Sesquilinear Functional and Jordan Derivation in Involutive Banach Algebras with Application to Tensor Product and Hyers-Ulam Stability

Goutam Das and Nilakshi Goswami

***Abstract*—In this paper, we extend Vukman's generalization of Kurepa's theorem on sesquilinear functional to the projective tensor product of two hermitian Banach *-algebras via sesquilinear functional. For a complex unital *-algebra $\mathbb{A}$ and two additive self mappings θ and ϕ as antihomomorphism and homomorphism respectively on $\mathbb{A}$, we define a generalized class of quadratic functional, viz., (θ,ϕ)-$\mathbb{A}$-quadratic functional. Using this, we give a characterization of sesquilinear functional on the projective tensor product in terms of Jordan (θ,ϕ)-derivation. The Hyers-Ulam stability of Jordan (θ,ϕ)-derivation is also discussed.**

***Index Terms*—Projective tensor product, quadratic functional, sesquilinear functional, Jordan derivation.**

I. Introduction

THE study of sesquilinear functionals has attained significant interest from numerous researchers due to its broad applicability across various domains. For a complex vector space X and a complex *-algebra $\mathbb{A}$ with X as a left $\mathbb{A}$-module, it is well known that each $\mathbb{A}$-sesquilinear functional

$$B : X \times X \to \mathbb{A}$$

gives rise to an $\mathbb{A}$-quadratic functional

$$Q : X \to \mathbb{A}$$

by the relation $Q(x) = B(x,x)$ for all $x \in X$. Kurepa [15] provided a positive response to the converse of this statement when examining the case of $\mathbb{A}$ being the field of complex numbers. In [28], Vroba obtained a simpler proof of Kurepa's result. In 1984, Vukman in [26] achieved a broader formulation of Kupera's theorem by replacing the complex field $\mathbb{C}$ with commutative hermitian Banach *-algebra. The generalization for noncommutative case was also done by Vukman in another paper [27] using a simpler approach.

Building upon the influence of these studies, in this paper we establish some results on sesquilinear functionals in Banach *-algebras. The novelty of our work lies in the fact that some existing works have been extended in the setting of projective tensor product of two hermitian Banach *-algebras. Furthermore, we have generalized the investigation conducted by Semrl [24] concerning Jordan *-derivation on the Banach *-algebra $\mathbb{A}$ to Jordan (θ,ϕ)-derivation. Jordan derivation was introduced by Herstein [13] in 1957, and he proved some results of Jordan derivation on prime rings. Subsequently, Semrl [24] demonstrated a series of findings pertaining to Jordan *-derivations, as well as exploring sesquilinear and quadratic functionals. In [1], Ashraf et al. discussed about Lie ideals and generalized Jordan (θ,ϕ)-derivations in prime rings. Different researchers (refer to [2], [8], [16], [19]) have established several interesting results considering different types of Jordan derivations which is the motivation to work in this topic.

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II. Preliminaries

In this section, we present some basic definitions necessary for the main results of the paper.

Definition 2.1 [5] In an algebra $\mathbb{A}$, for $x,x^* \in \mathbb{A}$, an involution is a self mapping on $\mathbb{A}$ with $x \to x^*$ such that

(i) $(x + y)^* = x^* + y^*$,
(ii) $(x^*)^* = x$,
(iii) $(xy)^* = y^*x^*$,
(iv) $(\alpha x^*) = \bar{\alpha}x^*$

for all $x,y \in \mathbb{A}$ and for all scalar α, where x^* is called the adjoint of x.

An algebra $\mathbb{A}$ with an involution is called a *-algebra. A Banach *-algebra is a Banach algebra $\mathbb{A}$ with an involution '*' defined on it. Let $\mathbb{A}$ be the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices and let $a = (a_{ij}) \in \mathbb{A}$. Then $\mathbb{A}$ is a Banach *-algebra, where $a^* = (\overline{a_{ji}})$.

If each hermitian element in a Banach *-algebra $\mathbb{A}$ has a real spectrum, then $\mathbb{A}$ is called a hermitian algebra. $\mathbb{B}$*-algebras are the most important hermitian Banach *-algebras. In a Banach *-algebra $\mathbb{A}$, for any hermitian element $h \in \mathbb{A}$, $h > 0$ ($h \geq 0$), if the spectrum of h is positive (nonnegative).

Definition 2.2 [24] Let X be a complex vector space and $\mathbb{A}$ be a complex *-algebra such that X is a left $\mathbb{A}$-module.
A mapping

$$B : X \times X \to \mathbb{A}$$

is an $\mathbb{A}$-sesquilinear functional if
(i) $B(a_1x_1 + a_2x_2,y) = a_1B(x_1,y) + a_2B(x_2,y)$ for all $x_1,x_2,y \in X; a_1,a_2 \in \mathbb{A}$,
(ii) $B(x,a_1y_1 + a_2y_2) = B(x,y_1)a_1^* + B(x,y_2)a_2^*$ for all $x,y_1,y_2 \in X; a_1,a_2 \in \mathbb{A}$.

For example, let H be a Hilbert space and $\beta(H)$ be the algebra of all bounded linear operators on H. Let the involution on $\beta(H)$ be the adjoint operation. The mapping

$$\phi : H \times H \to \beta(H)$$

defined by

$$(\phi(x,y))(z) =< z,y > x,$$

where $x,y,z \in H$ is a $\beta(H)$-sesquilinear functional.

A mapping $Q : X \to \mathbb{A}$ is said to be an $\mathbb{A}$-quadratic functional if the following conditions are satisfied:
(i) $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ for all $x,y \in X$,
(ii) $Q(ax) = aQ(x)a^*$ for all $x \in X$ and $a \in \mathbb{A}$.

In [27], Vukman proved the following result regarding sesquilinear functional in hermitian Banach *-algebras.

Lemma 2.3 [27] For a vector space X and a hermitian Banach *-algebra $\mathbb{A}$, let X be a unitary left $\mathbb{A}$-module. Suppose there exists an $\mathbb{A}$-quadratic functional $Q : X \to \mathbb{A}$. Then the mapping $B : X \times X \to \mathbb{A}$ defined by

$$B(x,y) = \frac{1}{4}(Q(x+y)-Q(x-y))+\frac{i}{4}(Q(x+iy)-Q(x-iy))$$

is an $\mathbb{A}$-sesquilinear functional. Moreover, for all $x \in X$ the relation $Q(x) = B(x,x)$ holds.

In the theory of Banach spaces, the tensor product serves as a tool to transform multilinear phenomena into linear ones, simplifying their analysis. In 1953, Grothendiek [10] developed the modern tensor product theory of Banach spaces. Various concepts linked to the tensor product have been explored in [7], [21], [23].

Definition 2.4 [5] Let $\mathbb{A}$ and $\mathbb{B}$ be two normed spaces over the field $\mathbb{F}$ with dual spaces $\mathbb{A}^*$ and $\mathbb{B}^*$. For $a \in \mathbb{A}$ and $b \in \mathbb{B}$, let $a \otimes b$ be the element of $BL(\mathbb{A}^*,\mathbb{B}^*;\mathbb{F})$ defined by

$$a \otimes b(f,g) = f(a)g(b),\ (f \in \mathbb{A}^*,g \in \mathbb{B}^*).$$

The algebraic tensor product of $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A}\otimes\mathbb{B}$ is defined as the linear span of $\{a \otimes b : a \in \mathbb{A},b \in \mathbb{B}\}$ in $BL(\mathbb{A}^*,\mathbb{B}^*;\mathbb{F})$, where $BL(\mathbb{A}^*,\mathbb{B}^*;\mathbb{F})$ is the set of all bounded bilinear mappings from $\mathbb{A}^* \times \mathbb{B}^*$ to $\mathbb{F}$.

For example, if $\mathbb{A}$ is a Banach *-algebra, then $M_n(\mathbb{C})\otimes\mathbb{A}$ is isomorphic to $M_n(\mathbb{A})$, which is the set of all $n \times n$ matrices over $\mathbb{A}$ (refer to [5]).

Definition 2.5 [6] For any two normed spaces $\mathbb{A}$ and $\mathbb{B}$, the projective tensor norm γ on $\mathbb{A} \otimes \mathbb{B}$ is defined by

$$\gamma(u) = \inf\{\sum_{i=1}^{n} ||a_i||.||b_i|| : u = \sum_{i=1}^{n} a_i \otimes b_i\},$$

where the infimum is taken over all finite representations of u. The completion of $\mathbb{A} \otimes \mathbb{B}$ with respect to γ is called the projective tensor product of $\mathbb{A}$ and $\mathbb{B}$ and it is denoted by $\mathbb{A} \otimes_\gamma \mathbb{B}$.

For example, for the sequence space l^1 over $\mathbb{R}$, there exists an isometric linear isomorphism of $l^1 \otimes_\gamma \mathbb{R}$ to $l^1(\mathbb{R})$.

Lemma 2.6 [5] Let $\mathbb{A}$ and $\mathbb{B}$ be two normed algebras over $\mathbb{F}$. There exists a unique product on $\mathbb{A}\otimes\mathbb{B}$ with respect to which $\mathbb{A} \otimes \mathbb{B}$ is an algebra and

$$(a \otimes b)(c \otimes d) = ac \otimes bd,\ (a,c \in \mathbb{A}\ and\ b,d \in \mathbb{B}).$$

If $\mathbb{A}$ and $\mathbb{B}$ are two hermitian Banach *-algebras, then $\mathbb{A} \otimes_\gamma \mathbb{B}$ is also a hermitian Banach *-algebra.

Definition 2.7 [24] For a *-algebra $\mathbb{A}$, a mapping $D : \mathbb{A} \to \mathbb{A}$ is a Jordan *-derivation if for all $u,v \in \mathbb{A}$,
(i) $D(u + v) = D(u) + D(v)$,
(ii) $D(u^2) = uD(u) + D(u)u^*$.

Definition 2.8 [1] Let $\mathbb{A}$ be a complex unital Banach *-algebra with unit element e. For two endomorphisms θ and ϕ on $\mathbb{A}$, a mapping $\Delta : \mathbb{A} \to \mathbb{A}$ is said to be a Jordan (θ,ϕ)-derivation if for all $u,v \in \mathbb{A}$,
(i) $\Delta(u + v) = \Delta(u) + \Delta(v)$,
(ii) $\Delta(u^2) = \Delta(u)\theta(u) + \phi(u)\Delta(u)$.

Example 2.9 For the C^*-algebra $\mathbb{A}$=$\{\begin{bmatrix} u & v \\ 0 & u \end{bmatrix}: u,v \in \mathbb{R}\}$ with usual matrix operations and the norm, let $\Delta : \mathbb{A} \to \mathbb{A}$ be defined by $\Delta(\begin{bmatrix} u & v \\ 0 & u \end{bmatrix}) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$. Let $\theta : \mathbb{A} \to \mathbb{A}$ and $\phi : \mathbb{A} \to \mathbb{A}$ be such that $\theta(\begin{bmatrix} u & v \\ 0 & u \end{bmatrix}) = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} = \phi(\begin{bmatrix} u & v \\ 0 & u \end{bmatrix})$. Then Δ is a Jordan (θ,ϕ)-derivation.

Definition 2.10 [12] A Banach *-algebra $\mathbb{A}$ is called a zero product determined Banach *-algebra if for every vector space X and every bilinear mapping

$$\Psi : \mathbb{A} \times \mathbb{A} \to X,$$

the following condition holds:
if $\Psi(u,v) = 0$ whenever $uv = 0$, then there exists a linear mapping

$$T : \mathbb{A}^2 \to X$$

such that $\Psi(u,v) = T(uv)$ for all $u,v \in \mathbb{A}$. [Here $\mathbb{A}^2$ denotes the complex linear span of all elements of the form xy where $x,y \in \mathbb{A}$].

If $\mathbb{A}$ has unit element e, and $\mathbb{A}$ is zero product determined Banach *-algebra then $\Psi(u,v) = \Psi(uv,e)$ for all $u,v \in \mathbb{A}$ and also $\Psi(u,e) = \Psi(e,u)$ for all $u \in \mathbb{A}$.

III. Main Results

We introduce the subsequent expansion of Vukman's findings to projective tensor product of two hermitian Banach *-algebras, $\mathbb{A}$ and $\mathbb{B}$. Starting with two quadratic functionals, Q_1 and Q_2 defined on the vector spaces X and Y, where X is a left $\mathbb{A}$-module, and Y is a left $\mathbb{B}$-module, we formulate an $\mathbb{A} \otimes_\gamma \mathbb{B}$-sesquilinear functional on $X \otimes Y$. Significantly, our work also finds a relationship between the norms of elements in $X \otimes Y$ via quadratic functionals and sesquilinear functionals in case of C^*-algebras.

Theorem 3.1 Let X, Y be two vector spaces and $\mathbb{A}$, $\mathbb{B}$ be two hermitian Banach *-algebras with unit elements e_1 and e_2 respectively. Let X be a unitary left $\mathbb{A}$- module and Y be a unitary left $\mathbb{B}$-module. Let $Q_1 : X \to \mathbb{A}$ be an $\mathbb{A}$-quadratic functional on X and $Q_2 : Y \to \mathbb{B}$ be a $\mathbb{B}$-quadratic functional on Y. Then corresponding to Q_1 and Q_2, there exists an $\mathbb{A} \otimes_\gamma \mathbb{B}$-sesquilinear functional

$$B : (X \otimes Y) \times (X \otimes Y) \to \mathbb{A} \otimes_\gamma \mathbb{B}$$

such that

$$B(x \otimes y,x \otimes y) = Q_1(x) \otimes Q_2(y)$$

for each $x \in X$ and $y \in Y$. Moreover, if X and Y are C^*-algebras and Q_1 and Q_2 are bounded, then for $u = x \otimes y \in X \otimes Y$,

$$||B(uu^*,uu^*)|| \leq 4||Q_1||.||Q_2||.||u||^2.$$

Proof: For the unitary left $\mathbb{A}$-module X, from the given $\mathbb{A}$-quadratic form $Q_1 : X \to \mathbb{A}$, by Lemma 2.3 we construct an $\mathbb{A}$-sesquilinear functional

$$B_1 : X \times X \to \mathbb{A}.$$

For fixed vectors $u_1,v_1 \in X$, we consider $f_1 : \mathbb{A} \to \mathbb{A}$ and $g_1 : \mathbb{A} \to \mathbb{A}$ defined by

$$f_1(w_1) = B_1(w_1u_1,v_1) \tag{1}$$

and

$$g_1(w_1) = B_1(u_1,w_1^*v_1), w_1 \in \mathbb{A}. \tag{2}$$

Again, for the unitary left $\mathbb{B}$-module Y, in a similar way we can construct the $\mathbb{B}$-sesquilinear functional

$$B_2 : Y \times Y \to \mathbb{B}.$$

For fixed vectors $u_2,v_2 \in Y$, we define $f_2 : \mathbb{B} \to \mathbb{B}$ and $g_2 : \mathbb{B} \to \mathbb{B}$ by the relation

$$f_2(w_2) = B_2(w_2u_2,v_2) \tag{3}$$

and

$$g_2(w_2) = B_2(u_2,w_2^*v_2),\ \ for\ w_2 \in \mathbb{B}. \tag{4}$$

Let $B : (X \otimes Y) \times (X \otimes Y) \to \mathbb{A} \otimes_\gamma \mathbb{B}$ be defined by

$$B(\sum_{i=1}^n u_{1_i} \otimes u_{2_i}, \sum_{j=1}^m v_{1_j} \otimes v_{2_j}) = \sum_{i=1}^n \sum_{j=1}^m B_1(u_{1_i},v_{1_j}) \otimes B_2(u_{2_i},v_{2_j}),$$

where $\sum_{i=1}^n u_{1_i} \otimes u_{2_i}, \sum_{j=1}^m v_{1_j} \otimes v_{2_j} \in X \otimes Y$.
Now,

$$\begin{aligned} B(iu_1 \otimes u_2,v_1 \otimes v_2) &= B_1(iu_1,v_1) \otimes B_2(u_2,v_2) \\ &= iB_1(u_1,v_1) \otimes B_2(u_2,v_2) \\ &= iB(u_1 \otimes u_2,v_1 \otimes v_2). \end{aligned} \tag{5}$$

Similarly,

$$B(u_1 \otimes u_2,iv_1 \otimes v_2) = -iB(u_1 \otimes u_2,v_1 \otimes v_2). \tag{6}$$

On the projective tensor product $\mathbb{A} \otimes_\gamma \mathbb{B}$, we consider the function $f : \mathbb{A} \otimes_\gamma \mathbb{B} \to \mathbb{A} \otimes_\gamma \mathbb{B}$ such that

$$f(\sum_k w_{1_k} \otimes w_{2_k}) = \frac{1}{2}\sum_k \{f_1(w_{1_k}) \otimes f_2(w_{2_k}) + g_1(w_{1_k}^*) \otimes g_2(w_{2_k}^*)\}, \tag{7}$$

where $\sum_i w_{1_k} \otimes w_{2_k} \in \mathbb{A} \otimes_\gamma \mathbb{B}$.
Now using (1), (2), (3) and (4), from (7) we have,

$$\begin{aligned} &f(\sum_k w_{1_k} \otimes w_{2_k}) \\ &= \frac{1}{2}\sum_k \{B_1(w_{1_k}u_1,v_1) \otimes B_2(w_{2_k}u_2,v_2) \\ &+ B_1(u_1,w_{1_k}v_1) \otimes B_2(u_2,w_{2_k}v_2)\} \\ &= \frac{1}{2}\sum_k \{B((w_{1_k} \otimes w_{2_k})(u_1 \otimes u_2),v_1 \otimes v_2) \\ &+ B(u_1 \otimes u_2,(w_{1_k} \otimes w_{2_k})(v_1 \otimes v_2))\} \\ &= \frac{1}{2}\{B((\sum_k w_{1_k} \otimes w_{2_k})(u_1 \otimes u_2),v_1 \otimes v_2) \\ &+ B(u_1 \otimes u_2,(\sum_k w_{1_k} \otimes w_{2_k})(v_1 \otimes v_2))\}. \end{aligned} \tag{8}$$

Since B_1 is $\mathbb{A}$-sesquilinear functional, so from (1) and (2) we obtain,

$$f_1(e_1) = B_1(u_1,v_1) = g_1(e_1) \tag{9}$$

and

$$\begin{aligned} f_1(w_1) &= B_1(w_1u_1,v_1) \\ &= w_1B_1(u_1,v_1) = w_1f_1(e_1)\ (using\ (9)). \end{aligned} \tag{10}$$

Similarly, we can show that

$$f_2(e_2) = B_2(u_2,v_2) = g_2(e_2),$$

$$f_2(w_2) = w_2f_2(e_2), \tag{11}$$

$$g_1(w_1^*) = g_1(e_1)w_1^* \tag{12}$$

and

$$g_2(w_2^*) = g_2(e_2)w_2^*. \tag{13}$$

Now using (10), (11), (12) and (13), from (7), we get,

$$\begin{aligned} &f(\sum_k w_{1_k} \otimes w_{2_k}) \\ &= \frac{1}{2}\sum_i \{f_1(w_{1_k}) \otimes f_2(w_{2_k}) + g_1(w_{1_k}^*) \otimes g_2(w_{2_k}^*)\} \\ &= \frac{1}{2}\sum_k \{w_{1_k}f_1(e_1) \otimes w_{2_k}f_2(e_2) + g_1(e_1)w_{1_k}^* \otimes g_2(e_2)w_{2_k}^*\} \\ &= \frac{1}{2}\sum_k \{(w_{1_k} \otimes w_{2_k})(B_1(u_1,v_1) \otimes B_2(u_2,v_2)) \\ &+ (B_1(u_1,v_2) \otimes B_2(u_2,v_2))(w_{1_k}^* \otimes w_{2_k}^*)\} \\ &= \frac{1}{2}\sum_k \{(w_{1_k} \otimes w_{2_k})B(u_1 \otimes u_2,v_1 \otimes v_2) \\ &+ B(u_1 \otimes u_2,v_1 \otimes v_2)(w_{1_k}^* \otimes w_{2_k}^*)\} \\ &= \frac{1}{2}\{\sum_k (w_{1_k} \otimes w_{2_k})B(u_1 \otimes u_2,v_1 \otimes v_2) \\ &+ B(u_1 \otimes u_2,v_1 \otimes v_2)\sum_k (w_{1_k}^* \otimes w_{2_k}^*)\}. \end{aligned} \tag{14}$$

Now comparing (8) and (14) we obtain,

$$\begin{aligned} &B((\sum_k w_{1_k} \otimes w_{2_k})(u_1 \otimes u_2),v_1 \otimes v_2) \\ &+ B(u_1 \otimes u_2,(\sum_k w_{1_k} \otimes w_{2_k})(v_1 \otimes v_2)) \\ &= (\sum_k w_{1_k} \otimes w_{2_k})B(u_1 \otimes u_2,v_1 \otimes v_2) \\ &+ B(u_1 \otimes u_2,v_1 \otimes v_2)(\sum_k w_{1_k}^* \otimes w_{2_k}^*). \end{aligned} \tag{15}$$

Replacing $\sum_k w_{1_k} \otimes w_{2_k}$ by $\sum_k iw_{1_k} \otimes w_{2_k}$ and using (5) and (6), we get,

$$B((\sum_k w_{1_k} \otimes w_{2_k})(u_1 \otimes u_2),v_1 \otimes v_2)$$

$$
\begin{aligned}
&- B(u_1 \otimes u_2,(\sum_k w_{1_k} \otimes w_{2_k})(v_1 \otimes v_2)) \\
&= (\sum_k w_{1_k} \otimes w_{2_k})B(u_1 \otimes u_2,v_1 \otimes v_2) \\
&- B(u_1 \otimes u_2,v_1 \otimes v_2)(\sum_k w^*_{1_k} \otimes w^*_{2_k}). \qquad (16)
\end{aligned}
$$

Thus for $u = \sum_{i=1}^n u_{1_i} \otimes u_{2_i}, v = \sum_{j=1}^m v_{1_j} \otimes v_{2_j} \in X \otimes Y$ and $w = \sum_k w_{1_k} \otimes w_{2_k} \in \mathbb{A} \otimes_\gamma \mathbb{B}$, comparing (15) and (16) we get,

$$B(wu,v) = wB(u,v)$$

and

$$B(u,wv) = B(u,v)w^*.$$

Thus B is an $\mathbb{A} \otimes_\gamma \mathbb{B}$-sesquilinear functional.
For $x \in X$, $y \in Y$,

$$B(x \otimes y,x \otimes y) = B_1(x,x) \otimes B_2(y,y) = Q_1(x) \otimes Q_2(y).$$

Now, we consider X and Y as C^*-algebras and let Q_1 and Q_2 be bounded. Then for $u_1,v_1 \in X$,

$$
\begin{aligned}
||B_1(u_1,v_1)|| &= ||\frac{1}{4}(Q_1(u_1+v_1) - Q_1(u_1-v_1)) \\
&+ \frac{i}{4}(Q_1(u_1+iv_1) - Q_1(u_1-iv_1))|| \\
&\leq \frac{1}{4}(||Q_1||.||u_1+v_1|| + ||Q_1||.||u_1-v_1||) \\
&+ \frac{1}{4}(||Q_1||.||u_1+iv_1|| + ||Q_1||.||u_1-iv_1||) \\
&\leq ||Q_1||(||u_1|| + ||v_1||).
\end{aligned}
$$

Similarly, $||B_2(u_2,v_2)|| \leq ||Q_2||(||u_2|| + ||v_2||)$ for all $u_2,v_2 \in Y$.
Now, for $u = x \otimes y \in X \otimes Y$,

$$
\begin{aligned}
&||B(uu^*,uu^*)|| \\
&= ||B_1(xx^*,xx^*)||.||B_2(yy^*,yy^*)|| \\
&\leq ||Q_1||(||xx^*|| + ||xx^*||).||Q_2||(||yy^*|| + ||yy^*||) \\
&= 4||Q_1||.||Q_2||.||xx^*||.||yy^*|| \\
&= 4||Q_1||.||Q_2||.||x||^2.||y||^2 \\
&= 4||Q_1||.||Q_2||.||x \otimes y||^2 \\
&= 4||Q_1||.||Q_2||.||u||^2.
\end{aligned}
$$

■

Example 3.2: Let $X = \mathbb{A} = l^1$ and $Y = \mathbb{B} = \mathbb{R}$. Let the mappings $Q_1 : l^1 \to l^1$ be defined by

$$Q_1(\{x_1,x_2,x_3,.....\}) = \{x_1^2,x_2^2,0,0,....\} \ for \ \{x_n\} \in l^1$$

and $Q_2 : \mathbb{R} \to \mathbb{R}$ by $Q_2(u) = u^2$, for $u \in \mathbb{R}$. Clearly, Q_1 and Q_2 are $\mathbb{A}$-quadratic functionals. Now, by Lemma 2.3, we can construct two $\mathbb{A}$-sesquilinear functionals $B_1 : l^1 \times l^1 \to l^1$ and $B_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$B_1(\{x_1,x_2,x_3,...\},\{y_1,y_2,y_3,...\}) = \{x_1y_1,x_2y_2,0,0,...\}$$

and $B_2(u,v) = uv$ where $\{x_n\},\{y_n\} \in l^1$ and $u,v \in \mathbb{R}$. Since, $l^1 \otimes_\gamma \mathbb{R} \cong l^1(\mathbb{R})$ so, by Theorem 3.1, there exists

$$B : (l^1 \otimes \mathbb{R}) \times (l^1 \otimes \mathbb{R}) \to l^1(\mathbb{R})$$

such that

$$
\begin{aligned}
&B(\sum_{i=1}^n u_i \otimes v_i, \sum_{j=1}^m r_j \otimes s_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m \{p_{i_1}q_{j_1}v_is_j,p_{i_2}q_{j_2}v_is_j,0,0,....\}
\end{aligned}
$$

where $u_i = \{p_{i_k}\}_k, r_j = \{q_{j_k}\}_k \in l^1$ and $v_i, s_j \in \mathbb{R}$.
Now,

$$
\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^m B_1(u_i,r_j) \otimes B_2(v_i,s_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m \{p_{i_1}q_{j_1},p_{i_2}q_{j_2},0,0,...\} \otimes v_is_j \\
&= \sum_{i=1}^n \sum_{j=1}^m \{p_{i_1}q_{j_1}v_is_j,p_{i_2}q_{j_2}v_is_j,0,0,....\} \\
&= B(\sum_{i=1}^n u_i \otimes v_i, \sum_{j=1}^m r_j \otimes s_j),
\end{aligned}
$$

which exhibits the content of the Theorem 3.1.

The following result deals with zero product determined Banach *-algebras.

Theorem 3.3 Let X, Y be two unital zero product determined Banach *-algebras with unit elements e_1', e_2' respectively and $\mathbb{A}$, $\mathbb{B}$ be two hermitian Banach *-algebras with unit elements e_1, e_2 respectively. Let X be a unitary left $\mathbb{A}$-module and Y be a unitary left $\mathbb{B}$-module. Let $Q_1 : X \to \mathbb{A}$ be a bounded $\mathbb{A}$-quadratic functional on X and $Q_2 : Y \to \mathbb{B}$ be a bounded $\mathbb{B}$-quadratic functional on Y satisfying $x_iy_i = 0$ implies $Q_i(x_i + y_i) = 0$ (for $i = 1,2$), $x_1,y_1 \in X$ and $x_2,y_2 \in Y$. Then there exists a bounded linear mapping

$$L : X \otimes Y \to \mathbb{A} \otimes_\gamma \mathbb{B}$$

such that

$$L(\sum_{i=1}^n x_i \otimes y_i) = B(\sum_{i=1}^n x_i \otimes y_i, e_1' \otimes e_2')$$

and $||L|| \leq ||B||$, where B is the $\mathbb{A} \otimes_\gamma \mathbb{B}$-sesquilinear functional as defined in Theorem 3.1.

Proof: Let B_1, B_2 be the sesquilinear functionals determined by Q_1 and Q_2 respectively. Let $x_1,y_1 \in X$ with $x_1y_1 = 0$. Now,

$$
\begin{aligned}
B_1(x_1,y_1) &= \frac{1}{4}(Q_1(x_1+y_1) - Q_1(x_1-y_1)) \\
&+ \frac{i}{4}(Q_1(x_1+iy_1) - Q_2(x_1-iy_1)) \\
&= 0,
\end{aligned}
$$

Thus, $x_1y_1 = 0$ implies $B_1(x_1,y_1) = 0$. So, there exists a linear mapping $L_1 : X^2 \to \mathbb{A}$ such that

$$B_1(u_1,v_1) = L_1(u_1v_1), u_1,v_1 \in X.$$

Similarly, we have a linear mapping $L_2 : Y^2 \to \mathbb{B}$ with

$$B_2(u_2,v_2) = L_2(u_2v_2), u_2,v_2 \in Y.$$

Now, we define $L : X \otimes Y \to \mathbb{A} \otimes_\gamma \mathbb{B}$ such that

$$L(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n L_1(x_ie_1') \otimes L_2(y_ie_2')$$

$$= \sum_{i=1}^{n} B_1(x_i,e_1') \otimes B_2(y_i,e_2')$$
$$= B(\sum_{i=1}^{n} x_i \otimes y_i,e_1' \otimes e_2').$$

Also it is easy to see that $||L|| \leq ||B||$. ■

Now we establish a relation between the $\mathbb{A} \otimes_\gamma \mathbb{B}$-sesquilinear functional and Jordan (θ,ϕ)-derivation. For this, we introduce a new class of $\mathbb{A}$-quadratic functionals with respect to the mappings θ and ϕ, denoted as (θ,ϕ)-$\mathbb{A}$-quadratic functional, and represent such quadratic functional using a given Jordan (θ,ϕ)-derivation.

Definition 3.4: (**(θ,ϕ)-$\mathbb{A}$-quadratic functional**) Let X be a vector space and $\mathbb{A}$ be a unital *-algebra with unit element e such that X is a left $\mathbb{A}$-module. For two additive self mappings θ and ϕ as antihomomorphism and homomorphism respectively on $\mathbb{A}$ and $\theta(e) = \phi(e) = e$, a mapping $Q : X \to \mathbb{A}$ is said to be a (θ,ϕ)-$\mathbb{A}$-quadratic functional if the following conditions hold:
(i) $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$,
(ii) $Q(ax) = \phi(a)Q(x)\theta(a)$ for all $x,y \in X$, $a \in \mathbb{A}$.

Example 3.5: Let $X = \mathbb{A} = M_n(\mathbb{R})$ with usual matrix operations. We define $Q : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by

$$Q(M) = MM^T$$

for all $M \in M_n(\mathbb{R})$, where M^T denotes the transpose of M. Let the self mappings θ and ϕ on $M_n(\mathbb{R})$ be defined by $\theta(M) = M^T$ and $\phi(M) = M$ for all $M \in M_n(\mathbb{R})$. Then Q is a (θ,ϕ)-$\mathbb{A}$-quadratic functional.

Example 3.6 Let $X = \mathbb{A} = l^1$. Let the mapping $Q : l^1 \to l^1$ be defined by

$$Q(\{x_1,x_2,x_3,....\}) = \{x_1x_2,x_1x_2,0,0,....\} \ for \ \{x_n\} \in l^1.$$

Let θ and ϕ be two self mappings on l^1 such that

$$\theta(\{x_1,x_2,x_3,...\}) = \{x_2,x_1,0,0,...\}$$
$$and \ \phi(\{x_1,x_2,x_3,...\}) = \{x_1,x_2,0,0,...\}.$$

Then Q is a (θ,ϕ)-$\mathbb{A}$-quadratic functional.

Remark 3.7 It becomes evident that when ϕ is the indentity mapping on a Banach *-algebra $\mathbb{A}$ and θ is an involution on $\mathbb{A}$, the class of all (θ,ϕ)-$\mathbb{A}$-quadratic functionals contains the class of $\mathbb{A}$-quadratic functionals.

Following the Theorem 2.1 of [24], some equivalent characterization for Jordan (θ,ϕ)-derivation can be obtained as follows:

Lemma 3.8 Let $\mathbb{A}$ be a unital Banach *-algebra with unit element e, and $\Delta : \mathbb{A} \to \mathbb{A}$ be an additive mapping. Let θ and ϕ be two additive self mappings on $\mathbb{A}$ with $\theta(uv) = \theta(v)\theta(u)$, $\phi(uv) = \phi(u)\phi(v)$ and $\theta(e) = \phi(e) = e$. Then the following conditions are equivalent:
(i) Δ is a Jordan (θ,ϕ)-derivation,
(ii) $\Delta(u) = -\phi(u)\Delta(u^{-1})\theta(u)$ for all invertible $u \in \mathbb{A}$,
(iii) $\Delta(uvu) = \phi(uv)\Delta(u) + \phi(u)\Delta(v)\theta(u) + \Delta(u)\theta(uv)$ for all $u,v \in \mathbb{A}$.

Proof: (ii) $\Longrightarrow$ (i):
For invertible $u \in \mathbb{A}$, $\Delta(u) = -\phi(u)\Delta(u^{-1})\theta(u)$. So $\Delta(e) = 0$.
Let u be invertible and $||u|| < 1$. Then $e+u$, $e-u$, $e-u^2$ are also invertible, and $(u - e)^{-1} - (u^2 - e)^{-1} = (u^2 - e)^{-1}u$. We have to show that $\Delta(u^2) = \phi(u)\Delta(u) + \Delta(u)\theta(u)$. Now,

$$\begin{aligned}
&\Delta(u) + \phi(u^{-1})\Delta(u)\theta(u^{-1})\\
&= \Delta(u) - \Delta(u^{-1}) = \Delta(u - u^{-1}) = \Delta(u^{-1}(u^2 - e))\\
&= -\phi(u^{-1}(u^2 - e))\Delta((u^2 - e)^{-1}u)\theta(u^{-1}(u^2 - e))\\
&= -\phi(u^{-1})\phi(u^2 - e)\Delta((u - e)^{-1})\theta(u^2 - e)\theta(u^{-1})\\
&+ \phi(u^{-1})\phi(u^2 - e)\Delta((u^2 - e)^{-1})\theta(u^2 - e)\theta(u^{-1})\\
&= -\phi(u^{-1})\phi(u + e)\phi(u - e)\Delta((u - e)^{-1})\\
&\theta(u - e)\theta(u + e)\theta(u^{-1}) - \phi(u^{-1})\Delta(u^2 - e)\theta(u^{-1})\\
&= \phi(u^{-1})\phi(u + e)\Delta(u - e)\theta(u + e)\theta(u^{-1})\\
&- \theta(u^{-1})\Delta(u^2)\theta(u^{-1})\\
&= \phi(e + u^{-1})\Delta(u)\theta(e + u^{-1}) - \phi(u^{-1})\Delta(u^2)\theta(u^{-1})\\
&= (\phi(e) + \phi(u^{-1}))\Delta(u)(\theta(e) + \theta(u^{-1}))\\
&- \phi(u^{-1})\Delta(u^2)\theta(u^{-1})\\
&= \Delta(u) + \Delta(u)\theta(u^{-1}) + \phi(u^{-1})\Delta(u) + \phi(u^{-1})\Delta(u)\theta(u^{-1})\\
&- \phi(u^{-1})\Delta(u^2)\theta(u^{-1}).
\end{aligned}$$

We finally get,

$$\phi(u^{-1})\Delta(u^2)\theta(u^{-1}) = \phi(u^{-1})\Delta(u) + \Delta(u)\theta(u^{-1}),$$
$$i.e., \Delta(u^2) = \phi(u)\Delta(u) + \Delta(u)\theta(u). \quad (17)$$

Thus, for $||u|| < 1$, Δ is a Jordan (θ,ϕ)-derivation.
Now, let $||u|| > 1$. Then $t^{-1}u$ is invertible for some positive integer t with $||t^{-1}u|| < 1$. Then by (17),

$$\Delta((t^{-1}u)^2) = \phi(t^{-1}u)\Delta(t^{-1}u) + \Delta(t^{-1}u)\theta(t^{-1}u).$$

Multiplying both sides of the above equation by t^2 and using the additivity of Δ we get,

$$\Delta(u^2) = \phi(u)\Delta(u) + \Delta(u)\theta(u).$$

Again let u be an arbitrary element. Then for some integer t, $||u|| < t$, i.e., $||t^{-1}u|| < 1$. So, $e - t^{-1}u$ is invertible and hence $u - te$ is also invertible. Then

$$\begin{aligned}
\Delta((u - te)^2) &= \phi(u - te)\Delta(u - te)\\
&+ \Delta(u - te)\theta(u - te),\\
i.e., \Delta(u^2) - 2t\Delta(u) &= \phi(u - te)\Delta(u) + \Delta(u)\theta(u - te)\\
&= (\phi(u) - \phi(te))\Delta(u)\\
&+ \Delta(u)(\theta(u) - \theta(te))\\
&= (\phi(u) - t)\Delta(u) + \Delta(u)(\theta(u) - t)\\
&= \phi(u)\Delta(u) + \Delta(u)\theta(u) - 2t\Delta(u),\\
i.e., \Delta(u^2) &= \phi(u)\Delta(u) + \Delta(u)\theta(u).
\end{aligned}$$

(i) $\Longrightarrow$ (iii):
Replacing u by $u + v$ in (17), for all $u,v \in \mathbb{A}$ we get,

$$\Delta(uv) + \Delta(vu) = \phi(v)\Delta(u) + \phi(u)\Delta(v) + \Delta(u)\theta(v) + \Delta(v)\theta(u) \quad (18)$$

Taking $z = \Delta(u(uv+vu)+(uv+vu)u)$ and using (18), we get,

$$\begin{aligned} z &= \Delta(u(uv+vu)+(uv+vu)u) \\ &= \phi(u)\Delta(uv+vu)+\phi(uv+vu)\Delta(u) \\ &+ \Delta(uv+vu)\theta(u)+\Delta(u)\theta(uv+vu) \\ &= \phi(u)\{\phi(u)\Delta(v)+\Delta(u)\theta(v)\}+\phi(u)\{\phi(v)\Delta(u) \\ &+ \Delta(v)\theta(u)\}+\phi(uv)\Delta(u)+\phi(vu)\Delta(u)+\{\phi(u)\Delta(v) \\ &+ \Delta(u)\theta(v)\}\theta(u)+\{\phi(v)\Delta(u)+\Delta(v)\theta(u)\}\theta(u) \\ &+ \Delta(u)\theta(uv)+\Delta(u)\theta(vu) \\ &= \phi(u^2)\Delta(v)+\phi(u)\Delta(u)\theta(v)+\phi(uv)\Delta(u) \\ &+ \phi(u)\Delta(v)\theta(u)+\phi(uv)\Delta(u)+\phi(vu)\Delta(u) \\ &+ \phi(u)\Delta(v)\theta(u)+\Delta(u)\theta(uv)+\phi(v)\Delta(u)\theta(u) \\ &+ \Delta(v)\theta(u^2)+\Delta(u)\theta(uv)+\Delta(u)\theta(vu) \\ &= 2\phi(uv)\Delta(u)+\phi(u^2)\Delta(v)+\phi(u)\Delta(u)\theta(v) \\ &+ 2\phi(u)\Delta(v)\theta(u)+\phi(vu)\Delta(u)+\phi(v)\Delta(u)\theta(u) \\ &+ 2\Delta(u)\theta(uv)+\Delta(v)\theta(u^2)+\Delta(u)\theta(vu). \end{aligned} \quad (19)$$

Again,

$$\begin{aligned} z &= 2\Delta(uvu)+\Delta(u^2v)+\Delta(vu^2) \\ &= 2\Delta(uvu)+\phi(v)\Delta(u^2)+\phi(u^2)\Delta(v) \\ &+ \Delta(u^2)\theta(v)+\Delta(v)\theta(u^2) \\ &= 2\Delta(uvu)+\phi(vu)\Delta(u)+\phi(v)\Delta(u)\theta(u) \\ &+ \phi(u^2)\Delta(v)+\phi(u)\Delta(u)\theta(v) \\ &+ \Delta(u)\theta(vu)+\Delta(v)\theta(u^2). \end{aligned} \quad (20)$$

Comparing (19) and (20) we get,

$$\Delta(uvu) = \phi(uv)\Delta(u)+\phi(u)\Delta(v)\theta(u)+\Delta(u)\theta(uv).$$

(iii) $\Longrightarrow$ (ii) follows by putting $v = u^{-1}$ in (iii).

■

Following a similar way as Semrl [24], we present the following two lemmas which will help to give a representation of (θ,ϕ)-$\mathbb{A}$-quadratic functional via Jordan (θ,ϕ)-derivation.

Lemma 3.9 Let $\mathbb{A}$ be a unital Banach *-algebra with unit element e and $\Delta : \mathbb{A} \to \mathbb{A}$ a Jordan (θ,ϕ)-derivation. Let θ and ϕ be two additive self mappings on $\mathbb{A}$ with $\theta(uv) = \theta(v)\theta(u)$, $\phi(uv) = \phi(u)\phi(v)$ and $\theta(e) = \phi(e) = e$. Then for all u,v,w and invertible $z \in \mathbb{A}$,
(i) $\phi(z)\Delta(z^{-1}u)\theta(z) = \Delta(uz)-\phi(u)\Delta(z)-\Delta(z)\theta(u)$,
(ii) $\Delta(wvwu) = \phi(w)\Delta(vu)\theta(w)+\phi(wv)\Delta(wu)-\phi(wv)\Delta(u)\theta(w)+\Delta(wu)\theta(wv)-\phi(w)\Delta(u)\theta(wv)$.

Proof: (i) Let $uz = e$. So, $u = ez^{-1}$. Now using the conditions (ii) and (iii) of the Lemma 3.8 we get,

$$\begin{aligned} &\phi(z)\Delta(z^{-1}u)\theta(z) \\ &= \phi(z)\Delta(z^{-1}ez^{-1})\theta(z) \\ &= \phi(e)\Delta(z^{-1})\theta(z)+\Delta(e)+\phi(z)\Delta(z^{-1})\theta(e) \\ &= \phi(uz)\Delta(z^{-1})\theta(z)+\Delta(uz)+\phi(z)\Delta(z^{-1})\theta(uz) \\ &= \phi(u)\phi(z)\Delta(z^{-1})\theta(z)+\Delta(uz)+\phi(z)\Delta(z^{-1})\theta(z)\theta(u) \\ &= \Delta(uz)-\phi(u)\Delta(z)-\Delta(z)\theta(u). \end{aligned}$$

(ii) Using the Lemma 3.8 we have,

$$\begin{aligned} &\Delta(wvwu) \\ &= \Delta(wu(u^{-1}v)wu) \\ &= \phi(wuu^{-1}v)\Delta(wu)+\phi(wu)\Delta(u^{-1}v)\theta(wu) \\ &+ \Delta(wu)\theta(wuu^{-1}v) \\ &= \phi(wv)\Delta(wu)+\phi(w)\phi(u)\Delta(u^{-1}v)\theta(u)\theta(w) \\ &+ \Delta(wu)\theta(wv) \\ &= \phi(wv)\Delta(wu)+\phi(w)\{\Delta(vu)-\phi(v)\Delta(u) \\ &- \Delta(u)\theta(v)\}\theta(w)+\Delta(wu)\theta(wv) \\ &= \phi(wv)\Delta(wu)+\phi(w)\Delta(vu)\theta(w)-\phi(w)\phi(v)\Delta(u)\theta(w) \\ &- \phi(w)\Delta(u)\theta(v)\theta(w)+\Delta(wu)\theta(wv) \\ &= \phi(wv)\Delta(wu)+\phi(w)\Delta(vu)\theta(w)-\phi(wv)\Delta(u)\theta(w) \\ &- \phi(w)\Delta(u)\theta(wv)+\Delta(wu)\theta(wv). \end{aligned}$$

■

Lemma 3.10 Let $\mathbb{A}$ be a unital Banach *-algebra with unit element e. Let ϕ and θ be two additive self mappings on $\mathbb{A}$ such that $\theta(uv) = \theta(v)\theta(u)$, $\phi(uv) = \phi(u)\phi(v)$ and $\theta(e) = \phi(e) = e$. Suppose that the mappings ψ_1,$\psi_2 : \mathbb{A} \to \mathbb{A}$ satisfy the conditions:
(i) $2\psi_1(u)+2\psi_1(v) = 4\psi_1(\frac{1}{2}(u+v))+\phi(u-v)\psi_2(0)\theta(u-v)$,
(ii) $2\psi_2(u)+2\psi_2(v) = 4\psi_2(\frac{1}{2}(u+v))+\phi(u-v)\psi_1(0)\theta(u-v)$,
and
(iii)$\psi_1(w) = \phi(w)\psi_2(w^{-1})\theta(w)$
for all u,$v \in \mathbb{A}$ and all invertible $w \in \mathbb{A}$. Then there exists an element $z \in \mathbb{A}$ and a Jordan (θ,ϕ)-derivation Δ on $\mathbb{A}$ such that

$$\psi_1(u) = \phi(u)\psi_2(0)\theta(u)+\phi(u)z+z\theta(u)+\psi_1(0)+\Delta(u)$$

for all $u \in \mathbb{A}$.

Proof: Suppose that

$$2z = \psi_1(e)-\psi_1(0)-\psi_2(0) = \psi_2(e)-\psi_1(0)-\psi_2(0). \quad (21)$$

Let Δ,$\tilde{\Delta} : \mathbb{A} \to \mathbb{A}$ be such that

$$\psi_1(u) = \phi(u)\psi_2(0)\theta(u)+\phi(u)z+z\theta(u)+\psi_1(0)+\Delta(u), \quad (22)$$

$$\psi_2(u) = \phi(u)\psi_1(0)\theta(u)+\phi(u)z+z\theta(u)+\psi_2(0)+\tilde{\Delta}(u). \quad (23)$$

From condition (iii), using (22) and (23) we get, for all invertible $u \in \mathbb{A}$,

$$\begin{aligned} \psi_1(u) &= \phi(u)\psi_2(u^{-1})\theta(u) \\ &= \phi(u)\{\phi(u^{-1})\psi_1(0)\theta(u^{-1})+\phi(u^{-1})z+z\theta(u^{-1}) \\ &+ \psi_2(0)+\tilde{\Delta}(u^{-1})\}\theta(u) \\ &= \psi_1(0)+z\theta(u)+\phi(u)z+\phi(u)\psi_2(0)\theta(u) \\ &+ \phi(u)\tilde{\Delta}(u^{-1})\theta(u), \end{aligned}$$

$$i.e., \Delta(u) = \phi(u)\tilde{\Delta}(u^{-1})\theta(u). \quad (24)$$

Now, putting $v = 0$ in condition (i), we get,

$$2\psi_1(u)+2\psi_1(0) = 4\psi_1(\frac{1}{2}u)+\phi(u)\psi_2(0)\theta(u). \quad (25)$$

Using (23), from (22) we get,

$$\psi_1(\frac{1}{2}u) = \frac{1}{4}\phi(u)\psi_2(0)\theta(u) + \frac{1}{2}\phi(u)z + \frac{1}{2}z\theta(u) + \psi_1(0) + \Delta(\frac{1}{2}u),$$
$$i.e., 2\psi_1(u) + 2\psi_1(0) = 2\phi(u)\psi_2(0)\theta(u) + 2\phi(u)z + 2z\theta(u) + 4\psi_1(0) + 4\Delta(\frac{1}{2}u),$$
$$i.e., \frac{1}{2}\Delta(u) = \Delta(\frac{1}{2}u). \quad (26)$$

Now from condition (i), using (22) we get,

$$\begin{aligned} 2\psi_1(u) + 2\psi_1(v) &= 4\{\phi(\frac{1}{2}(u+v))\psi_2(0)\theta(\frac{1}{2}(u+v)) \\ &+ \phi(\frac{1}{2}(u+v))z + z\theta(\frac{1}{2}(u+v)) + \psi_1(0) \\ &+ \Delta(\frac{1}{2}(u+v))\} + \phi(u-v)\psi_2(0)\theta(u-v) \\ &= \phi(u+v)\psi_2(0)\theta(u+v) + 2\phi(u+v)z \\ &+ 2z\theta(u+v) + 4\psi_1(0) + 4\Delta(\frac{1}{2}(u+v)) \\ &+ \phi(u)\psi_2(0)\theta(u) - \phi(u)\psi_2(0)\theta(v) \\ &- \phi(v)\psi_2(0)\theta(u) + \phi(v)\psi_2(0)\theta(v) \\ &= 2\psi_1(u) - 2\Delta(u) + 2\psi_1(v) - 2\Delta(v) \\ &+ 4\Delta(\frac{1}{2}(u+v)), \end{aligned}$$
$$i.e., \Delta(u) + \Delta(v) = \Delta(u+v) \ (using (26)).$$

Hence Δ is additive.
Now let $u \in \mathbb{A}$ be invertible with $||u|| < 1$. Then $e + u$ is also invertible and

$$(e+u)^{-1} = e - (e+u)^{-1}u. \quad (27)$$

From (22),

$$\begin{aligned} \psi_1(e) &= \phi(e)\psi_2(0)\theta(e) + \phi(e)z + z\theta(e) + \psi_1(0) + \Delta(e) \\ &= \psi_2(0) + 2z + \psi_1(0) + \Delta(e) \ (\phi(e) = \theta(e) = e), \end{aligned}$$
$$i.e., \Delta(e) = 0 \ (by \ (21)). \quad (28)$$

Similarly,

$$\tilde{\Delta}(e) = 0. \quad (29)$$

Now using (27), (28), (29) and the additivity of Δ, from (24) we get,

$$\begin{aligned} \Delta(u) = \Delta(e+u) &= \phi(e+u)\tilde{\Delta}((e+u)^{-1})\theta(e+u) \\ &= -\phi(e+u)\tilde{\Delta}((e+u)^{-1}u)\theta(e+u) \\ &= -\phi(e+u)\phi((e+u)^{-1})\phi(u)\Delta(u^{-1} \\ &+ e)\theta(u)\theta((e+u)^{-1})\theta(e+u) \\ &= -\phi(u)\Delta(u^{-1})\theta(u). \end{aligned}$$

Using additivity of Δ, it is easy to see that $\Delta(u) = -\phi(u)\Delta(u^{-1})\theta(u)$ holds for each invertible $u \in \mathbb{A}$. Now applying Lemma 3.8 we get, Δ is a Jordan (θ,ϕ)-derivation. ■

Theorem 3.11 Let X be a vector space and $\mathbb{A}$ be a unital Banach *-algebra with unit element e such that X is a left $\mathbb{A}$-module. Let θ and ϕ be two additive self mappings on $\mathbb{A}$ with $\theta(uv) = \theta(v)\theta(u)$, $\phi(uv) = \phi(u)\phi(v)$ and $\theta(e) = \phi(e) = e$. For a Jordan (θ,ϕ)-derivation Δ on $\mathbb{A}$, let a mapping $Q : X \to \mathbb{A}$ satisfy

$$\begin{aligned} Q(ux+vy) &= \phi(u)Q(x)\theta(u) + \phi(u)w\theta(v) + \phi(v)w\theta(u) \\ &+ \phi(v)Q(y)\theta(v) + \Delta(vu) - \phi(v)\Delta(u) \\ &- \Delta(u)\theta(v) \end{aligned} \quad (30)$$

for all $x,y \in X$ and $u,v,w \in \mathbb{A}$ with u invertible. Then Q is a (θ,ϕ)-$\mathbb{A}$-quadratic functional. Moreover, when ϕ is the identity mapping on $\mathbb{A}$ and θ is an involution on $\mathbb{A}$, then Q becomes an $\mathbb{A}$-quadratic functional.

Proof: Using Lemma 3.9 in (30) we get,

$$\begin{aligned} Q(ux+vy) &= \phi(u)Q(x)\theta(u) + \phi(u)w\theta(v) + \phi(v)w\theta(u) \\ &+ \phi(v)Q(y)\theta(v) + \phi(u)\Delta(u^{-1}v)\theta(u). \end{aligned} \quad (31)$$

Substituting u^{-1} for u and putting $v = e$, and applying Lemma 3.8 in (31) we get,

$$\begin{aligned} &Q(u^{-1}x+y) \\ &= \phi(u^{-1})Q(x)\theta(u^{-1}) + w\theta(u^{-1}) + \phi(u^{-1})w + Q(y) \\ &+ \phi(u^{-1})\Delta(u)\theta(u^{-1}) \\ &= \phi(u^{-1})Q(x)\theta(u^{-1}) + w\theta(u^{-1}) + \phi(u^{-1})w \\ &+ Q(y) - \Delta(u^{-1}). \end{aligned} \quad (32)$$

Again putting $u = e$ and substituting $v = u$ in (31) we get,

$$\begin{aligned} Q(x+uy) &= Q(x) + w\theta(u) + \phi(u)w \\ &+ \phi(u)Q(y)\theta(u) + \Delta(u). \end{aligned} \quad (33)$$

Using (32) and (33) we get,

$$\begin{aligned} &\phi(u)Q(u^{-1}x+y)\theta(u) \\ &= \phi(u)\{\phi(u^{-1})Q(x)\theta(u^{-1}) + w\theta(u^{-1}) + \phi(u^{-1})w \\ &+ Q(y) - \Delta(u^{-1})\}\theta(u) \\ &= Q(x) + w\theta(u) + \phi(u)w + \phi(u)Q(y)\theta(u) \\ &- \phi(u)\Delta(u^{-1})\theta(u) \\ &= Q(x) + w\theta(u) + \phi(u)w + \phi(u)Q(y)\theta(u) + \Delta(u) \\ &= Q(x+uy) \\ &= Q(u(u^{-1}x+y)). \end{aligned} \quad (34)$$

Taking $x = uz$, $z \in \mathbb{A}$ and $y = 0$ in (34) we get,

$$\phi(u)Q(z)\theta(u) = Q(uz). \quad (35)$$

Again $\Delta(e) = 0$. So, from (31) we get,

$$\begin{aligned} Q(x+y) + Q(x-y) &= Q(x) + w + w + Q(y) + Q(x) \\ &- w - w + Q(y) \\ &= 2Q(x) + 2Q(y). \end{aligned}$$

This shows that Q is a (θ,ϕ)-$\mathbb{A}$-quadratic functional.
In (35), taking ϕ as the identity mapping on $\mathbb{A}$ and θ as an involution on $\mathbb{A}$, we get,

$$Q(uz) = uQ(z)u^*.$$

Hence Q becomes an $\mathbb{A}$-quadratic functional. ■

The following theorem gives a characterization of $\mathbb{A}\otimes_\gamma\mathbb{B}$-sesquilinear functional in terms of Jordan (θ,ϕ)-derivations on the individual hermitian Banach *-algebras $\mathbb{A}$ and $\mathbb{B}$.

Theorem 3.12 Let X, Y be two vector spaces and $\mathbb{A}$, $\mathbb{B}$ be two unital hermitian Banach *-algebras with unit elements e_1 and e_2 respectively. Let X be a unitary left $\mathbb{A}$- module and Y

be a unitary left $\mathbb{B}$-module. For two additive self mappings ϕ_1 and θ_1 on $\mathbb{A}$, let Δ_1 be a Jordan (θ_1,ϕ_1)-derivation on $\mathbb{A}$, and for two additive self mappings ϕ_2 and θ_2 on $\mathbb{B}$, let Δ_2 be a Jordan (θ_2,ϕ_2)-derivation on $\mathbb{B}$. If
(i) ϕ_1 and ϕ_2 are identity mappings on $\mathbb{A}$ and $\mathbb{B}$ respectively,
(ii) θ_1 and θ_2 are involutions on $\mathbb{A}$ and $\mathbb{B}$ respectively, and
(iii) $Q_1 : X \to \mathbb{A}$ and $Q_2 : Y \to \mathbb{B}$ be two mappings satisfying

$$\begin{aligned}&Q_i(u_ix_i + v_iy_i)\\ &= \phi_i(u_i)Q_i(x_i)\theta_i(u_i) + \phi_i(u_i)w_i\theta_i(v_i) + \phi_i(v_i)w_i\theta_i(u_i)\\ &+ \phi_i(v_i)Q_i(y_i)\theta_i(v_i) + \Delta_i(v_iu_i) - \phi_i(v_i)\Delta_i(u_i)\\ &- \Delta_i(u_i)\theta_i(v_i)\end{aligned}$$

for $(i = 1,2)$ and for all $u_1,v_1,w_1 \in \mathbb{A}$ with u_1 invertible, $u_2,v_2,w_2 \in \mathbb{B}$ with u_2 invertible, $x_1,y_1 \in X$ and $x_2,y_2 \in Y$, then there exists an $\mathbb{A}\otimes_\gamma\mathbb{B}$-sesquilinear functional on $X\otimes Y$.

The proof follows from Theorem 3.12 and then Theorem 3.1.

IV. Hyers-Ulam stability of Jordan (θ,ϕ)-derivation

In this section, we undertake an analysis of the Hyers-Ulam stability concerning Jordan (θ,ϕ)-derivation. In 1940, Ulam [25] introduced the stability problem of functional equations involving group homomorphism.

Let G_1 be a group and (G_2,d) be a metric group and ϵ is a positive number. Does there exists a number $\delta > 0$, such that if a mapping f from G_1 to G_2 satisties the following inequality

$$d(f(uv),f(u)f(v)) \le \delta$$

for each $u,v \in G_1$, then there exists a homomorphism h from G_1 to G_2 such that

$$d(f(u),h(u)) \le \epsilon$$

for every $u \in G_1$?.

The homomorphism from G_1 to G_2 are stable if this problem has a solution. Hyers [14] gave the same concept of this Ulam's problem for Banach spaces using norm in place of metric. There are many interesting results on stability analysis considering different systems (refer to [4], [17], [18], [22]).

Lemma 4.1 [20] Let Δ be an additive mapping from a vector space X to a vector space Y such that $\Delta(\lambda u) = \lambda\Delta(u)$ for every $u \in X$ and $\lambda \in \mathbb{C}^1$ where $\mathbb{C}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, then Δ is a linear mapping.

Lemma 4.2 [11] Let X be a Banach space and $(G,+)$ be an abelian group. Let $T : G \times G \to [0,\infty)$ be such that

$$T(u,v) = 2^{-1}\sum_{j=0}^{\infty} 2^{-j}T(2^ju,2^jv) \le \infty$$

for each $u,v \in G$. If Δ is a mapping from G into X such that

$$||\Delta(u + v) - \Delta(u) - \Delta(v)|| \le T(u,v)$$

for each $u,v \in G$, then there exists a unique additive mapping h from G into X such that

$$||\Delta(u) - h(u)|| \le T(u,u)$$

for every $u \in G$.

Let $\mathbb{A}$ be a normed algebra and $\mathbb{M}$ be a Banach $\mathbb{A}$-bimodule. The mapping $T : \mathbb{A} \times \mathbb{A} \to (0,\infty]$ is said to have property P if

$$T(u,v) = 2^{-1}\sum_{j=0}^{\infty} 2^{-j}T(2^ju,2^jv) < \infty \qquad (36)$$

for each $u,v \in \mathbb{A}$ (refer to [4]). For two additive self mappings θ and ϕ on $\mathbb{A}$, a mapping $\Delta : \mathbb{A} \to \mathbb{M}$ is said to have the property Q-(θ,ϕ) if

(i) $||\Delta(\lambda u + v) - \lambda\Delta(u) - \Delta(v)|| \le T(u,v)$,
(ii) $||\Delta(u^2 + v^2) - \Delta(u)\theta(u) - \phi(u)\Delta(u) - \Delta(v)\theta(v) - \phi(v)\Delta(v)|| \le T(u,v)$

for each $u,v \in \mathbb{A}$ and every $\lambda \in \mathbb{C}^1$.

A mapping $f_\Delta : \mathbb{A} \to \mathbb{M}$ is defined by

$$f_\Delta(u) = \lim_{j\to\infty} 2^{-j}\Delta(2^ju) \qquad (37)$$

for every $u \in \mathbb{A}$ (refer to [4]).

Theorem 4.3 Let $\mathbb{A}$ be a normed algebra and $\mathbb{M}$ be a Banach $\mathbb{A}$-bimodule. Let θ and ϕ be two additive self mappings on $\mathbb{A}$. Suppose that T is a mapping from $\mathbb{A} \times \mathbb{A}$ into $(0,\infty]$ which satisfies the property P and Δ is a mapping from $\mathbb{A}$ into $\mathbb{M}$ satisfying the property Q-(θ,ϕ). Then there exists a unique Jordan (θ,ϕ)-derivation f_Δ such that

$$||\Delta(u) - f_\Delta(u)|| \le T(u,u)$$

for every $u \in \mathbb{A}$.

Proof: Define f_Δ as in (37). Proceeding similar to Theorem 2.3 of [4], and applying Lemma 4.1 and Lemma 4.2, it can be shown that f_Δ is a linear mapping.
Now we show that f_Δ is Jordan (θ,ϕ)-derivation.
Since Δ satisfies the property Q-(θ,ϕ), replacing u,v by $2^ju,2^jv$ in (ii) we get,

$$\begin{aligned}&||\Delta(2^{2j}(u^2 + v^2)) - \Delta(2^ju)\theta(2^ju) - \phi(2^ju)\Delta(2^ju)\\ &- \Delta(2^jv)\theta(2^jv) - \phi(2^jv)\Delta(2^jv)|| \le T(2^ju,2^jv).\end{aligned}$$

Since θ and ϕ are additive mappings, so, $\theta(2^ju) = 2^j\theta(u)$ and $\phi(2^jv) = 2^j\phi(v)$. So the above equation becomes

$$\begin{aligned}&||\Delta(2^{2j}(u^2 + v^2)) - 2^j\Delta(2^ju)\theta(u) - 2^j\phi(u)\Delta(2^ju)\\ &- 2^j\Delta(2^jv)\theta(v) - 2^j\phi(v)\Delta(2^jv)|| \le T(2^ju,2^jv).\end{aligned}$$

Multiplying the above equation by 2^{-2j} we get,

$$\begin{aligned}&||2^{-2j}\Delta(2^{2j}(u^2 + v^2)) - 2^{-j}\Delta(2^ju)\theta(u) - 2^{-j}\phi(u)\Delta(2^ju)\\ &- 2^{-j}\Delta(2^jv)\theta(v) - 2^{-j}\phi(v)\Delta(2^jv)|| \le 2^{-2j}T(2^ju,2^jv).\end{aligned}$$

Using (37) and taking limit as $j \to \infty$, from the above equation we get,

$$\begin{aligned}f_\Delta(u^2 + v^2) &= f_\Delta(u)\theta(u) + \phi(u)f_\Delta(u)\\ &+ f_\Delta(v)\theta(v) + \phi(v)f_\Delta(v).\end{aligned}$$

Hence f_Δ is a Jordan (θ,ϕ)-derivation. ■

Remark 4.4 In 2017, Dar et al. [9] explored the concept of generalized derivations within rings equipped with an involution, showing their resemblance to mappings that strongly preserve commutativity. In this context, investigation can be done considering generalized (θ,ϕ)-derivation in the tensor product spaces. In [3], Ashraf discussed commutativity of a 2-torsion free prime ring in terms of Jordan left (θ,θ)-derivation with an application. Investigating the commutativity of the tensor product of prime Banach *-algebras through sesquilinear functionals represents another scope of research in this domain. Moreover, investigation on the characteristics of Lie ideals of a Banach *-algebra $\mathbb{A}$ with the help of (θ,ϕ)-$\mathbb{A}$-quadratic functionals is also an interesting topic for further discussion.

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