii) $\|x + y\| \leq max\{\|x\|, \|y\|\}$ for all x,y $\in$ X

and the space (X,$\|.\|$) is called a non-Archimedean normed space.

III. Main Results

Theorem 3.1. Let $\beta : X \times X \to [0, \infty)$ be a function so that

$$\lim_{t\to\infty} \frac{1}{|a|^t} \beta(\sqrt[n]{a^t}x, 0) = 0 \tag{3}$$

$$\lim_{t\to\infty} \frac{1}{|a|^t} \beta(\sqrt[n]{a^t}x, \sqrt[n]{a^t}y) = 0 \tag{4}$$

and let for each x$\in$ X then the limit

$$max\{\frac{1}{|a|^j} \beta(\sqrt[n]{a^j}x, 0) : 0 \leq j < t\} \tag{5}$$

denoted by $\tilde{\beta}(x)$ exist. Suppose $g : X \to Y$ is a mapping satisfies

$$\|G(x, y)\| \leq \beta(x, y) \tag{6}$$

then there is a map $K : X \to Y$ so that

$$\|g(x) - K(x)\| \leq \frac{1}{|2a|} \tilde{\beta}(x) \tag{7}$$

Moreover if,

$$\lim_{m\to\infty} \lim_{t\to\infty} max\{\frac{1}{|a|^j} \beta(\sqrt[n]{a^j}x, 0) : m \leq j < t + m\} = 0 \tag{8}$$

then K is unique.

Proof: Put (x, y) as $(x, 0)$ in (6)

$$\|g(\sqrt[n]{ax^n}) - ag(x)\| \leq \frac{1}{|2|} \beta(x, 0) \tag{9}$$

Giving x by $\sqrt[n]{a^{t-1}x}$,

$$\|\frac{g(\sqrt[n]{a^t}x)}{a^t} - \frac{g(\sqrt[n]{a^{t-1}}x)}{a^{t-1}}\| \leq \frac{1}{|2a^t|} \beta(\sqrt[n]{a^{t-1}}x, 0) \tag{10}$$

Hence $\{\frac{1}{a^t} g(\sqrt[n]{a^t}x)\}$ is Cauchy.
Define the function,

$$K(x) = \lim_{t\to\infty} \frac{1}{a^t} g(\sqrt[n]{a^t}x). \tag{11}$$

By using induction,

$$\|\frac{g(\sqrt[n]{a^t}x)}{a^t} - g(x)\|$$

$$\leq max\{\|\frac{g(\sqrt[n]{a^t}x)}{a^t} - \frac{g(\sqrt[n]{a^{t-1}}x)}{a^{t-1}} + \frac{g(\sqrt[n]{a^{t-1}}x)}{a^{t-1}} - \frac{g(\sqrt[n]{a^{t-2}}x)}{a^{t-2}}, ..., \frac{g(\sqrt[n]{a}x)}{a} - g(x)\|\}$$
$$\leq max\{\|\frac{g(\sqrt[n]{a^t}x)}{a^t} - \frac{g(\sqrt[n]{a^{t-1}}x)}{a^{t-1}}\|, \|\frac{g(\sqrt[n]{a^{t-1}}x)}{a^{t-1}} - \frac{g(\sqrt[n]{a^{t-2}}x)}{a^{t-2}}\|, ..., \|\frac{g(\sqrt[n]{a}x)}{a} - g(x)\|\}$$
$$\leq max\{\frac{1}{|a|^{t-1}} \beta(\sqrt[n]{a}^{t-1}x, 0), \frac{1}{|a|^{t-2}} \beta(\sqrt[n]{a}^{t-2}x, 0), ..., \beta(x, 0)\}$$
$$\leq \frac{1}{|2a|} max\{\frac{1}{|a|^j} \beta(\sqrt[n]{a^j}x, 0) : 0 \leq j < t\} \tag{12}$$

By taking t tends to infinity in equation (12), we get (7).
To show that K is additive

$$\begin{aligned}&\|K(\sqrt[n]{a}x) - aK(x)\| \\ &= |a| \lim_{t\to\infty} \| \frac{1}{|a|^{t+1}} g(\sqrt[n]{a^{t+1}}x) \frac{1}{|a|^t} g(\sqrt[n]{a^t}x)\| \\ &\leq \frac{1}{|2a^t|}\beta(\sqrt[n]{a^t}x, 0)\end{aligned} \tag{13}$$

$$K(\sqrt[n]{a}x) = aK(x) \tag{14}$$

Hence equation (14) implies K is additive.
By using equation (11),

$$\|G_K(x, y)\| \leq \lim_{t\to\infty} \frac{1}{|a|^t}\beta(a^t x, a^t y) = 0 \tag{15}$$

which implies K satisfies $G(x, y)$.
Next we prove uniqueness, let K' be another function satisfying (7)

$$\|K(x) - K'(x)\| = \lim_{m\to\infty} \frac{1}{|a|^m} \|K(\sqrt[n]{a^m}x - K'(\sqrt[n]{a^m}x\|$$

$$\leq \frac{1}{|2a|} \lim_{m\to\infty} \lim_{t\to\infty} max\{\frac{1}{|a|^j}\beta(\sqrt[n]{a^j}x, 0) : m \leq j < t + m\} \tag{16}$$

Therefore, $K = K'$.
Hence the proof completes. ■

Corollary 3.1. Let s, γ are positive real numbers and $s > n$, if a mapping $g : X \to Y$ satisfies

$$\|G(x, y)\| \leq \gamma(\|x\|^s + \|y\|^s) \tag{17}$$

then there is a unique mapping $K : X \to Y$ so that

$$\|g(x) - K(x)\| \leq \frac{\gamma}{|2a|}\|x\|^s. \tag{18}$$

Proof: Consider

$$\|G(x, y)\| \leq \gamma(\|x\|^s + \|y\|^s).$$

Given

$$\beta(x, y) = \gamma(\|x\|^s + \|y\|^s).$$

Substituting (x, y) as $(\sqrt[n]{a^j}x, 0)$ in (17), we have,

$$\begin{aligned}\beta(\sqrt[n]{a^j}x, 0) &= \gamma(\|\sqrt[n]{a^j}x\|^s) \\ &= \gamma|a|^{\frac{js}{n}}\|x\|^s\end{aligned}$$

From Theorem 3.1,

$$\begin{aligned}\|g(x) - K(x)\| &\leq \frac{1}{|2a|}\tilde{\beta}(x) \\ &\leq \frac{1}{|2a|} max\{\frac{a}{|a|^j}\beta(\sqrt[n]{a^j}x, 0) : 0 \leq j < t\} \\ &= \frac{\gamma}{|2a|}\|x\|^s max\{|a|^{j(\frac{s}{n}-1)} : 0 \leq j < t\}\end{aligned}$$

If $s > n$, then we get

$$\|g(x) - K(x)\| \leq \frac{\gamma}{|2a|}\|x\|^s.$$

Hence the proof completes. ■

Example 3.1. Let $p > 2$ be a prime number $g : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g(x) = x^n + 1$ let $|2|_p^t = 1$, $\gamma > 1, a = 2, t \in \mathbb{Z}$, $s > n$ and if

$$\|G(x, y)\| = 1 \leq \gamma(\|x\|^s + \|y\|^s)$$

then

$$\|g(x) - K(x)\| = 1 \leq \frac{\gamma}{|4|}\|x\|^s$$

For the case $s = n$, we have following counterexample,

Example 3.2. Let $p > 2$ be a prime number $g : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g(x) = 4$ let $|2|_p^t = 1$, $\gamma > 0, a = 4, t \in \mathbb{Z}$ we have

$$\|G(x, y)\| = 0 \leq \gamma(\|x\|^s + \|y\|^s)$$

so,

$$\lim_{t\to\infty} \|\frac{1}{a^t}g(\sqrt[n]{a^t}x) - \frac{1}{a^{t-1}}g(\sqrt[n]{a^{t-1}})\| = |4|_p^{1-t}|3| \neq 0.$$

Hence $\{\frac{1}{a^t}g(\sqrt[n]{a^t}x)\}$ is not Cauchy.

Corollary 3.2. Let r, s, γ are positive real numbers and $r + s > n$, if a mapping $g : X \to Y$ satisfies

$$\|G(x, y)\| \leq \gamma(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s) \tag{19}$$

then there is a unique mapping $K : X \to Y$ so that

$$\|g(x) - K(x)\| \leq \frac{\gamma}{|2a|}\|x\|^{r+s}. \tag{20}$$

Proof: Consider

$$\|G(x, y)\| \leq \gamma(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s).$$

Given

$$\beta(x, y) = \gamma(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s).$$

Substituting (x, y) as $(\sqrt[n]{a^j}x, 0)$ in (19), we have,

$$\begin{aligned}\beta(\sqrt[n]{a^j}x, 0) &= \gamma(\|\sqrt[n]{a^j}x\|^{r+s}) \\ &= \gamma|a|^{\frac{j(r+s)}{n}}\|x\|^{r+s}\end{aligned}$$

From Theorem 3.1,

$$\begin{aligned}\|g(x) - K(x)\| &\leq \frac{1}{|2a|}\tilde{\beta}(x) \\ &\leq \frac{1}{|2a|} max\{\frac{a}{|a|^j}\beta(\sqrt[n]{a^j}x, 0) : 0 \leq j < t\} \\ &= \frac{\gamma}{|2a|}\|x\|^{r+s} max\{|a|^{j(\frac{r+s}{n}-1)} : 0 \leq j < t\}\end{aligned}$$

If $s > n$, then we get

$$\|g(x) - K(x)\| \leq \frac{\gamma}{|2a|}\|x\|^{r+s}.$$

Hence the proof completes. ■

Theorem 3.2. Let $\beta : X \times X \to [0, \infty)$ be a function so that

$$\lim_{t\to\infty} |a|^t\beta(\frac{x}{\sqrt[n]{a^t}}, 0) = 0 \tag{21}$$

$$\lim_{t\to\infty} |a|^t\beta(\frac{x}{\sqrt[n]{a^t}}, \frac{x}{\sqrt[n]{a^t}}) = 0 \tag{22}$$

and let for each x∈ X then the limit

$$max\{|a|^j\beta(\frac{x}{\sqrt[n]{a^{j+1}}}, 0) : 0 \leq j < t\} \tag{23}$$

denoted by $\tilde{\beta}(x)$ exist. Suppose a mapping $g : X \to Y$ satisfies g(0)=0 and

$$\|G(x, y)\| \leq \beta(x, y) \tag{24}$$

then there is a map $K : X \to Y$ so that

$$\|g(x) - K(x)\| \leq \frac{1}{|2a|}\tilde{\beta}(x) \tag{25}$$

Moreover if,

$$\lim_{m\to\infty}\lim_{t\to\infty} max\{|a|^j\beta(\frac{x}{\sqrt[n]{a^{j+1}}},0) : m \le j < t+m\} = 0 \tag{26}$$

then K is unique.

Proof: Put (x, y) as $(x, 0)$ in (24)

$$\|g(\sqrt[n]{ax^n}) - ag(x)\| \le \frac{1}{|2|}\beta(x,0) \tag{27}$$

Giving x by $\frac{x}{\sqrt[n]{a^{t+1}}}$,

$$\|a^t g(\frac{x}{\sqrt[n]{a^t}}) - a^{t+1}g(\frac{x}{\sqrt[n]{a^{t+1}}})\| \le \frac{|a|^t}{|2|}\beta(\frac{y}{\sqrt[n]{a^{t+1}}},0) \tag{28}$$

Hence $\{a^t g(\frac{x}{\sqrt[n]{a^t}}\}$ is Cauchy.
Define the function,

$$K(x) = \lim_{t\to\infty} a^t g(\frac{x}{\sqrt[n]{a^t}}). \tag{29}$$

By using induction,

$$\|g(\frac{x}{\sqrt[n]{a^t}}) - g(x)\|$$

$$\le max\{\|a^t g(\frac{x}{\sqrt[n]{a^t}}) - a^{t-1}g(\frac{x}{\sqrt[n]{a^{t-1}}}) + a^{t-1}g(\frac{x}{\sqrt[n]{a^{t-1}}}) -$$
$$a^{t-2}g(\frac{x}{\sqrt[n]{a^{t-2}}}), ..., ag(\frac{x}{\sqrt[n]{a}}) - g(x)\|\}$$
$$\le max\{\|a^t g(\frac{x}{\sqrt[n]{a^t}}) - a^{t-1}g(\frac{x}{\sqrt[n]{a^{t-1}}})\|,$$
$$\|a^{t-1}g(\frac{x}{\sqrt[n]{a^{t-1}}}) - a^{t-2}g(\frac{x}{\sqrt[n]{a^{t-2}}})\|, ..., \|ag(\frac{x}{\sqrt[n]{a}}) - g(x)\|\}$$
$$\le max\{\frac{|a|^{t-1}}{|2|}\beta(\frac{x}{\sqrt[n]{a^t}},0), \frac{|a|^{t-2}}{|2|}\beta(\frac{x}{\sqrt[n]{a^{t-1}}},0),$$
$$..., \frac{1}{|2|}\beta(\frac{x}{\sqrt[n]{a}},0)\}$$
$$\le \frac{1}{|2|}max\{|a|^j\beta(\frac{x}{\sqrt[n]{a^{j+1}}},0) : 0 \le j < t\} \tag{30}$$

By taking t tends to infinity in equation (30), we get (25).
To show that K is additive.

$$\|K(\frac{x}{\sqrt[n]{a}}) - \frac{1}{a}K(x)\|$$
$$= |a|\lim_{t\to\infty}\||a|^{t-1}g(\frac{x}{\sqrt[n]{a^{t-1}}}) - a^t g(\frac{x}{\sqrt[n]{a^t}})\|$$
$$\le \frac{|a|^{t-1}}{|2|}\beta(\frac{x}{\sqrt[n]{a^{t+1}}},0) \tag{31}$$

$$K(\sqrt[n]{a}x) = aK(x). \tag{32}$$

Hence K is additive.
By using equation (29),

$$\|G_K(x,y)\| \le \lim_{t\to\infty}|a|^t\beta(\frac{x}{\sqrt[n]{a^t}},\frac{y}{\sqrt[n]{a^t}}) = 0 \tag{33}$$

which implies K satisfies $G(x, y)$.
Next we prove uniqueness, if K' be another function satisfying

$$\|K(x) - K'(x)\| = \lim_{m\to\infty}|a|^m\|K(\frac{x}{\sqrt[n]{a^m}}) - K'(\frac{x}{\sqrt[n]{a^m}})\|$$
$$\le \frac{1}{|2|}\lim_{m\to\infty}\lim_{t\to\infty} max\{|a|^j\beta(\frac{x}{\sqrt[n]{a^{j+1}}},0) : m \le j < t+m\}. \tag{34}$$

Therefore, $K = K'$.
Hence the proof completes. ■

Corollary 3.3. Let s, γ are positive real numbers and $s < n$, if a mapping $g : X \to Y$ satisfies

$$\|G(x,y)\| \le \gamma(\|x\|^s + \|y\|^s) \tag{35}$$

then there is a unique mapping $K : X \to Y$ so that

$$\|g(x) - K(x)\| \le \frac{\gamma}{|2a^{s/n}|}\|x\|^s. \tag{36}$$

Proof: Consider

$$\|G(x,y)\| \le \gamma(\|x\|^s + \|y\|^s).$$

Given

$$\beta(x,y) = \gamma(\|x\|^s + \|y\|^s).$$

Substituting (x, y) as $(\frac{x}{\sqrt[n]{a^{j+1}}}, 0)$ in (35), we have,

$$\beta(\frac{x}{\sqrt[n]{a^{j+1}}},0) = \gamma(\|\frac{x}{\sqrt[n]{a^{j+1}}}x\|^s)$$
$$= \frac{\gamma}{|a|^{\frac{(j+1)s}{n}}}\|x\|^s$$

From Theorem 3.2,

$$\|g(x) - K(x)\| \le \frac{1}{|2a|}\tilde{\beta}(x)$$
$$\le \frac{1}{|2a|}max\{|a|^j\beta(\frac{x}{\sqrt[n]{a^{j+1}}},0) : 0 \le j < t\}$$
$$= \frac{\gamma}{|2a^{\frac{s}{n}}|}\|x\|^s max\{|a|^{j(1-\frac{s}{n})} : 0 \le j < t\}$$

If $s < n$, then we get

$$\|g(x) - K(x)\| \le \frac{\gamma}{|2a^{\frac{s}{n}}|}\|x\|^s.$$

Hence the proof completes. ■

Example 3.3. Let $p > 2$ be a prime number $g : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g(x) = x^n + 1$ let $|2|_p^t = 1$, $\gamma > 1, a = 2, t \in \mathbb{Z}$, $s < n$ and if

$$\|G(x,y)\| = 1 \le \gamma(\|x\|^s + \|y\|^s)$$

then

$$\|g(x) - K(x)\| = 1 \le \frac{\gamma|2|^{n/s}}{|2|}\|x\|^s.$$

For the case $s = n$, we have following counterexample,

Example 3.4. Let $p > 2$ be a prime number $g : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g(x) = 4$ let $|2|_p^t = 1$, $\gamma > 0, a = 4, t \in \mathbb{Z}$ we have,

$$\|G(x,y)\| = 0 \le \gamma(\|x\|^s + \|y\|^s)$$

so,

$$\lim_{t\to\infty}\|a^t g(\frac{x}{\sqrt[n]{a}}) - a^{t+1}g(\frac{x}{\sqrt[n]{a^{t+1}}})\| = |4|_p^{t+1}|3| \ne 0$$

Hence $\{|a|^t g(\frac{x}{\sqrt[n]{a}})\}$ is not Cauchy.

Corollary 3.4. Let s, r, γ are positive real numbers and $r + s < n$, if a mapping $g : X \to Y$ satisfies

$$\|G(x,y)\| \le \gamma(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s) \tag{37}$$

then there is a unique mapping $K : X \to Y$ so that

$$\|g(x) - K(x)\| \le \frac{\gamma}{|2a^{\frac{r+s}{n}}|}\|x\|^{r+s}. \tag{38}$$

Proof: Consider

$$\|G(x,y)\| \leq \gamma(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s).$$

Given

$$\beta(x,y) = \gamma(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s).$$

Substituting (x,y) as $(\frac{x}{\sqrt[n]{a^{j+1}}}, 0)$ in (37), we have,

$$\begin{aligned}\beta(\frac{x}{\sqrt[n]{a^{j+1}}}, 0) &= \gamma(\|\frac{x}{\sqrt[n]{a^{j+1}}}x\|^{r+s}) \\ &= \frac{\gamma}{|a|^{\frac{(j+1)(r+s)}{n}}}\|x\|^{r+s}\end{aligned}$$

From Theorem 3.2,

$$\begin{aligned}\|g(x) - K(x)\| &\leq \frac{1}{|2a|}\tilde{\beta}(x) \\ &\leq \frac{1}{|2a|}max\{|a|^j\beta(\frac{x}{\sqrt[n]{a^{j+1}}}, 0) : 0 \leq j < t\} \\ &= \frac{\gamma}{|2a^{\frac{r+s}{n}}|}\|x\|^{r+s}max\{|a|^{j(1-\frac{r+s}{n})} : 0 \leq j < t\}\end{aligned}$$

If $r+s < n$, then we get

$$\|g(x) - K(x)\| \leq \frac{\gamma}{|2a^{\frac{r+s}{n}}|}\|x\|^{r+s}.$$

Hence the proof completes. ■

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