Periodicity of A Four-order Maximum Fuzzy Difference Equation

Changyou Wang, Qiyu Wang, Qimin Zhang, and Jingwei Meng

Abstract—**In this article, we investigate the periodicity of a four-order maximum fuzzy difference equation with the positive fuzzy numbers for the initial values and the parameter. Firstly, we transform the maximum fuzzy difference equation into a corresponding maximum ordinary difference equations with parameters by utilizing the fuzzy set theory. Secondly, we obtain the expressions of the eventual periodic solutions for the maximum ordinary difference equations by the inequality technique, the iteration method and the mathematical induction. Finally, we achieve the expressions of the eventual periodic solutions for the maximum fuzzy difference equation. In addition, we give some numerical simulations to verify the effectiveness of our theoretical analysis.**

Index Terms—**Difference equation, Maximum, Periodicity, Fuzzy parameter**

I. Introduction

It is well known that the difference equation theory is an important branch of dynamical system theory. Difference equation or discrete systems are one of the most important classical dynamical models in applied mathematics, which have wide applications including computer science, biology, economics, infectious disease, etc (see [1-6]). A difference equation is an equation represented by a recursive sequence. If we classify the difference equation according to the parameter and initial value, the difference equation with real parameter and initial value can be called ordinary difference equation, and the difference equation with fuzzy parameter and initial value can be called fuzzy difference equation. Because the fuzzy difference equation can describe natural phenomenon and objective laws with fuzzy uncertainty in the real world, it has attracted the attention of many scholars. For examples, Wang et al. [7] proposed a fuzzy neural network temperature prediction method by combining the advantages of fuzzy neural network and genetic algorithm. Hussain et al. [8] solved a second-order fuzzy ordinary differential equation by using the derivative self-starting block method of two orders. So far, people have done a lot of work on the ordinary difference equation, but research paper on fuzzy difference equation is still rare. Based on the importance of the maximum operator in the automatic control model (see [9, 10]), the maximum difference equation has attracted the concern of many scholars (see [11-14]).

In 2002, Voulov [15] studied the positive solutions of the following maximum difference equation

$$x_n = \max\left\{\frac{A}{x_{n-k}}, \frac{B}{x_{n-m}}\right\},\ n = 0,1,\cdots, \tag{1}$$

where $A, B \in R^+$, and showed that every solution of the above equation (1) is eventually periodic.

In 2010, Gelisken et al.[16] studied the asymptotic behavior and periodicity of positive solutions for the maximum difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-2}^{\alpha}}\right\},\quad n \ge 0, \tag{2}$$

where $A \ge 0,\ 0 \le \alpha \le 1$, and proved that every positive solution of the above difference equation (2) convergences to equilibrium point $\bar{x} = 1$ or is eventually periodic with period 2, 3, or 4.

Furthermore, in 2016, Gelisken et al. [17] researched the behavior of positive solutions for the system of the maximum difference equations

$$\begin{aligned} x_{n+1} &= \max\left\{A, \frac{c}{y_{n-k}}\right\}, \\ y_{n+1} &= \max\left\{B, \frac{c}{x_{n-m}}\right\}, n = 0,1,2,\cdots, \end{aligned} \tag{3}$$

where $k, m \in \mathbb{N}, A, B, C \in R^+$, and the initial values $x_{-i}, y_{-j} \in R^+,\ 0 \le i \le m,\ 0 \le j \le k$, and proved that every solution of the above system (3) is bounded and eventually constant or eventually periodic with period $k+m+2$. In the same year, Wang et al. [18] studied the following maximum difference equations

$$\begin{aligned} x_{n+1} &= \max\left\{c, \frac{y_n^p}{x_{n-1}^q}\right\}, \\ y_{n+1} &= \max\left\{c, \frac{x_n^p}{y_{n-1}^q}\right\}, n = 0,1,2,\cdots, \end{aligned} \tag{4}$$

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where the parameters c, p, q and the initial values x_0, x_{-1}, y_0, y_{-1} are positive real numbers, and obtained some sufficient conditions which ensure the boundedness character of the maximum system (4).

In 2020, Su et al. [19] studied the periodicity of positive solutions for the following maximum difference equations

$$\begin{gathered} x_n = \max\left\{A, \frac{y_{n-t}}{x_{n-s}}\right\}, \\ y_{n+1} = \max\left\{B, \frac{x_{n-t}}{y_{n-s}}\right\}, n \in \{0,1,2,\cdots\}, \end{gathered} \tag{5}$$

here $A, B \in R^+, t, s \in N^+, \gcd(s,t) = 1, d = \max\{t, s\}$, and the initial values conditions $x_{-k}, y_{-k} \in R^+, 0 \le k \le d$.

Moreover, in 2022, Sun et al. [20] researched the global behavior of the following maximum difference equations with four variables

$$\begin{gathered} x_n = \max\left\{A, \frac{z_{n-1}}{y_{n-2}}\right\}, y_n = \max\left\{B, \frac{w_{n-1}}{x_{n-2}}\right\}, \\ z_n = \max\left\{C, \frac{x_{n-1}}{w_{n-2}}\right\}, w_n = \max\left\{D, \frac{y_{n-1}}{z_{n-2}}\right\}, \end{gathered} \tag{6}$$

where $n \in N$, $A, B, C, D \in R^+$, $A \le B, C \le D$, and the initial values $x_{-i}, y_{-i}, z_{-i}, w_{-i} \in R^+, i = -2, -1$.

In the real world, the parameters of mathematical models usually come from statistical data and experimental data. These models, even in a classical way, are often affected by uncertainty (fuzzy uncertainty), which can be expressed in state variables, initial conditions, and system parameters. Up to now, the fuzzy set instituted by Zadeh [21] has become a powerful tool in various theories and many applications including fuzzy difference equations. In recent years, people are increasingly interested in the research of fuzzy difference equations and maximum fuzzy difference equations (see [22-27]).

Sun et al. [28] studied the periodicity of the positive solutions of the following maximum fuzzy difference equation in 2018

$$x_n = \max\left\{\frac{1}{x_{n-m}}, \frac{\alpha_n}{x_{n-r}}\right\}, n \in \mathrm{N}_0, \tag{7}$$

here $m, r \in N, d = \max\{m, r\}$, α_n is a positive periodic fuzzy number sequence and the initial values $x_{-i} (1 \le i \le d)$ are positive fuzzy numbers. The authors proved that if $\max(\operatorname{supp} \alpha_n) < 1$, then every positive solution of the above equation (6) is eventually periodic with period $2m$.

In 2020, Han et al. [29] researched the periodicity of the following maximum high-order fuzzy difference equation

$$x_n = \max\left\{\mathrm{C}, \frac{x_{n-m-k}}{x_{n-m}}\right\}, n \in \mathrm{N}_0, \tag{8}$$

here $m, k \in N$, the parameter C and the initial values $x_i (1 \le i \le m+k)$ are positive fuzzy numbers.

Inspired by the above scholars' works on fuzzy difference equations and maximum difference systems, we will studies the following maximum difference equation with the fuzzy parameter and initial values in this paper

$$z_{n+1} = \max\left\{\frac{H}{z_{n-1}}, \frac{H}{z_{n-2}}, z_{n-3}\right\}, \tag{9}$$

where the initial values $x_i (0 \le i \le 3)$ and parameter H are positive fuzzy numbers.

This article is summarized as follows. In Section 2, we introduce the basic concept of fuzzy sets used in this paper. In Section 3, we study the periodicity of the positive solution of (9) by using a new iterative method, inequality techniques and mathematical induction. In Section 4, we give some numerical simulations to illustrate the theoretical analysis of this paper.

II. Preliminaries and Notations

In this section, we first review some basic concepts used in this paper, for more detail, readers can refer to [21, 23, 26, 30].

Definition 1. A function defined as $A: R \to [0,1]$ is said to be a fuzzy number if it is normal, convex, upper semicontinuous, and compactly support on R.

For $\alpha \in (0,1]$, the $\alpha - cuts$ of the fuzzy number U on R is defined as $[U]^\alpha = \{x \in R : U(x) \ge \alpha\}$. It is clear that the $[U]^\alpha$ is a bounded closed interval in R, we say that a fuzzy number U is positive if $\operatorname{supp} U \subset (0, \infty)$. It is obvious that if U is a positive real number then U is a positive fuzzy number and $[U]^\alpha = [U, U], \alpha \in (0,1]$. At this time, we say that U is a trivial fuzzy number.

Definition 2. The persistence and boundedness of a positive fuzzy number sequence are defined as follows:

(i) A sequence of positive fuzzy numbers $\{z_n\}$ is persistent (resp. is bounded) if there exists a positive real numbers M $(resp.\ N)$ such that $\operatorname{supp} z_n \in [M, \infty)$ $(resp. \operatorname{supp} z_n \in [0, N))$, $n = 1, 2, \cdots$;

(ii) A sequence of positive fuzzy numbers $\{z_n\}$ is bounded and persistent if there exist some positive real numbers $M, N > 0$ such that $\operatorname{supp} z_n \in [M, N), n = 1, 2, \cdots$.

Definition 3. A sequence of positive fuzzy number $\{z_n\}$ is called a positive solution of equation (9) if it satisfies (9).

Definition 4. A sequence $\{z_n\}$ is called eventually periodic with period T if there exists $K \in N$ such that $z_n = z_{n+T}$ for all $n \ge K$.

Lemma 1. If the parameter H and the initial conditions $z_{-3}, z_{-2}, z_{-1}, z_0$ of the fuzzy difference equation (9) are positive fuzzy numbers, then the equation (9) has a unique positive fuzzy solution.

Proof. The proof is similar to **Proposition 3.1** [31], so we omit the proof of **Lemma 1.** □

Next, we will prove a simple auxiliary result which will be used many times in the rest of the paper.

Lemma 2. Assume that $\{z_n\}_{n=-3}^{+\infty}$ is solution of the equation (9) and there exists $k_0 \in N_0 \bigcup \{-3,-2,-1\}$ such that

$$\begin{aligned} z_{k_0} &= z_{k_0+4}, z_{k_0+1} = z_{k_0+5}, \\ z_{k_0+2} &= z_{k_0+6}, z_{k_0+3} = z_{k_0+7}. \end{aligned} \tag{10}$$

Then this solution is eventually periodic with period four.

Proof. From **Definition 4**, it is easy to know that we only need to prove that

$$\begin{aligned} z_{k_0} &= z_{k_0+4m}, z_{k_0+1} = z_{k_0+1+4m}, \\ z_{k_0+2} &= z_{k_0+2+4m}, z_{k_0+3} = z_{k_0+3+4m}, \end{aligned} \tag{11}$$

for every $m \in N$, from which the **Lemma 2** follows.

We use the mathematical induction. For $m=1$, the equation (11) becomes (10), that is (11) holds for $m=1$. Assume that (11) holds for $m=m_0$, then from (9) and (10) we can obtain that

$$\begin{aligned} z_{k_0+4(m_0+1)} &= \max\{\frac{H}{z_{k_0+4m_0+2}}, \frac{H}{z_{k_0+4m_0+1}}, z_{k_0+4m_0}\} \\ &= \max\{\frac{H}{z_{k_0+2}}, \frac{H}{z_{k_0+1}}, z_{k_0}\} = z_{k_0+4} = z_{k_0}, \\ z_{k_0+1+4(m_0+1)} &= \max\{\frac{H}{z_{k_0+4m_0+3}}, \frac{H}{z_{k_0+4m_0+2}}, z_{k_0+4m_0+1}\} \\ &= \max\{\frac{H}{z_{k_0+3}}, \frac{H}{z_{k_0+2}}, z_{k_0+1}\} = z_{k_0+5} = z_{k_0+1}, \\ z_{k_0+2+4(m_0+1)} &= \max\{\frac{H}{z_{k_0+4m_0+4}}, \frac{H}{z_{k_0+4m_0+3}}, z_{k_0+4m_0+2}\} \\ &= \max\{\frac{H}{z_{k_0+4}}, \frac{H}{z_{k_0+3}}, z_{k_0+2}\} = z_{k_0+6} = z_{k_0+2}, \\ z_{k_0+3+4(m_0+1)} &= \max\{\frac{H}{z_{k_0+4m_0+5}}, \frac{H}{z_{k_0+4m_0+4}}, z_{k_0+4m_0+3}\} \\ &= \max\{\frac{H}{z_{k_0+5}}, \frac{H}{z_{k_0+4}}, z_{k_0+3}\} = z_{k_0+7} = z_{k_0+3}. \end{aligned}$$

That is, the equation (11) holds for $m=m_0+1$. By the induction method, the proof of **Lemma 2** is completed. □

According to **Lemma 1** and the $\alpha-cuts$ theory of the fuzzy number, the maximum fuzzy difference equation (9) can be converted into the following difference system of two ordinary difference equations

$$\begin{aligned} x_{n+1} &= \max\left\{\frac{M}{y_{n-1}}, \frac{M}{y_{n-2}}, x_{n-3}\right\}, \\ y_{n+1} &= \max\left\{\frac{N}{x_{n-1}}, \frac{N}{x_{n-2}}, y_{n-3}\right\}, \quad n \in N_0, \end{aligned} \tag{12}$$

where the parameter0s M,N are positive real numbers, which are, respectively, the left and right endpoints of the $\alpha-cuts$ interval of the fuzzy parameter H in equation (9), and $\{x_n\},\{y_n\}$ are two positive real number sequences, which are respectively composed of the left and right endpoint of the $\alpha-cuts$ interval of the solution sequence $\{z_n\}$ of fuzzy equation (9).

III. Periodicity of positive solution

In this section, we discuss the eventual periodicity of positive solutions of the fuzzy difference equation (9). First, we study the periodicity of the positive solutions of the ordinary difference equations (12).

Theorem 1. If the parameters $M \le N$, and the initial values $x_{-i}, y_{-i} (0 \le i \le 3)$ are positive real numbers, then every positive solution of the ordinary difference equations (12) is eventually periodic with period four.

Proof. Let (x_n, y_n) be a positive solution of (12), then we have

$$\begin{aligned} x_1 &= \max\left\{\frac{M}{y_{-1}}, \frac{M}{y_{-2}}, x_{-3}\right\}, \\ y_1 &= \max\left\{\frac{N}{x_{-1}}, \frac{N}{x_{-2}}, y_{-3}\right\}. \end{aligned} \tag{13}$$

From (13) we can get

$$x_1 \ge \frac{M}{y_{-2}}, \ y_1 \ge \frac{N}{x_{-2}}.$$

Because the initial values $x_{-i}, y_{-i} \left(0 \le i \le 3\right)$ are positive real numbers, so it holds that

$$y_{-2} \ge \frac{M}{x_1}, \ x_{-2} \ge \frac{N}{y_1}. \tag{14}$$

From (12), one has

$$x_2 = \max\left\{\frac{M}{y_0}, \frac{M}{y_{-1}}, x_{-2}\right\}, \ y_2 = \max\left\{\frac{N}{x_0}, \frac{N}{x_{-1}}, y_{-2}\right\}. \tag{15}$$

And then

$$x_2 \ge x_{-2}, \ y_2 \ge y_{-2}, x_2 \ge \frac{M}{y_0}, \ y_2 \ge \frac{N}{x_0}. \tag{16}$$

In view of (14) and (16), we have $x_2 \ge \frac{N}{y_1}, \ y_2 \ge \frac{M}{x_1}$.

Moreover, it holds that

$$x_1 \ge \frac{M}{y_2}, \ y_1 \ge \frac{N}{x_2}. \tag{17}$$

From (16), we have

$$x_0 \ge \frac{N}{y_2}. \tag{18}$$

Since $N > M$, so one has

$$x_0 \ge \frac{N}{y_2} \ge \frac{M}{y_2}. \tag{19}$$

From (19) and (12), it follows that

$$x_4 = \max\left\{\frac{M}{y_2}, \frac{M}{y_1}, x_0\right\} = \max\left\{\frac{M}{y_1}, x_0\right\},$$

$$y_4 = \max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_0\right\}.$$

From (12), if $n \geq 3$, then it holds that

$$y_n \geq \frac{N}{x_{n-3}},\, y_{n-1} \geq \frac{N}{x_{n-3}}. \tag{20}$$

So we have

$$x_{n-3} \geq \frac{N}{y_{n-1}},\, x_{n-3} \geq \frac{N}{y_{n-2}}. \tag{21}$$

Since $N \geq M$, we can obtain that

$$x_{n-3} \geq \frac{M}{y_{n-1}},\, x_{n-3} \geq \frac{M}{y_{n-2}}. \tag{22}$$

From (22), if $n \geq 3$, then it holds that

$$x_{n+1} = \max\left\{\frac{M}{y_{n-1}}, \frac{M}{y_{n-2}}, x_{n-3}\right\} = x_{n-3}.$$

Hence

$$\begin{aligned} &x_{4n} = x_0,\, x_{4n+1} = x_1, \\ &x_{4n+2} = x_2, x_{4n+3} = x_3,\ \ n \geq 1. \end{aligned} \tag{23}$$

Now, we consider the period nature of $\{y_n\}_{n=4}^{\infty}$:

(a_1) From (13), suppose that

$$\frac{M}{y_{-1}} \geq \frac{M}{y_{-2}}, \frac{M}{y_{-1}} \geq x_{-3} \text{ and } \frac{N}{x_{-1}} \geq \frac{N}{x_{-2}}, \frac{N}{x_{-1}} \geq y_{-3},$$

then it follows that

$$x_1 = \frac{M}{y_{-1}}, y_1 = \frac{N}{x_{-1}}.$$

(a_{1-1}) From (15), suppose that

$$\frac{M}{y_0} \geq \frac{M}{y_{-1}}, \frac{M}{y_0} \geq x_{-2} \text{ and } \frac{N}{x_0} \geq \frac{N}{x_{-1}}, \frac{N}{x_0} \geq y_{-2}$$

then it follows that

$$x_2 = \frac{M}{y_0}, y_2 = \frac{N}{x_0}. \tag{24}$$

From (12), we have

$$y_2 = \max\left\{\frac{N}{x_0}, \frac{N}{x_{-1}}, y_{-2}\right\}.$$

And then, we can get

$$y_2 \geq \frac{N}{x_0}, y_2 \geq \frac{N}{x_{-1}}. \tag{25}$$

Moreover, from (12), it holds that

$$x_3 = \max\left\{\frac{M}{y_1}, \frac{M}{y_0}, x_{-1}\right\},$$

and then, we can obtain

$$x_3 \geq x_{-1}. \tag{26}$$

From (25) and (26), it follows that

$$y_2 \geq \frac{N}{x_0}, y_2 \geq \frac{N}{x_{-1}} \geq \frac{N}{x_3},$$

and then, from (12), (23) and (24), we have

$$\begin{aligned} y_6 &= \max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_2\right\} \\ &= \max\left\{\frac{N}{x_0}, \frac{N}{x_3}, y_2\right\} = y_2 = \frac{N}{x_0}. \end{aligned}$$

In a similar way, we can get

$$y_7 = \max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_3\right\} = \max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_3\right\} = y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6}, \frac{N}{x_5}, y_4\right\} = \max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_4\right\} = y_4,$$

$$y_9 = \max\left\{\frac{N}{x_7}, \frac{N}{x_6}, y_5\right\} = \max\left\{\frac{N}{x_3}, \frac{N}{x_2}, y_5\right\} = y_5,$$

$$y_{10} = \max\left\{\frac{N}{x_8}, \frac{N}{x_7}, y_6\right\} = \max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_6\right\} = y_6,$$

$$y_{11} = \max\left\{\frac{N}{x_9}, \frac{N}{x_8}, y_7\right\} = \max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_7\right\} = y_7 = y_3.$$

According to the mathematical induction, we can prove that

$$y_{4n+2} = y_2 = \frac{N}{x_0},$$

$$y_{4n+3} = y_3 = \max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_{-1}\right\}$$

$$= \max\left\{\frac{N}{B} y_{-1}, \frac{N}{x_0}, y_{-1}\right\} = \max\left\{\frac{N}{B} y_{-1}, \frac{N}{x_0}\right\},$$

$$y_{4n} = y_4 = \max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_0\right\}$$

$$= \max\left\{\frac{N}{M} y_0, \frac{N}{M} y_{-1}, y_0\right\} = \max\left\{\frac{N}{M} y_0, \frac{N}{M} y_{-1}\right\},$$

$$y_{4n+1} = y_5 = \max\left\{\frac{N}{x_3}, \frac{N}{x_2}, y_1\right\} = \max\left\{\frac{N}{x_3}, \frac{N}{M} y_0, \frac{N}{x_{-1}}\right\}.$$

(a_{1-2}) From (15), suppose that $\frac{M}{y_0} \geq \frac{M}{y_{-1}}$, $\frac{M}{y_0} \geq x_{-2}$ and $\frac{N}{x_{-1}} \geq \frac{N}{x_0}$, $\frac{N}{x_{-1}} \geq y_{-2}$, then it follows that

$$x_2 = \frac{M}{y_0}, y_2 = \frac{N}{x_{-1}}.$$

Using a similar method to discussing (a_{1-1}), we can get

$$y_6 = \max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_2\right\} = \max\left\{\frac{N}{x_0}, \frac{N}{x_3}, y_2\right\}$$

$$= y_2 = \frac{N}{x_{-1}},$$

$$y_7 = \max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_3\right\} = \max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_3\right\} = y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6},\frac{N}{x_5},y_4\right\}=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_4\right\}=y_4,$$

$$y_9 = \max\left\{\frac{N}{x_7},\frac{N}{x_6},y_5\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_5\right\}=y_5,$$

$$y_{10} = \max\left\{\frac{N}{x_8},\frac{N}{x_7},y_6\right\}=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_6\right\}=y_6,$$

$$y_{11} = \max\left\{\frac{N}{x_9},\frac{N}{x_8},y_7\right\}=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be proved that

$$y_{4n+2} = y_2=\frac{N}{x_{-1}},$$

$$y_{4n+3} = y_3 = \max\left\{\frac{N}{x_1},\frac{N}{x_0},y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0},y_{-1}\right\},$$

$$y_{4n} = y_4=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_0\right\}=\max\left\{\frac{N}{M}y_0,\frac{N}{M}y_{-1},y_0\right\}$$

$$=\max\left\{\frac{N}{M}y_0,\frac{N}{M}y_{-1}\right\},$$

$$y_{4n+1} = y_5 = \max\left\{\frac{N}{x_3},\frac{N}{x_2},y_1\right\}=\max\left\{\frac{N}{x_3},\frac{N}{M}y_0,\frac{N}{x_{-1}}\right\}.$$

(a_{1-3}) From (15), suppose that $\frac{M}{y_0}\geq\frac{M}{y_{-1}},\frac{M}{y_0}\geq x_{-2}$ and $y_{-2}\geq\frac{N}{x_{-1}}, y_{-2}\geq\frac{N}{x_0},$ then it follows that

$$x_2=\frac{M}{y_0}, y_2=y_{-2}.$$

Using a similar method to discussing (a_{1-1}), we have

$$y_6 = \max\left\{\frac{N}{x_4},\frac{N}{x_3},y_2\right\}=\max\left\{\frac{N}{x_0},\frac{N}{x_3},y_2\right\}=y_2=y_{-2},$$

$$y_7 = \max\left\{\frac{N}{x_5},\frac{N}{x_4},y_3\right\}=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_3\right\}=y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6},\frac{N}{x_5},y_4\right\}=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_4\right\}=y_4,$$

$$y_9 = \max\left\{\frac{N}{x_7},\frac{N}{x_6},y_5\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_5\right\}=y_5,$$

$$y_{10} = \max\left\{\frac{N}{x_8},\frac{N}{x_7},y_6\right\}=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_6\right\}=y_6,$$

$$y_{11} = \max\left\{\frac{N}{x_9},\frac{N}{x_8},y_7\right\}=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be proved that

$$y_{4n+2} = y_2=y_{-2},$$

$$y_{4n+3} = y_3 = \max\left\{\frac{N}{x_1},\frac{N}{x_0},y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0},y_{-1}\right\},$$

$$y_{4n} = y_4=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_0\right\}=\max\left\{\frac{N}{M}y_0,\frac{N}{M}y_{-1},y_0\right\}$$

$$=\max\left\{\frac{N}{M}y_0,\frac{N}{M}y_{-1}\right\},$$

$$y_{4n+1} = y_5 = \max\left\{\frac{N}{x_3},\frac{N}{x_2},y_1\right\}=\max\left\{\frac{N}{x_3},\frac{N}{M}y_0,\frac{N}{x_{-1}}\right\}.$$

(a_{1-4}) From (15), suppose that $\frac{M}{y_{-1}}\geq\frac{M}{y_0},\frac{M}{y_{-1}}\geq x_{-2}$ and $\frac{N}{x_0}\geq\frac{N}{x_{-1}},\frac{N}{x_0}\geq y_{-2},$ then it follows that

$$x_2=\frac{M}{y_{-1}}, y_2=\frac{N}{x_0}.$$

Using a similar method to discussing (a_{1-1}), we can get

$$y_6 = \max\left\{\frac{N}{x_4},\frac{N}{x_3},y_2\right\}=\max\left\{\frac{N}{x_0},\frac{N}{x_3},y_2\right\}=y_2=\frac{N}{x_0},$$

$$y_7 = \max\left\{\frac{N}{x_5},\frac{N}{x_4},y_3\right\}=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_3\right\}=y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6},\frac{N}{x_5},y_4\right\}=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_4\right\}=y_4,$$

$$y_9 = \max\left\{\frac{N}{x_7},\frac{N}{x_6},y_5\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_5\right\}=y_5,$$

$$y_{10} = \max\left\{\frac{N}{x_8},\frac{N}{x_7},y_6\right\}=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_6\right\}=y_6,$$

$$y_{11} = \max\left\{\frac{N}{x_9},\frac{N}{x_8},y_7\right\}=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be proved that

$$y_{4n+2} = y_2=\frac{N}{x_0},$$

$$y_{4n+3} = y_3 = \max\left\{\frac{N}{x_1},\frac{N}{x_0},y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0},y_{-1}\right\},$$

$$y_{4n} = y_4=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_0\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{M}y_{-1},y_0\right\}$$

$$=\max\left\{y_0,\frac{N}{M}y_{-1}\right\},$$

$$y_{4n+1} = y_5 = \max\left\{\frac{N}{x_3},\frac{N}{x_2},y_1\right\}=\max\left\{\frac{N}{x_3},\frac{N}{M}y_{-1},\frac{N}{x_{-1}}\right\}.$$

$\left(a_{1-5}\right)$ From (15), suppose that $\frac{M}{y_{-1}} \ge \frac{M}{y_0}, \frac{M}{y_{-1}} \ge x_{-2}$ and $\frac{N}{x_{-1}} \ge \frac{N}{x_0}, \frac{N}{x_{-1}} \ge y_{-2}$, then it holds that

$$x_2=\frac{M}{y_{-1}}, y_2=\frac{N}{x_{-1}}.$$

Using a similar method to discussing $\left(a_{1-1}\right)$, we can get

$$y_6 = \max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_2\right\}=\max\left\{\frac{N}{x_0}, \frac{N}{x_3}, y_2\right\}= y_2 = \frac{N}{x_{-1}},$$

$$y_7 = \max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_3\right\}=\max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_3\right\}=y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6}, \frac{N}{x_5}, y_4\right\}=\max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_4\right\}=y_4,$$

$$y_9 = \max\left\{\frac{N}{x_7}, \frac{N}{x_6}, y_5\right\}=\max\left\{\frac{N}{x_3}, \frac{N}{x_2}, y_5\right\}=y_5,$$

$$y_{10} = \max\left\{\frac{N}{x_8}, \frac{N}{x_7}, y_6\right\}=\max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_6\right\}=y_6,$$

$$y_{11} = \max\left\{\frac{N}{x_9}, \frac{N}{x_8}, y_7\right\}=\max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be obtained that

$$y_{4n+2} = y_2=\frac{N}{x_{-1}},$$

$$y_{4n+3} = y_3 = \max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_{-1}\right\}$$

$$= \max\left\{\frac{N}{B}y_{-1}, \frac{N}{x_0}, y_{-1}\right\},$$

$$y_{4n} = y_4 = \max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_0\right\}$$

$$= \max\left\{\frac{N}{M}y_{-1}, \frac{N}{M}y_{-1}, y_0\right\} = \max\left\{y_0, \frac{N}{M}y_{-1}\right\},$$

$$y_{4n+1} = y_5 = \max\left\{\frac{N}{x_3}, \frac{N}{x_2}, y_1\right\}$$

$$= \max\left\{\frac{N}{x_3}, \frac{N}{M}y_{-1}, \frac{N}{x_{-1}}\right\}.$$

$\left(a_{1-6}\right)$ From (15), suppose that $\frac{M}{y_{-1}} \ge \frac{M}{y_0}, \frac{M}{y_{-1}} \ge x_{-2}$ and $y_{-2} \ge \frac{N}{x_{-1}}, y_{-2} \ge \frac{N}{x_0}$, then one has

$$x_2=\frac{M}{y_{-1}}, y_2=y_{-2}.$$

Using a similar method to discussing $\left(a_{1-1}\right)$, we can obtain

$$y_6 = \max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_2\right\}=\max\left\{\frac{N}{x_0}, \frac{N}{x_3}, y_2\right\}= y_2 = y_{-2},$$

$$y_7 = \max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_3\right\}=\max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_3\right\}=y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6}, \frac{N}{x_5}, y_4\right\}=\max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_4\right\}=y_4,$$

$$y_9 = \max\left\{\frac{N}{x_7}, \frac{N}{x_6}, y_5\right\}=\max\left\{\frac{N}{x_3}, \frac{N}{x_2}, y_5\right\}=y_5,$$

$$y_{10} = \max\left\{\frac{N}{x_8}, \frac{N}{x_7}, y_6\right\}=\max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_6\right\}=y_6,$$

$$y_{11} = \max\left\{\frac{N}{x_9}, \frac{N}{x_8}, y_7\right\}=\max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be proved that

$$y_{4n+2}= y_2 = y_{-2},$$

$$y_{4n+3} =y_3 = \max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1}, \frac{N}{x_0}, y_{-1}\right\}$$

$$=\max\left\{\frac{N}{M}y_{-1}, \frac{N}{x_0}\right\},$$

$$y_{4n}= y_4 = \max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_0\right\}=\max\left\{\frac{N}{M}y_{-1}, \frac{N}{M}y_{-1}, y_0\right\}$$

$$=\max\left\{\frac{N}{M}y_{-1}, y_0\right\},$$

$$y_{4n+1}= y_5 = \max\left\{\frac{N}{x_3}, \frac{N}{x_2}, y_1\right\}=\max\left\{\frac{N}{x_3}, \frac{N}{M}y_{-1}, y_1\right\}$$

$$=\max\left\{\frac{N}{x_3}, \frac{N}{M}y_{-1}, \frac{N}{x_{-1}}\right\}.$$

$\left(a_{1-7}\right)$ From (15), suppose that $x_{-2} \ge \frac{M}{y_{-1}}, x_{-2} \ge \frac{M}{y_0}$ and $\frac{N}{x_0} \ge \frac{N}{x_{-1}}, \frac{N}{x_0} \ge y_{-2}$, then we can obtain

$$x_2=x_{-2}, y_2=\frac{N}{x_0}.$$

Using a similar method to discussing $\left(a_{1-1}\right)$, we have

$$y_6 = \max\left\{\frac{N}{x_4}, \frac{N}{x_3}, y_2\right\}=\max\left\{\frac{N}{x_0}, \frac{N}{x_3}, y_2\right\}= y_2 = \frac{N}{x_0},$$

$$y_7 = \max\left\{\frac{N}{x_5}, \frac{N}{x_4}, y_3\right\}=\max\left\{\frac{N}{x_1}, \frac{N}{x_0}, y_3\right\}=y_3,$$

$$y_8 = \max\left\{\frac{N}{x_6}, \frac{N}{x_5}, y_4\right\}=\max\left\{\frac{N}{x_2}, \frac{N}{x_1}, y_4\right\}=y_4,$$

$$y_9=\max\left\{\frac{N}{x_7},\frac{N}{x_6},y_5\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_5\right\}=y_5,$$

$$y_{10}=\max\left\{\frac{N}{x_8},\frac{N}{x_7},y_6\right\}=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_6\right\}=y_6,$$

$$y_{11}=\max\left\{\frac{N}{x_9},\frac{N}{x_8},y_7\right\}=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_7\right\}=y_7=$$

According to mathematical induction, it can be proved that

$$y_{4n+2}=y_2=\frac{N}{x_0},$$

$$y_{4n+3}=y_3=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_{-1}\right\}$$

$$=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0},y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0}\right\},$$

$$y_{4n}=y_4=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_0\right\}=\max\left\{\frac{N}{x_{-2}},\frac{N}{M}y_{-1},y_0\right\},$$

$$y_{4n+1}=y_5=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_1\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_{-2}},\frac{N}{x_{-1}}\right\}.$$

(a_{1-8}) From (15), suppose that $x_{-2}\ge\frac{M}{y_{-1}},x_{-2}\ge\frac{M}{y_0}$ and $\frac{N}{x_{-1}}\ge\frac{N}{x_0},\frac{N}{x_{-1}}\ge y_{-2}$, then it follows that

$$x_2=x_{-2},y_2=\frac{N}{x_{-1}}.$$

Using a similar method to discussing (a_{1-1}), we can get

$$y_6=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_2\right\}=\max\left\{\frac{N}{x_0},\frac{N}{x_3},y_2\right\}=y_2=\frac{N}{x_{-1}},$$

$$y_7=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_3\right\}=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_3\right\}=y_3,$$

$$y_8=\max\left\{\frac{N}{x_6},\frac{N}{x_5},y_4\right\}=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_4\right\}=y_4,$$

$$y_9=\max\left\{\frac{N}{x_7},\frac{N}{x_6},y_5\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_5\right\}=y_5,$$

$$y_{10}=\max\left\{\frac{N}{x_8},\frac{N}{x_7},y_6\right\}=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_6\right\}=y_6,$$

$$y_{11}=\max\left\{\frac{N}{x_9},\frac{N}{x_8},y_7\right\}=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be proved that

$$y_{4n+2}=y_2=\frac{N}{x_{-1}},$$

$$y_{4n+3}=y_3=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_{-1}\right\}$$

$$=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0},y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0}\right\},$$

$$y_{4n}=y_4=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_0\right\}=\max\left\{\frac{N}{x_{-2}},\frac{N}{M}y_{-1},y_0\right\},$$

$$y_{4n+1}=y_5=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_1\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_{-2}},\frac{N}{x_{-1}}\right\}.$$

(a_{1-9}) From (15), suppose that $x_{-2}\ge\frac{M}{y_{-1}},x_{-2}\ge\frac{M}{y_0}$ and $y_{-2}\ge\frac{N}{x_{-1}},y_{-2}\ge\frac{N}{x_0}$, then it follows that

$$x_2=x_{-2},y_2=y_{-2}.$$

Using a similar method to discussing (a_{1-1}), we can get

$$y_6=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_2\right\}=\max\left\{\frac{N}{x_0},\frac{N}{x_3},y_2\right\}=y_2=y_{-2},$$

$$y_7=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_3\right\}=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_3\right\}=y_3,$$

$$y_8=\max\left\{\frac{N}{x_6},\frac{N}{x_5},y_4\right\}=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_4\right\}=y_4,$$

$$y_9=\max\left\{\frac{N}{x_7},\frac{N}{x_6},y_5\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_5\right\}=y_5,$$

$$y_{10}=\max\left\{\frac{N}{x_8},\frac{N}{x_7},y_6\right\}=\max\left\{\frac{N}{x_4},\frac{N}{x_3},y_6\right\}=y_6,$$

$$y_{11}=\max\left\{\frac{N}{x_9},\frac{N}{x_8},y_7\right\}=\max\left\{\frac{N}{x_5},\frac{N}{x_4},y_7\right\}=y_7=y_3.$$

According to mathematical induction, it can be proved that

$$y_{4n+2}=y_2=y_{-2},$$

$$y_{4n+3}=y_3=\max\left\{\frac{N}{x_1},\frac{N}{x_0},y_{-1}\right\}$$

$$=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0},y_{-1}\right\}=\max\left\{\frac{N}{M}y_{-1},\frac{N}{x_0}\right\},$$

$$y_{4n}=y_4=\max\left\{\frac{N}{x_2},\frac{N}{x_1},y_0\right\}=\max\left\{\frac{N}{x_{-2}},\frac{N}{M}y_{-1},y_0\right\},$$

$$y_{4n+1}=y_5=\max\left\{\frac{N}{x_3},\frac{N}{x_2},y_1\right\}=\max\left\{\frac{N}{x_3},\frac{N}{x_{-2}},\frac{N}{x_{-1}}\right\}.$$

(b_1) From (13), suppose that $\frac{M}{y_{-1}} \geq \frac{M}{y_{-2}}, \frac{M}{y_{-1}} \geq x_{-3}$ and $\frac{N}{x_{-2}} \geq \frac{N}{x_{-1}}, \frac{N}{x_{-2}} \geq y_{-3}$, then it follows that

$$x_1 = \frac{M}{y_{-1}}, y_1 = \frac{N}{x_{-2}}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(c_1) From (13), suppose that $\frac{M}{y_{-1}} \geq \frac{M}{y_{-2}}, \frac{M}{y_{-1}} \geq x_{-3}$ and $y_{-3} \geq \frac{N}{x_{-1}}, y_{-3} \geq \frac{N}{x_{-2}}$, then it holds that

$$x_1 = \frac{M}{y_{-1}}, y_1 = \mathrm{y}_{-3}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(d_1) From (13), suppose that $\frac{M}{y_{-2}} \geq \frac{M}{y_{-1}}, \frac{M}{y_{-2}} \geq x_{-3}$ and $\frac{N}{x_{-1}} \geq \frac{N}{x_{-2}}, \frac{N}{x_{-1}} \geq y_{-3}$, then it holds that

$$x_1 = \frac{M}{y_{-2}}, y_1 = \frac{N}{x_{-1}}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(e_1) From (13), suppose that $\frac{M}{y_{-2}} \geq \frac{M}{y_{-1}}, \frac{M}{y_{-2}} \geq x_{-3}$ and $\frac{N}{x_{-2}} \geq \frac{N}{x_{-1}}, \frac{N}{x_{-2}} \geq y_{-3}$, then we have

$$x_1 = \frac{M}{y_{-2}}, y_1 = \frac{N}{x_{-2}}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(f_1) From (13), suppose that $\frac{M}{y_{-2}} \geq \frac{M}{y_{-1}}, \frac{M}{y_{-2}} \geq x_{-3}$ and $y_{-3} \geq \frac{N}{x_{-1}}, y_{-3} \geq \frac{N}{x_{-2}}$, then we have

$$x_1 = \frac{M}{y_{-2}}, y_1 = \mathrm{y}_{-3}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(g_1) From (13), suppose that $x_{-3} \geq \frac{M}{y_{-1}}, x_{-3} \geq \frac{M}{y_{-2}}$ and $\frac{N}{x_{-1}} \geq \frac{N}{x_{-2}}, \frac{N}{x_{-1}} \geq y_{-3}$, then we have

$$x_1 = x_{-3}, y_1 = \frac{N}{x_{-1}}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(h_1) From (13), suppose that $x_{-3} \geq \frac{M}{y_{-1}}, x_{-3} \geq \frac{M}{y_{-2}}$ and $\frac{N}{x_{-2}} \geq \frac{N}{x_{-1}}, \frac{N}{x_{-2}} \geq y_{-3}$, then it follows that

$$x_1 = x_{-3}, y_1 = \frac{N}{x_{-2}}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

(i_1) From (13), suppose that $x_{-3} \geq \frac{M}{y_{-1}}, x_{-3} \geq \frac{M}{y_{-2}}$ and $y_{-3} \geq \frac{N}{x_{-1}}, y_{-3} \geq \frac{N}{x_{-2}}$, then we have

$$y_1 = y_{-3}, z_1 = \mathrm{z}_{-3}.$$

In this case, the nine kinds of discussions are similar to those of (a_1), so they are omitted.

The proof of **Theorem 1** is completed. □

It is obvious that when $M=N=H$, we have the following result.

Corollary 1. Consider the ordinary difference equations

$$\begin{aligned} x_{n+1} &= \max\left\{\frac{H}{y_{n-1}}, \frac{H}{y_{n-2}}, x_{n-3}\right\}, \\ y_{n+1} &= \max\left\{\frac{H}{x_{n-1}}, \frac{H}{x_{n-2}}, y_{n-3}\right\}, \quad n \geq 0, \end{aligned} \tag{27}$$

if the parameter H and the initial values y_{-i}, z_{-i} ($0 \leq i \leq 3$) are positive real numbers, then every positive solution of equations (27) is eventually periodic with period four.

Next, we study the periodicity of the positive solutions of fuzzy equation (9) when H is a trivial fuzzy number.

Theorem 2. If the parameter H is a positive trivial fuzzy number and the initial values $z_{-3}, z_{-2}, z_{-1}, z_0$ are positive fuzzy numbers, then every positive solution of the fuzzy difference equation (9) is eventually periodic with period four.

Proof. Let $\{z_n\}$ be a positive fuzzy solution of (9) with initial values $z_{-3}, z_{-2}, z_{-1}, z_0$, from $\alpha - cuts$, thus we have

$$\begin{aligned} [z_i]^\alpha &= [L_{i,\alpha}, R_{i,\alpha}],\ i = -3, -2, -1, 0,\ \alpha \in (0,1], \\ [z_n]^\alpha &= [L_{n,\alpha}, R_{n,\alpha}],\ i = 1, 2, 3, \cdots,\ \alpha \in (0,1]. \end{aligned} \tag{28}$$

In view of H is a positive trivial fuzzy number, one has

$$[H]^\alpha = [H_{l,\alpha}, H_{r,\alpha}] = [H, H],\ \alpha \in (0,1].$$

From **Theorem 1**, $(L_{n,\alpha}, R_{n,\alpha}), i = 1, 2, 3, \cdots, \alpha \in (0,1]$ satisfies the following system

$$L_{n+1,\alpha} = \max\left\{\frac{H}{R_{n-1,\alpha}}, \frac{H}{R_{n-2,\alpha}}, L_{n-3,\alpha}\right\},$$

$$R_{n+1,\alpha} = \max\left\{\frac{H}{L_{n-1,\alpha}}, \frac{H}{L_{n-2,\alpha}}, R_{n-3,\alpha}\right\}, \quad n \geq 0.$$

Using **Corollary 1**, it follows that

$$L_{4n,\alpha} = L_{0,\alpha},\quad L_{4n+1,\alpha} = L_{1,\alpha}, L_{4n+2,\alpha} = L_{2,\alpha},$$
$$L_{4n+3,\alpha} = L_{3,\alpha}, R_{4n,\alpha} = R_{0,\alpha},\quad R_{4n+1,\alpha} = R_{1,\alpha},$$
$$R_{4n+2,\alpha} = R_{2,\alpha}, R_{4n+3,\alpha} = R_{3,\alpha}, n \geq 1, \alpha \in (0,1].$$

Therefore, we have that $\{z_n\}$ is eventually periodic of period four. The proof of **Theorem 2** is completed. □

Next, we study the periodicity of the positive solutions of (9) when A is a nontrivial fuzzy number.

Theorem 3. If the parameter H and the initial values $z_{-3}, z_{-2}, z_{-1}, z_0$ are positive fuzzy numbers, then every positive solution of the fuzzy difference equation (9) is eventually periodic with period four.

Proof. Set $\{z_n\}$ be a positive solution of (9) with initial values $z_{-3}, z_{-2}, z_{-1}, z_0$, from $\alpha - cuts$, thus (28) hold. In view of H is a nontrivial fuzzy number, it holds that

$$[H]^\alpha = \left[H_{l,\alpha}, H_{r,\alpha}\right] = [M, N],\ \alpha \in (0,1].$$

From **Lemma 1**, $(L_{n,\alpha}, R_{n,\alpha}), i = 1,2,3,\cdots, \alpha \in (0,1]$ satisfies system

$$L_{n+1,\alpha} = \max\left\{\frac{M}{R_{n-1,\alpha}}, \frac{M}{R_{n-2,\alpha}}, L_{n-3,\alpha}\right\},$$

$$R_{n+1,\alpha} = \max\left\{\frac{N}{L_{n-1,\alpha}}, \frac{N}{L_{n-2,\alpha}}, R_{n-3,\alpha}\right\}, \quad n \geq 0.$$

Using **Theorem 1**, we can obtain that

$$L_{4n,\alpha} = L_{0,\alpha},\quad L_{4n+1,\alpha} = L_{1,\alpha}, L_{4n+2,\alpha} = L_{2,\alpha},$$
$$L_{4n+3,\alpha} = L_{3,\alpha}, R_{4n,\alpha} = R_{0,\alpha},\quad R_{4n+1,\alpha} = R_{1,\alpha},$$
$$R_{4n+2,\alpha} = R_{2,\alpha}, R_{4n+3,\alpha} = R_{3,\alpha}, n \geq 1, \alpha \in (0,1].$$

Therefore, we have that $\{z_n\}$ is eventually periodic of period four. The proof is completed. □

From **Theorem 3**, we can obtain the following result.

Corollary 2. Consider equation (9) where H is a positive fuzzy number, and the initial values $z_{-3}, z_{-2}, z_{-1}, z_0$ are positive fuzzy numbers. Then every positive solution of (9) is persisting and bounded.

IV. Numerical simulation

Example 1. Consider the maximum difference equation with the fuzzy parameter and initial values

$$z_{n+1} = \max\left\{\frac{H}{z_{n-1}}, \frac{H}{z_{n-2}}, z_{n-3}\right\}, \quad n = 0,1,2,\cdots, \tag{29}$$

where the parameter H is positive fuzzy number, its membership function is as follows

$$H(x) = \begin{cases} \dfrac{1}{4}x - \dfrac{1}{2}, & 2 \leq x \leq 6, \\ -\dfrac{1}{7}x + \dfrac{13}{7}, & 6 \leq x \leq 13. \end{cases} \tag{30}$$

In addition, the membership functions of the initial values $z_{-3}, z_{-2}, z_{-1}, z_0$ are defined as

$$z_0(x) = \begin{cases} x-1, & 1 \leq x \leq 2, \\ -x+3, & 2 \leq x \leq 3, \end{cases} \quad z_{-1}(x) = \begin{cases} \dfrac{1}{5}x - \dfrac{2}{5}, & 2 \leq x \leq 7, \\ -\dfrac{1}{6}x + \dfrac{13}{6}, & 7 \leq x \leq 13, \end{cases}$$
$$z_{-2}(x) = \begin{cases} \dfrac{1}{7}x - \dfrac{3}{7}, & 3 \leq x \leq 10, \\ -x+11, & 10 \leq x \leq 11, \end{cases} \quad z_{-3}(x) = \begin{cases} \dfrac{1}{3}x - \dfrac{2}{3}, & 2 \leq x \leq 5, \\ -\dfrac{1}{7}x + \dfrac{12}{7}, & 5 \leq x \leq 12. \end{cases} \tag{31}$$

From (30) and (31), one has

$$[H]^\alpha = [4\alpha + 2, 13 - 7\alpha],\ \ \alpha \in (0,1],$$

and

$$[z_0]^\alpha = [L_{0,\alpha}, R_{0,\alpha}] = [1+\alpha, 3-\alpha],$$
$$[z_{-1}]^\alpha = [L_{-1,\alpha}, R_{-1,\alpha}] = [2+5\alpha, 13-6\alpha],$$
$$[z_{-2}]^\alpha = [L_{-2,\alpha}, R_{-2,\alpha}] = [3+7\alpha, 11-\alpha],$$
$$[z_{-3}]^\alpha = [L_{-3,\alpha}, R_{-3,\alpha}] = [2+3\alpha, 12-7\alpha],\ \alpha \in (0,1].$$

From (29)-(31), we can obtain the following difference equations with parameter α ,

$$L_{n+1,\alpha} = \max\left\{\frac{2+4\alpha}{R_{n-1,\alpha}}, \frac{2+4\alpha}{R_{n-2,\alpha}}, L_{n-3,\alpha}\right\},$$
$$R_{n+1,\alpha} = \max\left\{\frac{13-7\alpha}{L_{n-1,\alpha}}, \frac{13-7\alpha}{L_{n-2,\alpha}}, R_{n-3,\alpha}\right\}, \tag{32}$$
$$\alpha \in (0,1],\quad n = 0,1,\cdots.$$

Through simple calculation, it can be proved that the systems (29)-(31) satisfy the conditions of **Theorem 3**. It follows from **Theorem 3** and **Corollary 2** that every positive solution of the system (29)-(31) is eventually periodic with period four, bounded and persists. (see **Figs**. 1-5).

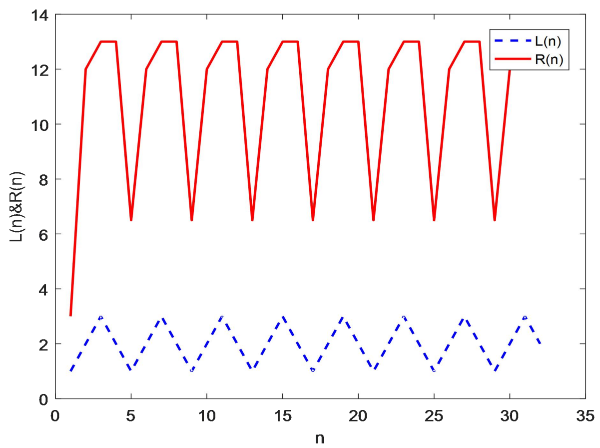

Fig.1. The solutions of difference equations (32) with $\alpha = 0$.

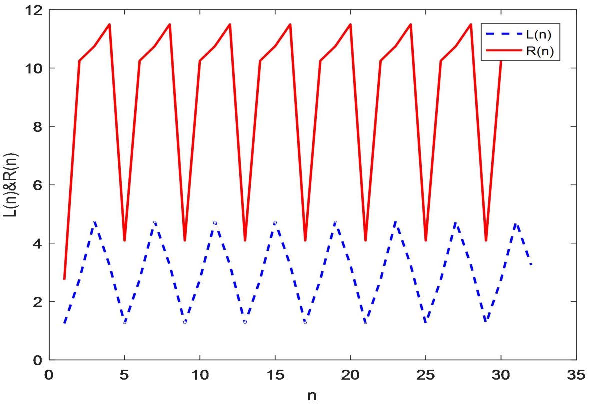

Fig.2. The solutions of difference equations (32) with $\alpha = 0.25$.

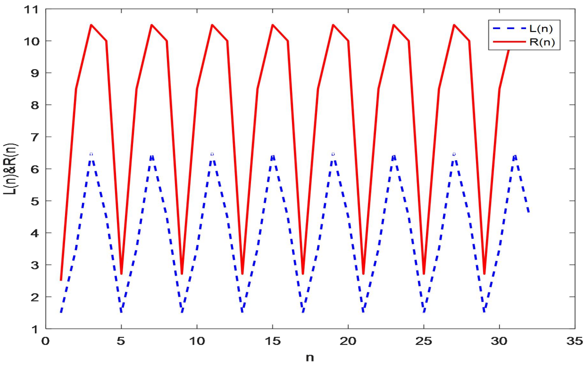

Fig.3. The solutions of difference equations (32) with $\alpha = 0.5$.

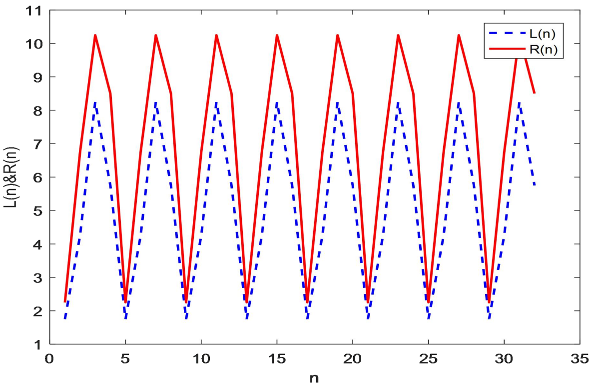

Fig.4. The solutions of difference equations (32) with $\alpha = 0.75$.

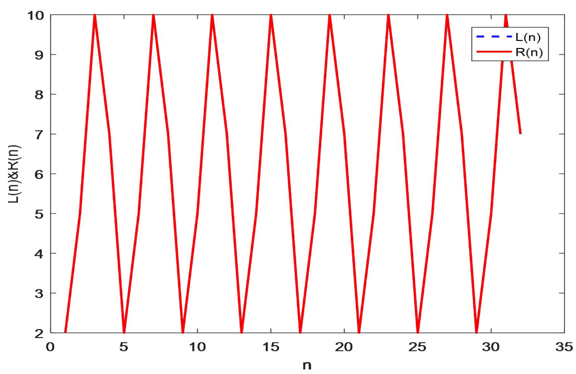

Fig.5. The solutions of difference equations (32) with $\alpha = 1$.

REFERENCES

[1] M. Kot, Elements of mathematical ecology, New York: Cambridge University Press, 2001.

[2] R. Beverton, S. Holt, On the dynamics of exploited fish populations, fishery investigations series II, volume XIX, London: Ministry of Agriculture, Fisheries and Food, UK, 1957.

[3] M. D. L. Sen, "The generalized Beverton-Holt equation and the control of populations," Applied Mathematical Modelling, vol.32, no.11, pp. 2312-2318, 2008.

[4] M. D. L Sen, S Alonso-Quesada, "Control issues for the Beverton-Holt equation in ecology by locally monitoring the environment carrying capacity: Nonadaptive and adaptive cases," Applied Mathematics and Computation, vol.215, pp.2616-2633, 2009.

[5] M. Bohner, S. Streipert, "Optimal harvesting policy for the Beverton-Holt model," Mathematical Biosciences and Engineering, vol.13, no.4, pp. 673-695, 2016.

[6] W. B. Yang,"Existence of Positive Steady-state Solution of a Predator-prey Dynamics with Dinosaur Functional Response and Heterogeneous Environment," Engineering Letters, vol.31, no.1, pp.77-81, 2023.

[7] L. Wang, W. Dai, J. Liu, X. Cui, B. Wang, "Research on the Prediction Model of Greenhouse Temperature Based on Fuzzy Neural Network Optimized by Genetic Algorithm," IAENG International Journal of Computer Science, vol.49, no.3, pp.828-832, 2022.

[8] K. Hussain, O. Adeyeye, N. Ahmad, R. Bibi, "Third-Fourth Derivative Three-Step Block Method for Direct Solution of Second-Order Fuzzy Ordinary Differential Equations," IAENG International Journal of Computer Science, vol.49, no.3, pp.856-863, 2022.

[9] E. P. Popov, Automatic Regulation and Control, Nauka, Moscow, Russia, 1966.

[10] Y. Gao, G. Zhang, "Oscillation of nonlinear first order neutral difference equations," Applied Mathematics E-Notes, vol.1, pp.5-10, 2001.

[11] X. F. Yang, X. F. Liao, C. D. Li, "On a difference equation with maximum," Applied Mathematics and Computation, vol.181, no.1, pp.1-5, 2006.

[12] F. K. Sun, "On the asymptotic behavior of a difference equation with maximum," Discrete Dynamics in Nature & Society, vol.2008, Article ID: 243291, 2008.

[13] A. Gelisken, C. Cinar, "On the global attractivity of a max-type difference equation," Discrete Dynamics in Nature and Society, vol.2009, Article ID: 812674, 2009.

[14] A. Gelisken, C. Cinar C, I. Yalcinkaya, "On a max-type difference equation," Advances in Difference Equations. Vol.2010, Article ID: 584890, 2010.

[15] H. D. Voulov, "On the periodic character of some difference equations," Journal of Difference Equations and Applications, vol.8, no.9, pp.799-810, 2002.

[16] A. Gelisken, C. Cengiz, A. S. Kurbanli, "On the asymptotic behavior and periodic nature of a difference equation with maximum," Computers & Mathematics with Applications, vol.59, no.2, pp.898-902, 2010.

[17] A. Gelişken, "Behavior of solutions of a system of max-type difference equations," Computational and Applied Mathematics Journal, vol.2, no.1, pp.34-37, 2016.

[18] C. Y. Wang, H. Liu, R. Li, X. H. Hu, Y. B. Shao, "Boundedness character of a symmetric system of max-type difference equations," IAENG International Journal of Applied Mathematics, vol.46, no.4, pp.505-511, 2016.

[19] G. W. Su, C. H. Han, T. X. Sun, L. Li, "On the solutions of a max-type system of difference equations of higher order," Advances in Difference Equations, vol.2020, Article ID: 213, 2020.

[20] T. X. Sun, G. W. Su, C. H. Han, L. Li, W. Z. Quan, "Global behavior of a max-type system of difference equations with four variables," Journal of Applied Mathematics and Computing, vo.68, pp.391-402, 2022.

[21] L.A. Zadeh, "Fuzzy set, Information and Control," vol.8, pp.338-353, 1965.

[22] D. P. Mishev, W. T. Patula, H. D. Voulov, "A reciprocal difference equation with maximum," Computers & Mathematics with Applications, vol.43, no.8-9, pp.1021-1026, 2002.

[23] G. Stefanidou, G. Papaschinopoulos, "The periodic nature of the positive solutions of a nonlinear fuzzy max-difference equation," Information Science, vol.176, no.24, pp. 3694-3710, 2006.

[24] Q. Zhang, L. Yang, D. Liao, "Behavior of solutions to a fuzzy nonlinear difference equation," Iranian Journal Fuzzy System, vol.9, no.2, pp.1-12, 2012.

[25] Q. He, C. Tao, T. Sun, X. Liu, D. Su, "Periodicity of the positive solutions of a fuzzy max-difference equation," Abstract and Applied Analysis, vol.2014, Article ID: 760247, 2014.

[26] C. Wang, X. Su, P. Liu, X. Hu, R. Li, "On the dynamics of a five-order fuzzy difference equation," Journal of Nonlinear Science and Applications, vol.10, pp.3303-3319, 2017.

[27] Q. H. Zhang, B. R. Pan, "Qualitative Analysis of k-order Rational Fuzzy Difference Equation," IAENG International Journal of Applied Mathematics, vol.53, no.3, pp.839-845, 2023.

[28] T. Sun, H. Xi, G. Su, B. Qin, "Dynamics of the fuzzy difference equation zn = max{1/zn−m,αn/zn−r}," Journal of Nonlinear Science and Applications, vol.11, pp.477-485, 2018.

[29] C. Han, G. Su, L. Li, G. Xia, G. Sun, “Eventual periodicity of the fuzzy max-difference equation,” Advances in Difference Equations, vol. 2020, Article ID: 673, 2022.
[30] E. Y. Deeba, A. De Korvin, E. L. Koh, “A fuzzy difference equation with an application,” Journal of Difference Equations and Applications, vol.2, pp.365-374, 1996.
[31] G. Stefanidou, G. Papaschinopoulos, “Behavior of the positive solutions of fuzzy max-difference equations,” Advances in Difference Equations, vol.2005, Article ID: 947038, 2005.

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