Homogeneous Quartic Equation $(x+y)(x^\wedge 3+y^\wedge 3)=kw^\wedge 2p^\wedge 2$

S. Muthuvel and R. Venkatraman*

***Abstract*—In this research, we applied the idea stemming from the equation $(x-y)(x^3+y^3)=4(w^2-p^2)T^6$ to generalize the homogeneous quartic equation $(x+y)(x^3+y^3)=kw^2p^2$, where $k=3+n^2, n\in\mathbb{Z}$. Through the application of appropriate transformations, it becomes a simplified equation of the form $u^2+3v^2=kp^2$, where varying equations are obtained by altering the values of n. We formulated equations using specific n values such as $4,8,11,13,15,16,17,18,19$ and 20. Solving these equations yielded a range of patterns, the solutions of which were determined to be non-zero integral solutions.**

***Index Terms*—Quartic Diophantine equations, integral solutions, figurate numbers.**

I. Introduction

THE diophantine equation was invented by Diophantus, recognized as the Father of Algebra. He devised the diophantine equation, which is a polynomial equation with two or more unknowns in which only the integer solutions matter. The enrichment of diophantine equation theory was achieved by the use of special problems. It is concerned with a range of equations produced from their transformations. In [5], [8], the authors analyzed a ternary quadratic equation with three unknowns for non-zero integral solutions. In [9], Bujačić Babić and Nabardi attempted to solve some Diophantine equations and present infinitely many positive integer solutions. Izadi and Nabardi [13], solved the biquadratic equation $X^4+Y^4=2(U^4+V^4)$ by using the theory of elliptic curve. Further, Janfada and Nabardi [14] generalized the biquadratic equation $x^4+y^4=n(u^4+v^4)$ with some condition on n and solved through the parametric method. The authors of [4], [6], [7], [12], [15], [17], [18] examined a sextic equation with five unknowns in search of non-zero integral solutions. A heptic equation with four unknowns was explored by the authors of [10], [11] in order to find non-zero integer solutions. In [16] Vidhyalakshmi et. al., investigated an octic equation $(x-y)(x^3+y^3)=4(w^2-p^2)T^6$ with five unknowns for non-zero integral solutions. We generalized the homogeneous quartic equation $(x+y)(x^3+y^3)=kw^2p^2$, where $k=3+n^2, n\in\mathbb{Z}$ and it reduces to $u^2+3v^2=kp^2$ with suitable transformations. The answers to $n=4,8,11,13,15,16,17,18,19$ and 20 are addressed in this paper and the solutions are related to special numbers.

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S. Muthuvel is a Research Scholar in the Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Vadapalani Campus, No.1 Jawaharlal Nehru Salai, Vadapalani, Chennai - 600026, Tamilnadu, India. E-mail: muthushan15@gmail.com, ms3081@srmist.edu.in

R. Venkatraman is an Assistant Professor in the Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Vadapalani Campus, No.1 Jawaharlal Nehru Salai, Vadapalani, Chennai - 600026, Tamilnadu, India. *Correspondence E-mail: venkatrr1@srmist.edu.in

II. Notations

$T_m(n)$ - n^{th} m-gonal number
$CS_m(n)$ - n^{th} centered m-gonal number
$P_m(n)$ - n^{th} m-gram number
$S(n)$ - n^{th} star number
$P(n)$ - n^{th} pronic number
$G(n)$ - n^{th} gnomonic number
$N^k(n)$ - n^{th} k-dimensional nexus number
$C^k(n)$ - n^{th} k-dimensional hypercube number

III. Method Of Analysis

A homogeneous quartic equation with four unknowns is considered as

$$(x+y)(x^3+y^3)=kw^2p^2 \quad \text{where } k=3+n^2,\ n\in\mathbb{Z} \tag{1}$$

Consider the linear transformation,

$$x=u+v, y=u-v, w=2u \quad \text{where } u,v\neq 0 \tag{2}$$

in (1) leads to

$$u^2+3v^2=kp^2 \tag{3}$$

Let

$$p=a^2+3b^2 \quad \text{where } a,b\neq 0, a\neq b \tag{4}$$

We put different values of n and we get different types of equations and solve these equations.

Case 1: If $n=4$ in Eq. (3), we get

$$u^2+3v^2=19p^2 \tag{5}$$

$$\text{Write} \quad 19=(4+i\sqrt{3})(4-i\sqrt{3}) \tag{6}$$

Using (4) and (6) in (5) and apply the factorization, we get

$$\begin{aligned} u+i\sqrt{3}v &= (4+i\sqrt{3})(a+ib\sqrt{3})^2 \\ &= (4a^2-12b^2-6ab)+i\sqrt{3}(a^2-3b^2+8ab) \end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 4a^2-12b^2-6ab \\ v &= a^2-3b^2+8ab \end{aligned} \tag{7}$$

Using (7) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 5a^2-15b^2+2ab \\ y(a,b) &= 3a^2-9b^2-14ab \\ w(a,b) &= 8a^2-24b^2-12ab \end{aligned} \tag{8}$$

Properties:

1) $x(1,n)+2CS_{15}(n)-2T_{17}(n)+15T_4(n)$ is a Zuckerman number.
2) $-x(n,1)+T_{12}(n)+3G(n)$ is a O'Halloran number.
3) $-y(1,n)-T_{20}(n)-11G(n)$ is an amenable number.

4) $-y(n,1)+3P(n)+T_{38}(n)-18T_4(n)$ is a Curzon number.
5) $-w(n,1)+2S(n)-4T_4(n)$ is an Ulam number.
6) $-w(1,n)-P_{24}(n)-18G(n)$ is a Cunningham number.
7) $-x(1,n)-y(n,1)-12P(n)$ is a Duffinian number.
8) $-x(n,1)-y(1,n)-T_{10}(n)+2T_{19}(n)-17T_4(n)$ is a Zumkeller number.
9) $-x(1,n)-w(n,1)-7T_4(n)-5G(n)$ is a practical number.
10) $y(n,1)+w(1,n)+7N^2(n)-2CS_5(n)+P_5(n)-2T_9(n)+7T_4(n)$ is a Chen prime.

Case 2: If $n=8$ in Eq. (3), we get

$$u^2+3v^2 = 67p^2 \tag{9}$$

$$\text{Write} \quad 67 = (8+i\sqrt{3})(8-i\sqrt{3}) \tag{10}$$

Using (4) and (10) in (9) and apply the factorization, we get

$$\begin{aligned}u+i\sqrt{3}v &= (8+i\sqrt{3})(a+ib\sqrt{3})^2\\ &= (8a^2-24b^2-6ab)+i\sqrt{3}(a^2-3b^2+16ab)\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned}u &= 8a^2-24b^2-6ab\\ v &= a^2-3b^2+16ab\end{aligned} \tag{11}$$

Using (11) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned}x(a,b) &= 9a^2-27b^2+10ab\\ y(a,b) &= 7a^2-21b^2-22ab\\ w(a,b) &= 16a^2-48b^2-12ab\end{aligned} \tag{12}$$

Properties:

1) $-x(n,1)+P_9(n)+19P(n)-19T_4(n)$ is a happy number.
2) $x(1,n)+9N^2(n)+2CS_{37}(n)-37T_4(n)$ is a magnanimous number.
3) $-y(n,1)+T_{16}(n)-8G(n)$ is a truncatable prime.
4) $-y(1,n)-21T_4(n)-11G(n)$ is a panconsummate number.
5) $-w(1,n)-8S(n)-30G(n)$ is a primorial.
6) $-w(n,1)+P_{16}(n)-T_{12}(n)+5T_4(n)$ is a Gilda number.
7) $x(n,1)+y(1,n)+12P(n)$ is a Pierpont prime.
8) $-x(1,n)-y(n,1)+2S(n)-32T_4(n)$ is an equidigital number.
9) $-x(n,1)-w(1,n)+2CS_{39}(n)-P_{37}(n)+37T_4(n)$ is an interprime number.
10) $-y(1,n)-w(n,1)+2CS_{34}(n)-39T_4(n)$ is a weak prime.

Case 3: If $n=11$ in Eq. (3), we get

$$u^2+3v^2 = 124p^2 \tag{13}$$

Observation-1

$$\text{Write} \quad 124 = (11+i\sqrt{3})(11-i\sqrt{3}) \tag{14}$$

Using (4) and (14) in (13) and apply the factorization, we get

$$\begin{aligned}u+i\sqrt{3}v &= (11+i\sqrt{3})(a+ib\sqrt{3})^2\\ &= (11a^2-33b^2-6ab)+i\sqrt{3}(a^2-3b^2+22ab)\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned}u &= 11a^2-33b^2-6ab\\ v &= a^2-3b^2+22ab\end{aligned} \tag{15}$$

Using (15) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned}x(a,b) &= 12a^2-36b^2+16ab\\ y(a,b) &= 10a^2-30b^2-28ab\\ w(a,b) &= 22a^2-66b^2-12ab\end{aligned} \tag{16}$$

Properties:

1) $-x(1,n)-4\left[T_{20}(n)+6G(n)\right]$ is an abundant number.
2) $-x(n,1)+4\left[T_8(n)-2P(n)+2T_4(n)\right]$ is a perfect square.
3) $-y(1,n)-2\left[CS_{30}(n)+14G(n)\right]$ is a refactorable number.
4) $-y(n,1)+2\left[P_5(n)+2T_{13}(n)-11T_4(n)\right]$ is a frugal number.
5) $w(1,n)+6\left[T_{24}(n)+6G(n)\right]$ is a deficient number.
6) $-w(n,1)+2\left[2T_{13}(n)-T_{10}(n)+4T_4(n)\right]$ is a sphenic number.
7) $-x(1,n)-y(n,1)+2S(n)-38T_4(n)$ is an admirable number.
8) $-x(n,1)-y(1,n)-18P(n)+3G(n)$ is a Sophie Germain prime.
9) $-x(1,n)-w(n,1)-CS_{28}(n)-5G(n)$ is an idoneal number.
10) $-y(n,1)-w(1,n)-9S(n)-2T_4(n)-47G(n)$ is an upside-down number.

Observation-2

$$\text{Write} \quad 124 = (7+i5\sqrt{3})(7-i5\sqrt{3}) \tag{17}$$

Using (4) and (17) in (13) and apply the factorization, we get

$$\begin{aligned}u+i\sqrt{3}v &= (7+i5\sqrt{3})(a+ib\sqrt{3})^2\\ &= (7a^2-21b^2-30ab)+i\sqrt{3}(5a^2-15b^2+14ab)\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned}u &= 7a^2-21b^2-30ab\\ v &= 5a^2-15b^2+14ab\end{aligned} \tag{18}$$

Using (18) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned}x(a,b) &= 12a^2-36b^2-16ab\\ y(a,b) &= 2a^2-6b^2-44ab\\ w(a,b) &= 14a^2-42b^2-60ab\end{aligned} \tag{19}$$

Properties:

1) $x(1,n)+4\left[2T_{11}(n)-2T_{15}(n)+13T_4(n)\right]$ is a pronic number.
2) $-x(n,1)+4\left[2T_5(n)+2T_7(n)-5T_4(n)\right]$ is an abundant number.
3) $-y(1,n)-2\left[T_8(n)+22G(n)\right]$ is a catalan number.
4) $-y(n,1)+2\left[T_4(n)-11G(n)\right]$ is a perfect number.
5) $w(1,n)+6\left[2T_9(n)-2T_{19}(n)+17T_4(n)\right]$ is a semiprime.
6) $-w(n,1)+2\left[T_{16}(n)-24P(n)+24T_4(n)\right]$ is a sphenic number.
7) $x(1,n)+y(n,1)-P_{60}(n)+94T_4(n)$ is a balanced prime.

8) $-x(n,1)-y(1,n)+2N^2(n)+T_{134}(n)-P(n)-65T_4(n)$ is a Jordan-Polya number.
9) $-x(1,n)-w(n,1)-44T_3(n)+9S(n)-54T_4(n)$ is a D-number.
10) $-y(n,1)-w(1,n)-CS_{80}(n)-73G(n)$ is an impolite number.

Case 4: If $n=13$ in Eq. (3), we get

$$u^2+3v^2=172p^2 \tag{20}$$

Observation-1

$$\text{Write}\quad 172=(13+i\sqrt{3})(13-i\sqrt{3}) \tag{21}$$

Using (4) and (21) in (20) and apply the factorization, we get

$$\begin{aligned}u+i\sqrt{3}v &= (13+i\sqrt{3})(a+ib\sqrt{3})^2\\ &= (13a^2-39b^2-6ab)+i\sqrt{3}(a^2-3b^2+26ab)\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned}u &= 13a^2-39b^2-6ab\\ v &= a^2-3b^2+26ab\end{aligned} \tag{22}$$

Using (22) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned}x(a,b) &= 14a^2-42b^2+20ab\\ y(a,b) &= 12a^2-36b^2-32ab\\ w(a,b) &= 26a^2-78b^2-12ab\end{aligned} \tag{23}$$

Properties:

1) $x(1,n)-2[-P_{21}(n)+2CS_{11}(n)-11T_4(n)]$ is an abundant number.
2) $-x(n,1)+2[2CS_7(n)-2T_{21}(n)+19T_4(n)]$ is a deficient number.
3) $y(1,n)+4[2T_{11}(n)-P_{15}(n)+15T_4(n)]$ is an almost perfect number.
4) $-y(n,1)+4[T_8(n)-3G(n)]\equiv 0 \pmod 2$
5) $-w(1,n)-6[T_{28}(n)+7G(n)]$ is a frugal number.
6) $-w(n,1)+2[P_{13}(n)-T_{18}(n)+8T_4(n)]$ is a refactorable number.
7) $-x(1,n)-y(n,1)-60T_3(n)+9G(n)$ is an emirp.
8) $-x(n,1)-y(1,n)-P_{22}(n)+CS_{68}(n)-34T_4(n)$ is an evil number.
9) $-x(1,n)-w(n,1)-T_{34}(n)+2T_{11}(n)-9T_4(n)$ is an ABA number.
10) $-y(n,1)-w(1,n)-11S(n)-110P(n)+36N^2(n)+2T_4(n)-54G(n)$ is a modest number.

Observation-2

$$\text{Write}\quad 172=(5+i7\sqrt{3})(5-i7\sqrt{3}) \tag{24}$$

Using (4) and (24) in (20) and apply the factorization, we get

$$\begin{aligned}u+i\sqrt{3}v &= (5+i7\sqrt{3})(a+ib\sqrt{3})^2\\ &= (5a^2-15b^2-42ab)+i\sqrt{3}(7a^2-21b^2+10ab)\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned}u &= 5a^2-15b^2-42ab\\ v &= 7a^2-21b^2+10ab\end{aligned} \tag{25}$$

Using (25) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned}x(a,b) &= 12a^2-36b^2-32ab\\ y(a,b) &= -2a^2+6b^2-52ab\\ w(a,b) &= 10a^2-30b^2-84ab\end{aligned} \tag{26}$$

Properties:

1) $x(1,n)+4[P_9(n)-2CS_{17}(n)+17T_{4,n}]$ is a deficient number.
2) $-x(n,1)+4[T_{20}(n)-6T_4(n)]$ is a perfect square.
3) $-y(1,n)+2[2CS_3(n)+2T_{27}(n)-25T_4(n)]$ is a semiprime.
4) $-y(n,1)+2[T_4(n)-13G(n)]$ is a Dedekind number.
5) $-w(1,n)-3[2CS_{10}(n)+19G(n)]$ is a prime.
6) $-w(n,1)+2[P_5(n)+2T_{41}(n)-39T_4(n)]$ is a frugal number.
7) $-x(n,1)-y(1,n)+4T_{11}(n)-35G(n)$ is an amenable number.
8) $-x(1,n)-w(n,1)-2CS_{26}(n)-142P(n)+142T_4(n)$ is a powerful number.
9) $-y(n,1)-w(1,n)-P_{32}(n)+28G(n)-168T_4(n)$ is an a-pointer prime.
10) $-x(n,1)-y(n,1)-w(n,1)+2CS_{20}(n)-94G(n)$ is a congruent number.

Observation-3

$$\text{Write}\quad 172 = (8+i6\sqrt{3})(8-i6\sqrt{3}) \tag{27}$$

Using (4) and (27) in (20) and apply the factorization, we get

$$\begin{aligned}u+i\sqrt{3}v &= (8+i6\sqrt{3})(a+ib\sqrt{3})^2\\ &= (8a^2-24b^2-36ab)+i\sqrt{3}(6a^2-18b^2+16ab)\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned}u &= 8a^2-24b^2-36ab\\ v &= 6a^2-18b^2+16ab\end{aligned} \tag{28}$$

Using (28) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned}x(a,b) &= 14a^2-42b^2-20ab\\ y(a,b) &= 2a^2-6b^2-52ab\\ w(a,b) &= 16a^2-48b^2-72ab\end{aligned} \tag{29}$$

Properties:

1) $x(1,n)+2[2T_{23}(n)-P_{29}(n)+29T_4(n)]\equiv 0 \pmod 2$
2) $-x(n,1)+2[2CS_7(n)+2T_{17}(n)-5T_4(n)]$ is a square free number.
3) $y(1,n)+2[CS_6(n)-P_{29}(n)+29T_4(n)]$ is a Bell number.
4) $-y(n,1)+2[T_4(n)-13G(n)]$ is an almost perfect number.
5) $-w(1,n)-24[P_2(n)-2CS_5(n)+5T_4(n)]$ is a deficient number.
6) $-w(n,1)+8[CS_4(n)-P_7(n)-7T_4(n)]$ is a perfect cube.
7) $-x(1,n)-y(n,1)-40P(n)-16G(n)$ is a strobogrammatic number.
8) $-x(n,1)-y(1,n)+T_{18}(n)+P_{63}(n)-63T_4(n)$ is a strong prime.
9) $-x(1,n)-w(n,1)-2CS_{26}(n)+59T_8(n)-177T_4(n)$ is a Leyland number.

10) $y(n,1)+w(1,n)-15N^2(n)+2CS_{79}(n)-80T_4(n)$ is a m-pointer prime.

Case 5: If $n=15$ in Eq. (3), we get

$$u^2+3v^2 = 228p^2 \quad (30)$$

Observation-1

$$\text{Write} \quad 228 = (15+i\sqrt{3})(15-i\sqrt{3}) \quad (31)$$

Using (4) and (31) in (30) and apply the factorization, we get

$$u+i\sqrt{3}v=(15+i\sqrt{3})(a+ib\sqrt{3})^2$$
$$=(15a^2-45b^2-6ab)+i\sqrt{3}(a^2-3b^2+30ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 15a^2-45b^2-6ab$$
$$v = a^2-3b^2+30ab \quad (32)$$

Using (32) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 16a^2-48b^2+24ab$$
$$y(a,b) = 14a^2-42b^2-36ab$$
$$w(a,b) = 30a^2-90b^2-12ab \quad (33)$$

Properties:

1) $x(1,n)-y(1,n)+4\left[T_{10}(n)-6G(n)\right]$ is a semiprime.
2) $y(n,1)-x(n,1)+2\left[-2CS_{30}(n)+T_4(n)\right]$ is a catalan number.
3) $x(1,n)+y(1,n)-w(1,n)=0$
4) $x(n,1)+y(n,1)-w(n,1)=0$
5) $-y(n,1)+2\left[T_{16}(n)-6G(n)\right]$ is an abundant number.
6) $x(1,n)+24T_6(n)$ is a perfect square.
7) $-x(n,1)+16P(n)+4G(n)$ is a tribonacci number.
8) $y(1,n)+7S(n)+26N^2(n)-78T_4(n)$ is a Carol number.
9) $-w(1,n)-2\left[15N^2(n)+13T_{10}(n)-52T_4(n)\right]$ is an Ulam number.
10) $-w(n,1)+2\left[2T_{17}(n)+CS_{14}(n)-7T_4(n)\right]$ is an arithmetic number.

Observation-2

$$\text{Write} \quad 228 = (9+i7\sqrt{3})(9-i7\sqrt{3}) \quad (34)$$

Using (4) and (34) in (30) and apply the factorization, we get

$$u+i\sqrt{3}v=(9+i7\sqrt{3})(a+ib\sqrt{3})^2$$
$$=(9a^2-27b^2-42ab)+i\sqrt{3}(7a^2-21b^2+18ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 9a^2-27b^2-42ab$$
$$v = 7a^2-21b^2+18ab \quad (35)$$

Using (35) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 16a^2-48b^2-24ab$$
$$y(a,b) = 2a^2-6b^2-60ab$$
$$w(a,b) = 18a^2-54b^2-84ab \quad (36)$$

Properties:

1) $y(n,1)-x(n,1)+2\left[P_7(n)-CS_{50}(n)+25T_4(n)\right]$ is a square free number.
2) $x(1,n)-y(1,n)+6\left[2CS_6(n)+T_4(n)\right]$ is a deficient number.
3) $-x(n,1)+8\left[T_6(n)-G(n)\right]$ is a refactorable number.
4) $-y(n,1)+2\left[P_{30}(n)-29T_4(n)\right]$ is an almost perfect number.
5) $x(1,n)+y(1,n)-w(1,n)=0$
6) $x(1,n)+8S(n)-P_{72}(n)+72T_4(n)$ is a lonely number.
7) $-y(1,n)-S(n)-33G(n)$ is a Curzon number.
8) $-w(n,1)+9N^2(n)-51G(n)$ is a Zumkeller number.
9) $w(1,n)+54P(n)+15G(n)$ is a cyclic number.
10) $-x(1,n)-w(n,1)-3T_{22}(n)+2CS_{135}(n)-135T_4(n)$ is a Harshad number.

Observation-3

$$\text{Write} \quad 228 = (6+i8\sqrt{3})(6-i8\sqrt{3}) \quad (37)$$

Using (4) and (37) in (30) and apply the factorization, we get

$$u+i\sqrt{3}v=(6+i8\sqrt{3})(a+ib\sqrt{3})^2$$
$$=(6a^2-18b^2-48ab)+i\sqrt{3}(8a^2-24b^2+12ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 6a^2-18b^2-48ab$$
$$v = 8a^2-24b^2+12ab \quad (38)$$

Using (38) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 14a^2-42b^2-36ab$$
$$y(a,b) = -2a^2+6b^2-60ab$$
$$w(a,b) = 12a^2-36b^2-96ab \quad (39)$$

Properties:

1) $-y(n,1)-w(n,1)+2\left[P_{78}(n)-73T_4(n)\right]$ is a Leyland number.
2) $w(1,n)+y(1,n)+2\left[15T_4(n)+78C(n)\right]\equiv 0 \pmod 5$
3) $x(1,n)+y(1,n)-w(1,n)=0$
4) $-x(n,1)+2\left[2CS_7(n)+P_{11}(n)\right]$ is an abundant number.
5) $y(n,1)+2\left[-CS_{60}(n)+31T_4(n)\right]$ is a semiprime.
6) $-x(1,n)-7S(n)-39G(n)$ is a harmonic number.
7) $-y(1,n)+2N^2(n)+22T_{10}(n)-88T_4(n)$ is a trimorphic number.
8) $-w(n,1)+P_{12}(n)-42G(n)$ is a Kynea number.
9) $-x(1,n)-w(n,1)-4T_{17}(n)+4CS_{79}(n)-79T_6(n)$ is a junction number.
10) $-x(n,1)-y(1,n)+T_{22}(n)+2CS_{87}(n)-87T_4(n)$ is a congruent number.

Case 6: If $n=16$ in Eq. (3), we get

$$u^2+3v^2 = 259p^2 \quad (40)$$

Observation-1

$$\text{Write} \quad 259 = (16+i\sqrt{3})(16-i\sqrt{3}) \quad (41)$$

Using (4) and (41) in (40) and apply the factorization, we get

$$u+i\sqrt{3}v=(16+i\sqrt{3})(a+ib\sqrt{3})^2$$
$$=(16a^2-48b^2-6ab)+i\sqrt{3}(a^2-3b^2+32ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 16a^2-48b^2-6ab$$
$$v = a^2-3b^2+32ab \quad (42)$$

Using (42) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 17a^2-51b^2+26ab \\ y(a,b) &= 15a^2-45b^2-38ab \\ w(a,b) &= 32a^2-96b^2-12ab \end{aligned} \quad (43)$$

Properties:

1) $x(1,n)+2CS_{51}(n)-P_{25}(n)+25T_4(n)$ is a Curzon number.
2) $-x(n,1)+2T_{19}(n)-2CS_{41}(n)+41T_4(n)$ is a Gilda number.
3) $y(1,n)-2CS_{38}(n)+83T_4(n)$ is an emirp.
4) $w(1,n)+12\left[P(n)+7T_4(n)\right]$ is a frugal number.
5) $x(1,n)-y(1,n)+2\left[3P(n)+2CS_{35}(n)-35T_4(n)\right]$ is an interprime number.
6) $y(n,1)-x(n,1)+2\left[32P(n)-31T_4(n)\right]$ is a harmonic number.
7) $-y(n,1)+5N^2(n)+2CS_{53}(n)-53T_4(n)$ is an amenable number.
8) $-w(n,1)+4\left[-3P(n)+11T_4(n)\right]$ is a strobogrammatic number.
9) $-x(1,n)-w(n,1)-2CS_{19}(n)-5P(n)+5T_4(n)$ is a panconsummate number.
10) $y(n,1)+w(1,n)+27N^2(n)+P_{31}(n)-31T_4(n)$ is a Sophie Germain prime.

Observation-2

$$\text{Write} \quad 259 = (4+i9\sqrt{3})(4-i9\sqrt{3}) \quad (44)$$

Using (4) and (44) in (40) and apply the factorization, we get

$$\begin{aligned} u+i\sqrt{3}v &= (4+i9\sqrt{3})(a+ib\sqrt{3})^2 \\ &= (4a^2-12b^2-54ab)+i\sqrt{3}(9a^2-27b^2+8ab) \end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 4a^2-12b^2-54ab \\ v &= 9a^2-27b^2+8ab \end{aligned} \quad (45)$$

Using (45) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 13a^2-39b^2-46ab \\ y(a,b) &= -5a^2+15b^2-62ab \\ w(a,b) &= 8a^2-24b^2-108ab \end{aligned} \quad (46)$$

Properties:

1) $-x(1,n)+CS_{92}(n)-85T_4(n)\equiv 0 \pmod 2$
2) $-y(1,n)+5N^2(n)-2CS_{47}(n)-47T_4(n)\equiv 0 \pmod 3$
3) $y(n,1)+2T_7(n)+65P(n)-65T_4(n)$ is Lucas Carmichael number.
4) $w(1,n)+12\left[3N^2(n)-7T_4(n)\right]$ is a deficient number.
5) $-w(n,1)-4\left[P_{27}(n)+25T_4(n)\right]$is a harmonic number.
6) $y(1,n)+w(1,n)+3N^2(n)-2CS_{161}(n)+161T_4(n)$ is a prime power.
7) $-x(n,1)+2CS_{13}(n)-11N^2(n)+33T_4(n)$ is an idoneal number.
8) $-x(1,n)-y(n,1)-8T_{13}(n)-72G(n)$ is an arithmetic number.
9) $-x(1,n)-w(n,1)-P_{31}(n)+37T_{14}(n)-222T_4(n)$ is a pernicious number.
10) $-x(n,1)+y(n,1)-w(n,1)+4T_{15}(n)-35G(n)$ is a truncatable prime.

Case 7: If $n=17$ in Eq. (3), we get

$$u^2+3v^2 = 292p^2 \quad (47)$$

Observation-1

$$\text{Write} \quad 292 = (17+i\sqrt{3})(17-i\sqrt{3}) \quad (48)$$

Using (4) and (48) in (47) and apply the factorization, we get

$$\begin{aligned} u+i\sqrt{3}v &= (17+i\sqrt{3})(a+ib\sqrt{3})^2 \\ &= (17a^2-51b^2-6ab)+i\sqrt{3}(a^2-3b^2+34ab) \end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 17a^2-51b^2-6ab \\ v &= a^2-3b^2+34ab \end{aligned} \quad (49)$$

Using (49) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 18a^2-54b^2+28ab \\ y(a,b) &= 16a^2-48b^2-40ab \\ w(a,b) &= 34a^2-102b^2-12ab \end{aligned} \quad (50)$$

Properties:

1) $x(1,n)+2\left[9N^2(n)+P_{41}(n)-41T_4(n)\right]$ is a square free number.
2) $-x(n,1)+2\left[9T_4(n)+7G(n)\right]$ is an abundant number.
3) $-y(n,1)+8\left[2P(n)+2CS_7(n)-7T_4(n)\right]$ is a deficient number.
4) $-w(n,1)+2\left[S(n)-11T_4(n)\right]$ is a polite number.
5) $x(1,n)-y(1,n)+S(n)-27G(n)$ is sphenic number.
6) $y(n,1)-x(n,1)+2\left[10N^2(n)-29T_4(n)\right]$ is a semiprime.
7) $-y(1,n)-8S(n)-44G(n)$ is a magnanimous number.
8) $w(1,n)+34N^2(n)-90P(n)+90T_4(n)$ is an amenable number.
9) $-x(1,n)-w(n,1)-P_{20}(n)+2T_8(n)-S(n)-3G(n)$ is a hoax number.
10) $-y(n,1)-w(1,n)-CS_{172}(n)-69G(n)$ is an upside-down number.

Observation-2

$$\text{Write} \quad 292 = (10+i8\sqrt{3})(10-i8\sqrt{3}) \quad (51)$$

Using (4) and (51) in (47) and apply the factorization, we get

$$\begin{aligned} u+i\sqrt{3}v &= (10+i8\sqrt{3})(a+ib\sqrt{3})^2 \\ &= (10a^2-30b^2-48ab)+i\sqrt{3}(8a^2-24b^2+20ab) \end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 10a^2-30b^2-48ab \\ v &= 8a^2-24b^2+20ab \end{aligned} \quad (52)$$

Using (52) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 18a^2-54b^2-28ab \\ y(a,b) &= 2a^2-6b^2-68ab \\ w(a,b) &= 20a^2-60b^2-96ab \end{aligned} \quad (53)$$

Properties:

1) $-x(n,1)+2\left[2T_{11}(n)+P_7(n)-7T_4(n)\right]$ is a pronic number.
2) $-y(n,1)+2\left[T_4(n)-17G(n)\right]\equiv 0 \pmod 5$
3) $y(1,n)+2\left[N^2(n)-2CS_{31}(n)+31T_4(n)\right]=0$
4) $-w(n,1)+4\left[2CS_5(n)+P_{19}(n)-19T_4(n)\right]$ is an abundant number.
5) $y(n,1)-x(n,1)+8\left[9T_4(n)-2T_9(n)\right]\equiv 0 \pmod 2$
6) $x(1,n)-y(1,n)+8\left[S(n)+P(n)-T_4(n)\right]$is a nasty number.
7) $x(1,n)+9S(n)+82P(n)-82T_4(n)$ is a Moran number.
8) $w(1,n)+20N^2(n)-3T_{28}(n)+39T_4(n)$ is a super Niven number.
9) $-x(1,n)-3w(n,1)+T_{14}(n)-107N^2(n)+321T_4(n)$ is a Kaprekar number.
10) $-50y(n,1)-9w(1,n)-80T_{13}(n)-2312G(n)$ is a pernicious number.

Observation-3

$$\text{Write} \quad 292 = (7+i9\sqrt{3})(7-i9\sqrt{3}) \tag{54}$$

Using (4) and (54) in (47) and apply the factorization, we get

$$u+i\sqrt{3}v = (7+i9\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (7a^2-21b^2-54ab)+i\sqrt{3}(9a^2-27b^2+14ab)$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 7a^2-21b^2-54ab \\ v &= 9a^2-27b^2+14ab \end{aligned} \tag{55}$$

Using (55) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 16a^2-48b^2-40ab \\ y(a,b) &= -2a^2+6b^2-68ab \\ w(a,b) &= 14a^2-42b^2-108ab \end{aligned} \tag{56}$$

Properties:

1) $x(1,n)+8\left[S(n)-2CS_{11}(n)+11T_4(n)\right]$ is a Cunningham number.
2) $-y(1,n)+2\left[N^2(n)+P_{37}(n)-3T_4(n)\right]$ is an idoneal number.
3) $-x(n,1)+8\left[-5P(n)+7T_4(n)\right]$ is a Zumkeller number.
4) $-(n,1)-2\left[T_4(n)+17G(n)\right]$ is a congruent number.
5) $-w(1,n)-6\left[T_{16}(n)+12G(n)\right]$ is a semiprime.
6) $-w(n,1)+2\left[9S(n)-47T_4(n)\right]$ is a O'Halloran number.
7) $x(1,n)-24y(n,1)-796G(n)$ is a pernicious number.
8) $-y(1,n)-w(n,1)+2T_{22}(n)-79G(n)$ is an arithmetic number.
9) $-x(n,1)-w(1,n)-2T_{28}(n)+4P_{43}(n)-172T_4(n)$ is a Jordan-Polya number.
10) $y(n,1)-x(n,1)+w(n,1)+2CS_{136}(n)-140T_4(n)$ is a Harshad number.

Case 8: If $n=18$ in Eq. (3), we get

$$u^2+3v^2 = 327p^2 \tag{57}$$

Observation-1

$$\text{Write} \quad 327 = (18+i\sqrt{3})(18-i\sqrt{3}) \tag{58}$$

Using (4) and (58) in (57) and apply the factorization, we get

$$u+i\sqrt{3}v = (18+i\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (18a^2-54b^2-6ab)+i\sqrt{3}(a^2-3b^2+36ab)$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 18a^2-54b^2-6ab \\ v &= a^2-3b^2+36ab \end{aligned} \tag{59}$$

Using (59) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 19a^2-57b^2+30ab \\ y(a,b) &= 17a^2-51b^2-42ab \\ w(a,b) &= 36a^2-108b^2-12ab \end{aligned} \tag{60}$$

Properties:

1) $x(1,n)+2CS_{57}(n)+9N^2(n)-27T_4(n)$ is an interprime number.
2) $-x(n,1)+2T_{21}(n)-P_{47}(n)+47T_4(n)$ is an admirable number.
3) $y(1,n)+17N^2(n)-9P(n)+9T_4(n)$ is a panconsummate number.
4) $-y(n,1)+2T_{19}(n)+2CS_{27}(n)-27T_4(n)$ is a Sophie Germain prime.
5) $w(1,n)-18S(n)-48G(n)$ is a sphenic number.
6) $-w(n,1)+6T_{14}(n)+9G(n)$ is a Cunningham number.
7) $-x(1,n)-y(n,1)-4T_{22}(n)-24G(n)$ is a congruent number.
8) $-x(n,1)-w(1,n)-P_{89}(n)+2CS_{71}(n)-71T_4(n)$ is a Smith number.
9) $-y(1,n)-w(n,1)-2T_{17}(n)+67P(n)-67T_4(n)$ is a deficient number.
10) $-x(1,n)-y(1,n)-w(1,n)-36S(n)-120G(n)$ is a Harshad number.

Case 9: If $n=19$ in Eq. (3), we get

$$u^2+3v^2 = 364p^2 \tag{61}$$

Observation-1

$$\text{Write} \quad 364 = (19+i\sqrt{3})(19-i\sqrt{3}) \tag{62}$$

Using (4) and (62) in (61) and apply the factorization, we get

$$u+i\sqrt{3}v = (19+i\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (19a^2-57b^2-6ab)+i\sqrt{3}(a^2-3b^2+38ab)$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} u &= 19a^2-57b^2-6ab \\ v &= a^2-3b^2+38ab \end{aligned} \tag{63}$$

Using (63) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 20a^2-60b^2+32ab \\ y(a,b) &= 18a^2-54b^2-44ab \\ w(a,b) &= 38a^2-114b^2-12ab \end{aligned} \tag{64}$$

Properties:

1) $x(1,n)+6T_{22}(n)+11G(n)$ is a Cullen number.
2) $-x(n,1)+4T_{12}(n)+24G(n)$ is a O'Halloran number.
3) $y(1,n)+9S(n)-14T_{18}(n)+112T_4(n)$ is an astonishing number.

4) $-y(n,1)+6N^2(n)-31G(n)$ is a deceptive number.
5) $-w(1,n)-2CS_{114}(n)-63G(n)$ is a truncatable prime.
6) $-w(n,1)+4T_{21}(n)+11G(n)$ is a congruent number.
7) $x(1,n)+y(n,1)+7S(n)+27G(n)=0$
8) $-x(n,1)-w(1,n)-4T_6(n)+57G(n)$ is a panconsummate number.
9) $-y(1,n)-w(n,1)-2T_{18}(n)+P_{70}(n)-70T_4(n)$ is a strong prime.
10) $-x(1,n)-y(1,n)-w(n,1)-8T_{21}(n)-46G(n)$ is a deficient number.

Observation-2

$$\text{Write} \quad 364 = (17+i5\sqrt{3})(17-i5\sqrt{3}) \qquad (65)$$

Using (4) and (65) in (61) and apply the factorization, we get

$$u+i\sqrt{3}v = (17+i5\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (17a^2-51b^2-30ab)+i\sqrt{3}(5a^2-15b^2+34ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 17a^2-51b^2-30ab$$
$$v = 5a^2-15b^2+34ab \qquad (66)$$

Using (66) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 22a^2-66b^2+4ab$$
$$y(a,b) = 12a^2-36b^2-64ab$$
$$w(a,b) = 34a^2-102b^2-60ab \qquad (67)$$

Properties:

1) $-x(n,1)+2P_{11}(n)+13G(n)$ is a Duffinian number.
2) $x(1,n)+11S(n)-31T_8(n)+93T_4(n)$ is a semiprime.
3) $-y(n,1)+4N^2(n)+4P_{19}(n)-76T_4(n)$ is a tribonacci number.
4) $-y(1,n)-2CS_{36}(n)-50G(n)$ is an amenable number.
5) $w(1,n)+17S(n)-21G(n)$ is an abundant number.
6) $-w(n,1)+4T_{19}(n)+3T_{24}(n)-33T_4(n)$ is an admirable number.
7) $x(1,n)+y(n,1)+9S(n)+3G(n)$ is a harmonic number.
8) $-y(1,n)-w(n,1)-2P(n)+4CS_{61}(n)-122T_4(n)$ is a magnanimous number.
9) $-x(n,1)-w(1,n)-P_{80}(n)+16CS_{17}(n)-136T_4(n)$ is a Carol number.
10) $-x(n,1)-y(n,1)-w(1,n)-8T_{19}(n)+30S(n)-180T_4(n)$ is a polite number.

Observation-3

$$\text{Write} \quad 364 = (16+i6\sqrt{3})(16-i6\sqrt{3}) \qquad (68)$$

Using (4) and (68) in (61) and apply the factorization, we get

$$u+i\sqrt{3}v = (16+i6\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (16a^2-48b^2-36ab)+i\sqrt{3}(6a^2-18b^2+32ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 16a^2-48b^2-36ab$$
$$v = 6a^2-18b^2+32ab \qquad (69)$$

Using (69) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 22a^2-66b^2-4ab$$
$$y(a,b) = 10a^2-30b^2-68ab$$
$$w(a,b) = 32a^2-96b^2-72ab \qquad (70)$$

Properties:

1) $-x(1,n)-11S(n)-35G(n)$ is a Sophie Germain prime.
2) $-x(n,1)+4T_{13}(n)+7G(n)$ is a truncatable prime.
3) $y(1,n)+10N^2(n)+38P(n)-38T_4(n)$ is a magnanimous number.
4) $-y(n,1)+T_{22}(n)+P_{59}(n)-59T_4(n)$ is a Chen prime.
5) $w(1,n)+32N^2(n)+2CS_{24}(n)-24T_4(n)$ is an admirable number.
6) $-w(n,1)+4T_{18}(n)+8T_{15}(n)-52T_4(n)$ is an amenable number.
7) $-x(1,n)-y(n,1)-2CS_{56}(n)-64G(n)$ is a sphenic number.
8) $-y(1,n)-w(n,1)+2T_4(n)-70G(n)$ is a O'Halloran number.
9) $-x(n,1)-w(1,n)+2CS_{76}(n)-150T_4(n)$ is a Duffinian number.
10) $x(1,n)+y(1,n)+w(1,n)+8P_{24}(n)-56S(n)+336T_4(n)$ is an idoneal number.

Observation-4

$$\text{Write} \quad 364 = (11+i9\sqrt{3})(11-i9\sqrt{3}) \qquad (71)$$

Using (4) and (71) in (61) and apply the factorization, we get

$$u+i\sqrt{3}v = (11+i9\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (11a^2-33b^2-54ab)+i\sqrt{3}(9a^2-27b^2+22ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 11a^2-33b^2-54ab$$
$$v = 9a^2-27b^2+22ab \qquad (72)$$

Using (72) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 20a^2-60b^2-32ab$$
$$y(a,b) = 2a^2-6b^2-76ab$$
$$w(a,b) = 22a^2-66b^2-108ab \qquad (73)$$

Properties:

1) $-x(n,1)+8T_{12}(n)-20T_4(n)$ is a Saint-Exupery number.
2) $-x(1,n)-10S(n)-46G(n)$ is an Ulam number.
3) $-y(n,1)+2CS_{76}(n)-74T_4(n)$ is a strobogrammatic number.
4) $y(1,n)-8T_{23}(n)+90T_4(n)$ is a double factorial.
5) $w(1,n)+11S(n)-6P_{29}(n)+174T_4(n)$ is an astonishing number.
6) $-x(n,1)-y(1,n)+2CS_{14}(n)+47T_8(n)-141T_4(n)$ is a Zumkeller number.
7) $-y(n,1)-w(1,n)-8T_{18}(n)-120G(n)$ is an odious number.
8) $-x(1,n)-w(n,1)-2CS_{38}(n)+2P_{89}(n)-178T_4(n)$ is a modest number.
9) $-w(n,1)+4T_{13}(n)-45G(n)$ is an insolite number.
10) $-x(n,,1)-w(n,1)-y(n,1)-72N^2(n)+260T_4(n)$ is an arithmetic number.

Observation-5

$$\text{Write}\quad 364 = (8+i10\sqrt{3})(8-i10\sqrt{3}) \quad (74)$$

Using (4) and (74) in (61) and apply the factorization, we get

$$u+i\sqrt{3}v = (8+i10\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (8a^2-24b^2-60ab)+i\sqrt{3}(10a^2-30b^2+16ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 8a^2-24b^2-60ab$$
$$v = 10a^2-30b^2+16ab \quad (75)$$

Using (75) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 18a^2-54b^2-44ab$$
$$y(a,b) = -2a^2+6b^2-76ab$$
$$w(a,b) = 16a^2-48b^2-120ab \quad (76)$$

Properties:

1) $-x(1,n)-9S(n)-49G(n)$ is a self-describing number.
2) $-x(n,1)+6N^2(n)+2P_{31}(n)-62T_4(n)$ is an inconsummate number.
3) $y(n,1)-8T_{23}(n)+86T_4(n)$ is a narcissistic number.
4) $-y(1,n)+S(n)-35G(n)$ is an alternating number.
5) $-w(1,n)-16N^2(n)-36G(n)$ is a compositorial.
6) $-w(n,,1)+16T_4(n)-60G(n)$ is a self number.
7) $-x(n,1)-y(1,n)+4T_{14}(n)-50G(n)$ is an ABA number.
8) $-y(n,1)-w(1,n)-2CS_{50}(n)+41P_6(n)-246T_4(n)$ is a good prime.
9) $-x(1,n)-w(n,1)-38P(n)+2CS_{126}(n)-126T_4(n)$ is an equidigital number.
10) $-x(n,1)-y(n,,1)-w(n,1)+4T_{18}(n)+106CS_4(n)-212T_4(n)$ is a Smith number.

Observation-6

$$\text{Write}\quad 364 = (1+i11\sqrt{3})(1-i11\sqrt{3}) \quad (77)$$

Using (4) and (77) in (61) and apply the factorization, we get

$$u+i\sqrt{3}v = (1+i11\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (a^2-3b^2-66ab)+i\sqrt{3}(11a^2-33b^2+2ab)$$

Equating real and imaginary parts on both sides, we get

$$u = a^2-3b^2-66ab$$
$$v = 11a^2-33b^2+2ab \quad (78)$$

Using (78) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 12a^2-36b^2-64ab$$
$$y(a,b) = -10a^2+30b^2-68ab$$
$$w(a,b) = 2a^2-6b^2-132ab \quad (79)$$

Properties:

1) $-x(n,1)+4N^2(n)-38G(n)$ is a Gilda number.
2) $x(1,n)+6S(n)+100P(n)-100T_4(n)$ is a pernicious number.
3) $-y(n,1)-P_{10}(n)-39G(n)$ is a Harshad number.
4) $-y(1,n)+3T_{22}(n)+2CS_{41}(n)-41T_4(n)$ is a super Niven number.
5) $-w(1,n)-S(n)-69G(n)$ is a nude number.
6) $-w(n,1)+2P(n)+P_{134}(n)-134T_4(n)$ is a Pierpont prime.
7) $-x(n,1)-y(1,n)+14N^2(n)-87G(n)$ is a straight-line number.
8) $-y(n,1)-w(1,n)-2T_{18}(n)-107G(n)$ is a fibodiv number.
9) $x(1,n)+w(n,1)+4T_{19}(n)-2CS_{226}(n)+226T_4(n)$ is a Zuckerman number.
10) $-x(1,n)-y(1,n)-w(1,n)+44S(n)-276T_4(n)$ is a pseudoperfect number.

Case 10: If $n=20$ in Eq. (3), we get

$$u^2+3v^2 = 403p^2 \quad (80)$$

Observation-1

$$\text{Write}\quad 403 = (20+i\sqrt{3})(20-i\sqrt{3}) \quad (81)$$

Using (4) and (81) in (80) and apply the factorization, we get

$$u+i\sqrt{3}v = (20+i\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (20a^2-60b^2-6ab)+i\sqrt{3}(a^2-3b^2+40ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 20a^2-60b^2-6ab$$
$$v = a^2-3b^2+40ab \quad (82)$$

Using (82) in (2), we get the values of $x,\ y,$ and w

$$x(a,b) = 21a^2-63b^2+34ab$$
$$y(a,b) = 19a^2-57b^2-46ab$$
$$w(a,b) = 40a^2-120b^2-12ab \quad (83)$$

Properties:

1) $x(1,n)+14T_{11}(n)+5N^2(n)-15T_4(n)$ is a pernicious number.
2) $-x(n,1)+21T_4(n)+17G(n)$ is an arithmetic number.
3) $-y(n,1)+2CS_{19}(n)+P_{27}(n)-27T_4(n)$ is a Saint-Exupery number.
4) $y(1,n)+6T_{21}(n)+97P(n)-97T_4(n)$ is a Pierpont prime.
5) $w(1,n)+120T_4(n)+6G(n)$ is a magnanimous number.
6) $-w(n,1)+4T_{22}(n)-4S(n)+24T_4(n)$ is a Curzon number.
7) $-x(n,1)-y(1,n)-6S(n)-24G(n)$ is a Ulam number.
8) $-y(n,1)-w(1,n)+2CS_{58}(n)-159T_4(n)$ is a Chen prime.
9) $-x(1,n)-w(n,1)-23P(n)-5T_{22}(n)+50T_4(n)$ is a Kaprekar number.
10) $x(1,n)+y(1,n)+w(1,n)+24[P_{10}(n)-2T_{15}(n)+13T_4(n)]$ is an amenable number.

Observation-2

$$\text{Write}\quad 403 = (16+i7\sqrt{3})(16-i7\sqrt{3}) \quad (84)$$

Using (4) and (84) in (80) and apply the factorization, we get

$$u+i\sqrt{3}v = (16+i7\sqrt{3})(a+ib\sqrt{3})^2$$
$$= (16a^2-48b^2-42ab)+i\sqrt{3}(7a^2-21b^2+32ab)$$

Equating real and imaginary parts on both sides, we get

$$u = 16a^2-48b^2-42ab$$
$$v = 7a^2-21b^2+32ab \quad (85)$$

Using (85) in (2), we get the values of $x,\ y,$ and w

$$\begin{aligned} x(a,b) &= 23a^2 - 69b^2 - 10ab \\ y(a,b) &= 9a^2 - 27b^2 - 74ab \\ w(a,b) &= 32a^2 - 96b^2 - 84ab \end{aligned} \quad (86)$$

Properties:

1) $x(1,n)+23N^2(n)+2CS_{59}(n)-59T_4(n)$ is a Jordan-Polya number.
2) $-x(n,1)+2T_{15}(n)-P_{11}(n)+11T_4(n)$ is a happy number.
3) $y(1,n)+2T_{29}(n)+99P(n)-99T_4(n)$ is a Cullen number.
4) $-y(n,1)+3N^2(n)+P_{83}(n)-83T_4(n)$ is a truncatable prime.
5) $-w(n,1)+CS_{64}(n)-26G(n)$ is a Lucas number.
6) $w(1,n)+16S(n)-18T_{24}((n)+198T_4(n)$ is an idoneal number.
7) $-x(n,1)-y(1,n)-4T_4(n)-42G(n)$ is a primitive abundant number.
8) $y(n,1)+w(1,n)+29N^2(n)+71P(n)-71T_4(n)$ is an interprime number.
9) $-x(1,n)-w(n,1)-2CS_{37}(n)-131P(n)+131T_4(n)$ is a strong prime.
10) $-x(1,n)-y(n,1)-w(n,1)-28T_4(n)-84G(n)$ is a congruent number.

IV. Conclusion

In this paper, a non-zero integral solution for the fourth degree homogeneous equation has been identified. In the future, it is possible to determine an alternative form of the equation involving more than four variables.

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