Equilibrium Analysis of the M/M/1 Unreliable Queue with Two-phase Vacations and Vacation Interruption

Ruiling Tian, Yuting Song, Yadan Xue, Yibo Hao

Abstract—**This paper considers the equilibrium behavior of M/M/1 unreliable queue which has two-phase vacations and vacation interruption. It enters buffer period, when the system becomes empty. If customers arrive within this time, the server transitions into the working state; otherwise, the server enters the working vacation, their service rates become slower. In addition, the working vacation can be end because of the vacation interruption. We discuss two information cases: the fully visible queue and the fully invisible queue. The strategy of equilibrium threshold and social benefit are considered in the first case. In the other case, we get the probability of the servers by using generating function. Then we calculate some performance measures and research the individual equilibrium strategy of customers and equilibrium social benefit. Moreover, we use numerical examples to demonstrate how information levels and system parameters impact equilibrium strategy and social benefit.**

Index Terms—**Queueing system; Equilibrium behavior; Unreliable queue; Vacation interruption; Two-phase vacations**

I. Introduction

UNRELIABLE queues have been studied in various fields, such as network data, communication systems, and regionalized production systems. Meanwhile many scholars have done related research, theoretical derivation and analysis of this model. The M/G/1 retrial queue's policy for server failures was described by Falin [1]. They used the Bessel function to derive the system performance measures. Lv [2] considered a repairable system which has two maintenance men and limited repairable machines. He derived the steady-state and transientstate indexes of the system. Then he studied the system performances. Tsai et al. [3] studied an opening queueing network with operating service stations that experience breakdowns while in operation. The usual assumption is that the server stops its operation during breakdowns, but in some special queueing systems, the server cannot be stopped completely without causing significant production losses, such as computer network systems. This phenomenon is called a working breakdown. Kim and Lee [4] got the length distribution and average sojourn time of an M/G/1 model, which has disasters and working breakdowns. This system also has standby servers, they obtained the length distribution and average sojourn time. Yang and Wu [5] examined working breakdowns and reneging in a Markovian queue.

The vacation is an extension of classical queueing theory and has wide applications in computer, communication, management engineering and manufacturing systems. This mechanism can reduce resource loss and system costs by making full use of idle periods. Working vacation policies were incorporated into queueing models by Servi and Finn [6], where customers arriving during vacations are still served. The introduction about working vacations, orbit search in the M/M/1 retrial queue was proposed by Li and Li [7]. Gao and Wang [8] introduced vacation policy into queue model to analyze the behavior of customers when only knew the status of the servers. Ye and Liu [9] studied the GI/M/1 queue with two vacation policies. Ye [10] considered the queueing systems with two-stage vacation policy and got the probabilities of steady-state. Anshul and Madhu [11] studied system which has bi-level vacations and two-phase service. Then they performed cost analysis and derived performance measures. However, in many practical applications, the system is shut down immediately after all customers are served, the server doesn't go on vacation immediately. They do some vacation preparations before the vacation, which is called "delayed vacation time". Leung [12] was the first to propose the delayed vacation strategy. Zhang [13] added the delayed vacation strategy to the multiple vacation queueing model and derived a stochastic decomposition. Yang et al. [14] studied working breakdowns in Markovian queues which also had delayed vacations. Then they obtained the steady-state probabilities using spectral expansion.

In real life, there may be unexpected events in some service systems, and at this time, the servers on vacation should resume work instead of continuing their vacation. This phenomenon is known as vacation interruption policy in queueing theory. If there are any customers left in the system, the vacation stops immediately to start normal work. Li et al. [15] discussed working vacations and interruption of vacation in the queueing model with four information levels. The random arrival strategy in queueing systems with vacation strategy was considered by Shekhar et al. [16]. Laxmi and Jyothsna [17] examined Bernoulli vacation interruption in the M/M/1 queues. They derived the steady-state probabilities and performance measures using the generating function.

In recent years, economic analysis in queues among customers has been studied by many scholars. Wang and Zhang [18] calculated the strategic analysis of the system with

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delayed repairs under observable conditions and breakdowns. The queue which has two types of breakdowns was studied in an economic perspective by Zhang and Xu [19]. Recently, an analysis of customer behavior in M/M/1 queues with double adaptable working vacations was conducted by Sun et al. [20]. Li et al. [21] considered M/M/1 queues with breakdowns and studied equilibrium strategies. Yu et al. [22] analyzed strategies of equilibrium in invisible queues with delayed repairs and balking. Liu and Wang [23] analyzed joining strategic in the queue which has only one server with Bernoulli vacations. Hao et al. [24] considered joining strategies in the Markovian queue which has setup times and N-policy. Sun and Li [25] investigated equilibrium and optimal social strategies under four information levels.

The structure of this paper is as follows. The strategies of equilibrium threshold for customers and the expected social benefit are discussed in Section 3. In Section 4, we get the stationary probability through generating function and develop several important performance measures. In addition, we consider the customers' individual strategy. Then the expected social benefit is considered in fully unobservable model. Next, numerical analysis are presented which in order to determine the influence of the system parameters. Then, the social benefit of two cases are compared. In Section 6, we draw conclusions based on our findings.

II. Model Description

The Markovian queue with two-phase vacations, unreliable machine and vacation interruptions is considered. Arriving customers obey Poisson process with λ. The rate of the service time is expected to be exponentially distributed with the value of μ. Only in normal busy period the machine maybe break down. The probability distribution of the machine's lifetime is given by an exponential distribution with the parameter η. When the server fails, it will be repaired at once. The repair time is an exponentially distributed random variable with ξ. When the system is empty, the system enters a buffer period, which is characterized by an exponential distribution with γ. The server transitions into a working vacation state if no customer comes during the buffer time. The working vacation duration is modeled by an exponential distribution with θ_1. Customers receive service at μ_v $(\mu_v < \mu)$ during the working vacation period. If there are people during the working vacation, the server transitions back to the normal condition. If not, the server proceeds with the classical vacation phase, which is determined by θ_2. At the end of vacation comes, if there are customers in the system, the normal busy period starts. Otherwise, another vacation is continued.

Furthermore, the assumption is made that the inter-arrival times, service times and vacation times are separate variables that do not affect each other. Let $(I(t), N(t))$ represent the system stste at time t, where $I(t)$ and $N(t)$ denote the server's state and the customers' number. Define

$$I(t) = \begin{cases} 0, & \text{the server is broken,} \\ 1, & \text{the server is busy,} \\ 2, & \text{the server is taking a working vacation,} \\ 3, & \text{the server is taking a vacation.} \end{cases}$$

Clearly, the process $\{(I(t), N(t)), t \geq 0\}$ is a continuous-time Markov chain. $\Omega = \{(i,n) \,|i = 0,1,2, n \geq 0\}$ is the stste space.

Assume that the arriving customers are identical and that the service is completed with a reward R. The system incurs a waiting cost of C units per time. Expected benefit is maximized by customers. In addition, the customer's decision is irrevocable.

In the following, we study fully observable and fully unobservable queues. In the first case, arriving customers know $I(t)$ and $N(t)$, while in another case, customers get no information at all.

III. Fully Obeservable Queue

We first consider the threshold strategy in the fully observable case. Assuming that the customer arrives at the state (i,n) and makes the decision whether to join the queue, they receives the benefit upon completion of the service is

$$S_{fo}(i,n) = R - CT(i,n),$$

where $T(i,n)$ is the expected sojourn time of a customer joining the system at the state (i,n). When the customer is over the threshold, the arriving customers refuse to join the queue. The equilibrium threshold strategy of customers is $n_e(i)$. The combined strategy can be represented as $(n_e(0), n_e(1), n_e(2), n_e(3))$.

Theorem 1 In the fully visible M/M/1 unreliable queue with two-phase vacations and vacation interruption, there exists an equilibrium threshold

$$(n_e(0), n_e(1), n_e(2), n_e(3)),$$

where

$$n_e(0) = \left\lfloor \tfrac{R\mu\xi}{C(\eta+\xi)} \right\rfloor - 1,$$

$$n_e(1) = \left\lfloor \tfrac{\mu\xi(R\xi - C)}{C\xi(\eta+\xi)} \right\rfloor - 1,$$

$$n_e(2) = \left\lfloor \tfrac{\mu\xi}{\eta+\xi}\left(\tfrac{R}{C} - \tfrac{1}{\mu_v+\theta} - \tfrac{\theta(\eta+\xi)}{\mu\xi(\mu_\nu+\theta)}\right) \right\rfloor,$$

$$n_e(3) = \left\lfloor \tfrac{\mu\xi(R\theta_v - C)}{C\theta_v(\eta+\xi)} \right\rfloor - 1.$$

This strategy satisfies the unique Nash equilibrium strategy.

Proof.

From assumption, we have

$$T(0,n) = \frac{1}{\xi} + T(1,n), n \geq 0, \tag{1}$$

$$T(1,0) = \frac{1}{\mu+\eta} + \frac{\eta}{\mu+\eta}T(0,0), \tag{2}$$

$$\begin{aligned} T(1,n) = &\frac{1}{\mu+\eta} + \frac{\eta}{\mu+\eta}T(0,0) \\ &+ \frac{\mu}{\mu+\eta}T(1,n-1), n \geq 1, \end{aligned} \tag{3}$$

$$T(2,0) = \frac{1}{\mu_v+\theta} + \frac{\theta}{\mu_v+\theta}T(1,0), \tag{4}$$

$$\begin{aligned} T(2,n) = &\frac{\mu_v}{\mu_v+\theta}T(1,n-1+)\frac{\theta}{\mu_v+\theta}T(1,n) \\ &+ \frac{1}{\mu_v+\theta}, n \geq 1, \end{aligned} \tag{5}$$

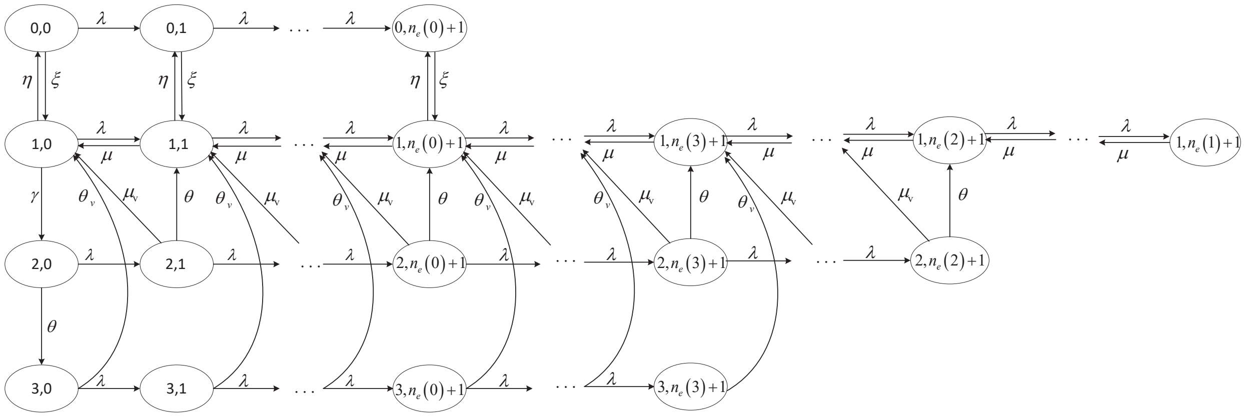

Fig. 1: State transition diagram for fully observable case

$$T\left(3,n\right)=\frac{1}{\theta_v}+T\left(1,n\right),n\geq 0. \tag{6}$$

Bringing (1) into (3) by iteration, we get

$$T\left(1,n\right)=\frac{\left(n+1\right)\left(\eta+\xi\right)}{\mu\xi},n\geq 0, \tag{7}$$

Plugging (7) into (1), (5) and (6), we have

$$T\left(0,n\right)=\frac{\left(n+1\right)\left(\eta+\xi\right)}{\mu\xi}+\frac{1}{\xi},n\geq 0, \tag{8}$$

$$T\left(2,n\right)=\frac{n\left(\eta+\xi\right)}{\mu\xi}+\frac{1}{\mu_v+\theta}+\frac{\theta\left(\eta+\xi\right)}{\mu\xi\left(\mu_v+\theta\right)},n\geq 0, \tag{9}$$

$$T\left(3,n\right)=\frac{\left(n+1\right)\left(\eta+\xi\right)}{\mu\xi}+\frac{1}{\theta_v},n\geq 0. \tag{10}$$

$T\left(i,n\right)$ is a function that monotonically increasing with respect to n. So $S_{fo}\left(i,n\right)$ is a monotonically decreasing function. If $S_{fo}\left(i,n\right)>0$, the customer joins the line. If $S_{fo}\left(i,n\right)=0$, it makes no difference whether the customer chooses to join or balk. By (7)-(10) and $S_{fo}\left(i,n\right)\geq 0$ solving for n, we get that if $n\leq n_e\left(I\left(t\right)\right)$ then the queue is joined by arriving customers.

If the threshold strategy described above are followed by all customers, the system can be treated as a Markov chain with a state space in steady condition. The state space is:

$$Q_{fo}=\left\{\left(i,n\right)|0\leq n\leq n_e\left(\mathrm{i}\right)+1,i=0,1,2,3\right\}.$$

The state transition diagram of the fully observable case is shown in Figure 1. The corresponding stationary distribution $\left\{P\left(i,n\right):\left(i,n\right)\in Q_{fo}\right\}$ are the following solutions.

$$\left(\lambda+\xi\right)P\left(0,0\right)=\eta P\left(1,0\right), \tag{11}$$

$$\begin{aligned}\left(\lambda+\xi\right)P\left(0,n\right)&=\eta P\left(1,n\right)+\lambda P\left(0,n-1\right),\\ n&=1,2,3,...,n_e\left(0\right),\end{aligned} \tag{12}$$

$$\xi P\left(0,n_e\left(0\right)+1\right)=\eta P\left(1,n_e\left(0\right)+1\right)+\lambda P\left(0,n_e\left(0\right)\right), \tag{13}$$

$$\begin{aligned}\left(\lambda+\eta+\gamma\right)P\left(1,0\right)=&\ \xi P\left(0,0\right)+\mu P\left(1,1\right)+\mu_v P\left(2,1\right)\\ &+\theta_v P\left(3,0\right),\end{aligned} \tag{14}$$

$$\begin{aligned}\left(\lambda+\eta+\mu\right)P\left(1,n\right)=&\ \xi P\left(0,n\right)+\mu P\left(1,n+1\right)\\ &+\mu_v P\left(2,n+1\right)+\theta_v P\left(3,n\right)\\ &+\lambda P\left(1,n-1\right)+\theta P\left(2,n\right),\\ n=&\ 1,2,...,3,n_e\left(0\right)+1,\end{aligned} \tag{15}$$

$$\begin{aligned}\left(\lambda+\mu\right)P\left(1,n\right)=&\ \mu P\left(1,n+1\right)+\mu_v P\left(2,n+1\right)\\ &+\theta_v P\left(3,n\right)+\lambda P\left(1,n-1\right)\\ &+\theta P\left(2,n\right),\\ n=&\ n_e\left(0\right)+2,\cdots,n_e\left(3\right)+1,\end{aligned} \tag{16}$$

$$\begin{aligned}\left(\lambda+\mu\right)P\left(1,n_e\left(2\right)+1\right)=&\ \mu P\left(1,n_e\left(2\right)+1\right)\\ &+\lambda P\left(1,n_e\left(2\right)\right)\\ &+\theta P\left(2,n_e\left(2\right)+1\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\left(\lambda+\mu\right)P\left(1,n\right)&=\mu P\left(1,n+1\right)+\lambda P\left(1,n-1\right),\\ n&=n_e\left(2\right)+2,\cdots,n_e\left(1\right),\end{aligned} \tag{18}$$

$$\mu P\left(1,n_e\left(1\right)+1\right)=\lambda P\left(1,n_e\left(1\right)\right), \tag{19}$$

$$\mu P\left(1,n_e\left(1\right)+1\right)=\lambda P\left(1,n_e\left(1\right)\right), \tag{20}$$

$$P\left(2,n\right)=\gamma P\left(2,n-1\right),n=1,2,\cdots,n_e\left(2\right), \tag{21}$$

$$\left(\theta+\mu_v\right)P\left(2,n_e\left(2\right)+1\right)=\gamma P\left(2,n_e\left(2\right)\right), \tag{22}$$

$$\left(\lambda+\theta_v\right)P\left(3,0\right)=\theta P\left(2,0\right), \tag{23}$$

$$\begin{aligned}\left(\lambda+\theta_v\right)P\left(3,n\right)&=\lambda P\left(3,n-1\right),\\ n&=1,2,\cdots,n_e\left(3\right),\end{aligned} \tag{24}$$

$$\theta_v P\left(3,n_e\left(3\right)+1\right)=\lambda P\left(3,n_e\left(3\right)\right). \tag{25}$$

Normalized equation:

$$\begin{aligned}&\sum_{n=0}^{n_e(0)+1}P\left(0,n\right)+\sum_{n=0}^{n_e(1)+1}P\left(1,n\right)+\sum_{n=0}^{n_e(2)+1}P\left(2,n\right)\\ &+\sum_{n=0}^{n_2(3)+1}P\left(3,n\right)=1.\end{aligned} \tag{26}$$

All above, the unique solution of steady-state probability is obtained. We get P_{balk} and L_{fo}. P_{balk} is the customer balking probability. The queue length marked by L_{fo}.

$$\begin{aligned}P_{balk} = & P\left(0, n_e\left(0\right)+1\right) + P\left(1, n_e\left(0\right)+1\right) \\ & + P\left(2, n_e\left(0\right)+1\right) + P\left(3, n_e\left(0\right)+1\right),\end{aligned}$$

$$\begin{aligned}L_{fo} = & \sum_{n=0}^{n_e(0)+1} nP\left(0, n\right) + \sum_{n=0}^{n_e(1)+1} nP\left(1, n\right) \\ & + \sum_{n=2}^{n_e(2)+1} nP\left(2, n\right) + \sum_{n=0}^{n_e(3)+1} nP\left(3, n\right).\end{aligned}$$

$SW_{fo} = R\lambda\left(1 - P_{balk}\right) - CL_{fo}$ represents the average social benefit per unit of time. Plugging P_{balk} and L_{fo} into SW_{fo} we have

$$\begin{aligned}SW_{fo} = & R\lambda(1 - P_{fo}(0, n_e(0)+1 - P_{fo}(1, n_e(0)+1) \\ & - P_{fo}(2, n_e(0)+1) - P_{fo}(3, n_e(0)+1) \\ & - C\left[\sum_{n=0}^{n_e(0)+1} nP_{fo}(0, n) + \sum_{n=0}^{n_e(1)+1} nP_{fo}(1, n)\right. \\ & \left. + \sum_{n=0}^{n_e(2)+1} nP_{fo}(2, n) + \sum_{n-0}^{n_e(3)+1} nP_{fo}(3, n)\right].\end{aligned}$$

IV. Fully Unbeservable Queue

When dealing with fully invisible case, the arriving customer knows nothing about the system. They join the queue with probability q. Then, $\lambda_{eff} = \lambda q$ is defined as effective arrival rate. Figure 2 shows the state transition diagram.

Let $P_i\left(n\right) = \lim_{t\to\infty} P\left\{I(t) = i, N(t) = n\right\}, (i, n) \in \Omega$ denote the stationary probability. Then we can get the stability condition, which is $\mu\xi > \lambda q\left(\eta + \xi\right)$. The followings are the balance equations.

$$\left(\lambda q + \xi\right) P_0\left(0\right) = \eta P_1\left(0\right), \tag{27}$$

$$\left(\lambda q + \xi\right) P_0\left(n\right) = \eta P_1\left(n\right) + \lambda q P_0\left(n-1\right), n \geq 1, \tag{28}$$

$$\begin{aligned}\left(\eta + \lambda q + \gamma\right) P_1\left(0\right) = & \xi P_0\left(0\right) + \mu P_1\left(1\right) + \mu_v P_2\left(1\right) \\ & + \theta_v P_3\left(0\right), n \geq 1,\end{aligned} \tag{29}$$

$$\begin{aligned}\left(\eta + \lambda q + \mu\right) P_1\left(n\right) = & \xi P_0\left(n\right) + \mu P_1\left(n+1\right) \\ & + \mu_v P_2\left(n+1\right) + \theta_v P_3\left(n\right) \\ & + \theta P_2\left(n\right) \\ & + \lambda q P_1\left(n-1\right), n \geq 1,\end{aligned} \tag{30}$$

$$\left(\lambda q + \theta\right) P_2\left(0\right) = \gamma P_1\left(0\right), \tag{31}$$

$$\left(\lambda q + \theta + \mu_v\right) P_2\left(n\right) = \lambda q P_2\left(n-1\right), n \geq 1, \tag{32}$$

$$\left(\theta_v + \lambda q\right) P_3\left(0\right) = \theta P_2\left(0\right), \tag{33}$$

$$\left(\theta_v + \lambda q\right) P_3\left(n\right) = \lambda q P_3\left(n-1\right), n \geq 1. \tag{34}$$

Define the partial generating function:

$$G_i\left(z\right) = \sum_{n=0}^{\infty} z^n P\left(i, n\right), i = 0, 1, 2, 3. \tag{35}$$

Equations (27)-(34) are multiplied by z^n, and summed over all n to obtain:

$$\left(\lambda q + \xi\right) G_0\left(z\right) = \eta G_1\left(z\right) + \lambda q z G_0\left(z\right), \tag{36}$$

$$\begin{aligned}& \left(\eta + \lambda q\right) G_1\left(z\right) + \mu\left[G_1\left(z\right) - P_1\left(0\right)\right] \\ & = -\gamma P_1\left(0\right) + \xi G_0\left(z\right) + \frac{\mu}{z}\left[G_1\left(z\right) - P_1\left(0\right)\right] \\ & \quad + \frac{\mu_\nu}{z}\left[G_2\left(z\right) - P_2\left(0\right)\right] + \theta_v G_3\left(z\right) \\ & \quad + \theta\left[G_2\left(z\right) - P_2\left(0\right)\right] + \lambda q z G_1\left(z\right),\end{aligned} \tag{37}$$

$$\begin{aligned}& \left(\lambda q + \theta\right) G_2\left(z\right) + \mu_v\left[G_2\left(z\right) - P_2\left(0\right)\right] \\ & \qquad = \gamma P_1\left(0\right) + \lambda q z G_2\left(z\right),\end{aligned} \tag{38}$$

$$\left(\theta_v + \lambda q\right) G_3\left(z\right) = \lambda q z G_3\left(z\right) + \theta P_2\left(0\right). \tag{39}$$

Combining (36)-(39), we get

$$\begin{aligned}\left(\mu - \lambda q z\right) G_1\left(z\right) = & \left(\mu - \gamma\right) P_1\left(0\right) + \lambda q z G_0\left(z\right) \\ & + \lambda q G_2\left(z\right) + \lambda q z G_3\left(z\right) \\ & + \theta G_2\left(z\right).\end{aligned} \tag{40}$$

Combining (36), (38) and (39) after a series of calculations, we have

$$\begin{aligned}G_0\left(z\right) = & K\left[\frac{\left(\lambda q + \theta\right)\left(\mu - \gamma\right)}{\gamma} + \frac{\left(\lambda q + \theta\right)\left(\lambda q + \theta + \mu_v\right)}{\lambda q\left(1 - z\right) + \theta + \mu_v}\right. \\ & \left. + \frac{\lambda q z \theta}{\lambda q\left(1 - z\right) + \theta_v}\right] P_2\left(0\right),\end{aligned} \tag{41}$$

where

$$K = \frac{\eta}{\left(\mu - \lambda q z\right)\left[\lambda q\left(1 - z\right) + \xi\right] - \lambda q \eta z}.$$

Substituting (41) into (36), we get

$$G_1\left(z\right) = \frac{\lambda q\left(1 - z\right) + \xi}{\eta} G_0\left(z\right). \tag{42}$$

By (38) and (39), we obtain

$$G_2\left(z\right) = \frac{\lambda q + \theta + \mu_v}{\lambda q\left(1 - z\right) + \theta + \mu_v} P_2\left(0\right), \tag{43}$$

$$G_3\left(z\right) = \frac{\theta}{\lambda q\left(1 - z\right) + \theta_v} P_2\left(0\right). \tag{44}$$

Substituting $z = 1$ into (41)-(44), and using the normalization condition, we can get

$$P_2\left(0\right) = \frac{\theta_v \gamma\left(\theta + \mu_v\right)\left[\xi\left(\mu - \lambda q\right) - \lambda q \eta\right]}{A}, \tag{45}$$

where

$$\begin{aligned}A = & (\xi + \eta)\{\theta\theta_v[\mu(\lambda q + \theta_v + \mu_v) + \lambda q \gamma] \\ & + \lambda q[\gamma\left(\theta^2 + \theta\mu_v + \lambda q \theta_v\right) + \theta_v \mu \mu_v]\} \\ & + \gamma[\xi\left(\mu - \lambda q\right) - \lambda q \eta][\theta_v(\lambda q + \theta + \mu_v) + \theta(\theta + \mu_{\mathrm{v}})].\end{aligned}$$

Then some important performance measures are presented. First, we get the probability in different states.

(1) In breakdown state:

$$\begin{aligned}P_0 & = G_0\left(1\right) \\ & = \frac{\eta\left(\theta + \mu_v\right)\left[\theta_v\left(\mu - \gamma\right)\left(\lambda q + \theta\right) + \lambda q \theta \gamma\right]}{\theta_v \gamma\left(\theta + \mu_{\mathrm{v}}\right)\left[\xi\left(\mu - \lambda q\right) - \lambda q \eta\right]} P_2\left(0\right) \\ & = \frac{\eta \theta_v \gamma\left(\lambda q + \theta\right)\left(\lambda q + \theta + \mu_v\right)}{\theta_v \gamma\left(\theta + \mu_{\mathrm{v}}\right)\left[\xi\left(\mu - \lambda q\right) - \lambda q \eta\right]} P_2\left(0\right).\end{aligned} \tag{46}$$

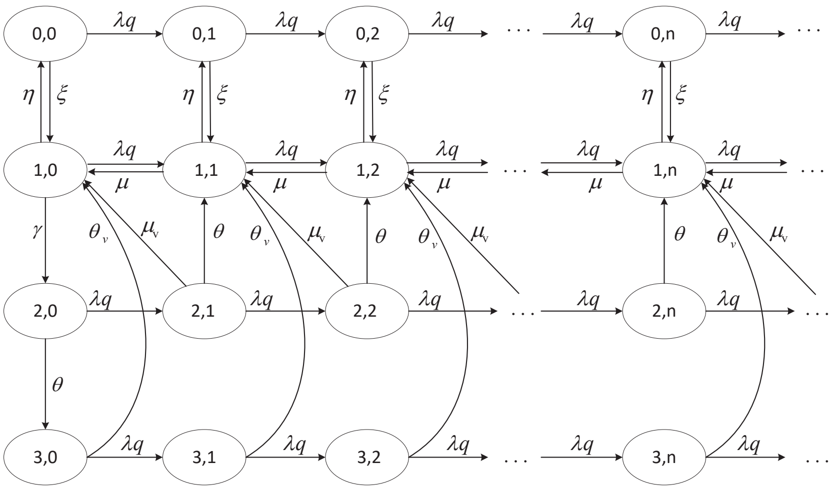

Fig. 2: State transition diagram for the fully unobservable case

(2) In normal busy state:

$$\begin{aligned} P_1 &= G_1(1) \\ &= \frac{\xi(\theta+\mu_v)\left[\theta_v(\mu-\gamma)(\lambda q+\theta)+\lambda q\theta\gamma\right]}{\theta_v\gamma(\theta+\mu_{\mathrm{v}})\left[\xi(\mu-\lambda q)-\lambda q\eta\right]}P_2(0) \\ &= \frac{\xi\theta_v\gamma(\lambda q+\theta)(\lambda q+\theta+\mu_v)}{\theta_v\gamma(\theta+\mu_{\mathrm{v}})\left[\xi(\mu-\lambda q)-\lambda q\eta\right]}P_2(0). \end{aligned} \tag{47}$$

(3) In working vacation state:

$$P_2 = G_2(1) = \frac{\lambda q+\theta+\mu_v}{\theta+\mu_v}P_2(0). \tag{48}$$

(4) In vacation state:

$$P_3 = G_3(1) = \frac{\theta}{\theta_v}P_2(0). \tag{49}$$

Subsequently, we calculate the mean number of customers.

(5) In broken state:

$$\begin{aligned} E[N_0] &= G_0'(1) \\ &= \frac{\eta\left[B+\lambda q\theta_v D(\theta+\mu_v)\left(\xi^2+\lambda q\eta+\eta\xi\right)\right]}{\xi\gamma\theta_v^2(\theta+\mu_v)^2[\xi(\mu-\lambda q)-\lambda q\eta]^2}P_2(0) \\ &\quad + \frac{\lambda q\eta\theta_v D(\mu+\theta_v)\left[\xi(\mu-\lambda q)-\lambda q\eta\right]}{\xi\gamma\theta_v^2(\theta+\mu_v)^2[\xi(\mu-\lambda q)-\lambda q\eta]^2}P_2(0). \end{aligned} \tag{50}$$

(6) In normal busy state:

$$\begin{aligned} E[N_1] &= G_1'(1) \\ &= \frac{\mathrm{B}+\lambda \mathrm{q}\theta_{\mathrm{v}}(\theta+\mu_{\mathrm{v}})\left(\xi^2+\lambda \mathrm{q}\eta+\eta\xi\right)\mathrm{D}}{\gamma\theta_{\mathrm{v}}^2(\theta+\mu_{\mathrm{v}})^2[\xi(\mu-\lambda \mathrm{q})-\lambda \mathrm{q}\eta]^2}\mathrm{P}_2(0). \end{aligned} \tag{51}$$

(7) In working vacation state:

$$E[N_2] = G_2'(1) = \frac{\lambda q(\lambda q+\theta+\mu_v)}{(\theta+\mu_v)^2}P_2(0). \tag{52}$$

(8) In vacation state:

$$E[N_3] = G_3'(1) = \frac{\lambda q\theta}{\theta_v^2}P_2(0). \tag{53}$$

(9) In the system:

$$\begin{aligned} E[N] &= E(N_0)+E(N_1)+E(N_2)+E(N_3) \\ &= \left\{\frac{\lambda q\theta_{\mathrm{v}}^2(\eta+\xi)(\lambda q+\theta)(\lambda q+\theta+\mu_{\mathrm{v}})}{\theta_v^2(\theta+\mu_v)^2\left[\xi(\mu-\lambda q)-\lambda q\eta\right]}\right. \\ &\quad + \frac{\lambda q\theta(\eta+\xi)(\lambda q+\theta)(\theta+\mu_v)^2}{\theta_v^2(\theta+\mu_v)^2\left[\xi(\mu-\lambda q)-\lambda q\eta\right]} \\ &\quad + \frac{\lambda q D\left[(\eta+\xi)^2+\eta\mu\right]}{\gamma\theta_v(\theta+\mu_v)\left[\xi(\mu-\lambda q)-\lambda q\eta\right]^2} \\ &\quad \left. + \frac{\lambda q\theta_v^2(\lambda q+\theta+\mu_v)+\lambda q\theta(\theta+\mu_v)^2}{\theta_v^2(\theta+\mu_v)^2}\right\}P_2(0). \end{aligned} \tag{54}$$

where

$$\begin{aligned} B &= \lambda q\gamma\xi(\lambda q+\theta)[\xi(\mu-\lambda q)-\lambda q\eta] \\ &\quad \times\left[\theta_v^2(\lambda q+\theta+\mu_v)+\theta(\theta+\mu_v)^2\right], \end{aligned}$$

$$\begin{aligned} D &= (\theta+\mu_v)\left[\theta_v(\mu-\gamma)(\lambda q+\theta)+\lambda q\theta\gamma\right] \\ &\quad + \theta_v\gamma(\lambda q+\theta)(\lambda q+\theta+\mu_{\mathrm{v}}). \end{aligned}$$

(10) The average sojourn time is given by:

$$\begin{aligned} E[W] &= \frac{E(N)}{\lambda q} \\ &= \left\{\frac{\theta_{\mathrm{v}}^2(\eta+\xi)(\lambda q+\theta)(\lambda q+\theta+\mu_{\mathrm{v}})}{\theta_v^2(\theta+\mu_v)^2\left[\xi(\mu-\lambda q)-\lambda q\eta\right]}\right. \\ &\quad + \frac{\theta(\eta+\xi)(\lambda q+\theta)(\theta+\mu_v)^2}{\theta_v^2(\theta+\mu_v)^2\left[\xi(\mu-\lambda q)-\lambda q\eta\right]} \\ &\quad + \frac{D\left[(\eta+\xi)^2+\eta\mu\right]}{\nu\theta_v(\theta+\mu_v)\left[\xi(\mu-\lambda q)-\lambda q\eta\right]^2} \\ &\quad \left. + \frac{\theta_v^2(\lambda q+\theta+\mu_v)+\theta(\theta+\mu_v)^2}{\theta_v^2(\theta+\mu_v)^2}\right\}P_2(0). \end{aligned} \tag{55}$$

(11) The balking probability of the system:

$$P_b = \lambda(1-q)\left[G_0(1)+G_1(1)+G_2(1)\right] = \frac{\gamma\lambda(1-q)(\theta_v-\theta)(\theta+\mu_v)\left[\xi(\mu-\lambda q)-\lambda\eta q\right]}{A}. \tag{56}$$

(12) The proportion that the server is working when it is in normal busy state:

$$\begin{aligned} P_N &= G_1(1)-P_1(1) \\ &= \frac{\xi(\theta+\mu_v)\left[\theta_v(\mu-\gamma)(\lambda q+\theta)+\lambda q\theta\gamma\right]}{A} \\ &\quad+\frac{\xi\theta_v\gamma(\lambda q+\theta)(\lambda q+\theta+\mu_v)}{A} \\ &\quad-\frac{\theta_v(\lambda \mathrm{q}+\theta)(\theta+\mu_v)\left[\xi(\mu-\lambda q)-\lambda q\eta\right]}{A}. \end{aligned} \tag{57}$$

(13) The proportion that the machine is working when it is in working vacation state:

$$P_w = G_2(1)-P_2(0) = \frac{\lambda q\theta_v\gamma(\theta+\mu_v)\left[\xi(\mu-\lambda q)-\lambda\eta q\right]}{A}. \tag{58}$$

(14) The probability that the server remains unoccupied:

$$\begin{aligned} P_I &= G_0(1)+G_3(1)+P_1(0)+P_2(0) \\ &= \frac{(\theta+\mu_v)\left[\xi(\mu-\lambda q)-\lambda q\eta\right]\left[\theta_v(\lambda q+\varsigma+\gamma)+\gamma\theta\right]}{A} \\ &\quad+\frac{\eta\theta_v\gamma(\lambda q+\theta)(\lambda q+\theta+\mu_v)}{A} \\ &\quad+\frac{\eta(\theta+\mu_v)\left[\theta_v(\mu-\gamma)(\lambda q+\theta)+\lambda q\theta\gamma\right]}{A}. \end{aligned} \tag{59}$$

Stochastic decomposition theory is an important part in vacation queues. It is often to decompose the stationary measures into two random variables. One part corresponds to the measures in the classical queueing model, and the other part is an additional variable caused by vacations. This theory plays a crucial role in the vacation model by illustrating how vacations affect performance measures. Next, we decompose two measures in our model.

Theorem 2 For $\rho<1$, the stationary queue length N can be decomposed into a sum of two variables as $N=N_c+N_d$. N_c is the length of the queue in the classical M/M/1 model. The other N_d is the queue length under the effect of the vacations policy. And N_d has the following PGF:

$$\begin{aligned} N_d(z) &= \frac{\left(\mu+\eta-\lambda q^2 z\right)(\mu-\lambda z)\left[\lambda q(1-z)+\xi\right]G_3(\mathrm{z})}{M} \\ &\quad+\frac{\left(\mu+\eta-\lambda q^2 z\right)(\mu-\lambda z)\left[\lambda q(1-z)+\xi\right]G_2(z)}{M} \\ &\quad+\frac{\left(\mu+\eta-\lambda q^2 z\right)(\lambda q\eta+\eta\theta-\lambda q\eta z)G_2(z)}{M} \\ &= \frac{(\lambda q+\theta)(\mu-\gamma)\left(\mu+\eta-\lambda q^2 z\right)}{N} \\ &\quad\times\frac{\left[\lambda q(1-z)+\xi+\eta\right]}{N}P_2(0), \end{aligned} \tag{60}$$

where

$$M = \left(\mu+\eta-\lambda q^2\right)(\mu-\lambda q z)\left[\lambda q(1-z)+\xi\right]- \lambda q\eta z\left(\mu+\eta-\lambda q^2\right),$$

$$N = \gamma\left(\mu+\eta-\lambda q^2\right)\left[\lambda q(1-z)-\gamma\right]- \lambda q\gamma\eta z\left(\mu+\eta-\lambda q^2\right).$$

Proof.

$$\begin{aligned} N(z) &= G_0(z)+G_1(z)+G_2(z)+G_3(z) \\ &= \frac{(\lambda q+\theta)(\mu-\gamma)\left[\lambda q(1-z)+\xi+\eta\right]}{\gamma(\mu-\lambda q z)\left[\lambda q(1-z)-\gamma\right]-\lambda q\gamma\eta z}P_2(0) \\ &\quad+\frac{\eta(\lambda q+\theta)+(\mu-\lambda q z)\left[\lambda q(1-z)+\xi\right]}{(\mu-\lambda q z)\left[\lambda q(1-z)+\xi\right]-\lambda q\eta z}G_2(z) \\ &\quad-\frac{\lambda q\eta z}{(\mu-\lambda q z)\left[\lambda q(1-z)+\xi\right]-\lambda q\eta z}G_2(z) \\ &\quad+\frac{\lambda q z\eta+(\mu-\lambda q z)\left[\lambda q(1-z)+\xi\right]-\lambda q\eta z}{(\mu-\lambda q z)\left[\lambda q(1-z)+\xi\right]-\lambda q\eta z}G_3(z) \\ &= \left(\frac{\mu+\eta-\lambda q^2}{\mu+\eta-\lambda q^2 z}\right)\times \\ &\quad\left\{\frac{\left(\mu+\eta-\lambda q^2 z\right)\left[\eta(\lambda q+\theta)-\lambda q\eta z\right]}{M}G_2(z)\right. \\ &\quad+\frac{\left(\mu+\eta-\lambda q^2 z\right)(\mu-\lambda q z)(\lambda q-\lambda q z+\xi)}{M}G_2(z) \\ &\quad+\frac{\left(\mu+\eta-\lambda q^2 z\right)(\mu-\lambda q z)(\lambda q-\lambda q z+\xi)}{M}G_3(z) \\ &\quad+\frac{\left(\mu+\eta-\lambda q^2 z\right)(\lambda q+\theta)(\mu-\gamma)}{N} \\ &\quad\left.\times\frac{(\lambda q-\lambda q z+\xi+\eta)}{N}P_2(0)\right\} \\ &= \left(\frac{1-\rho}{1-\rho z}\right)N_d(z) \end{aligned}$$

Theorem 3 If $\rho<1$, the stationary waiting time can be decomposed a sum of two variables as $W=W_c+W_d$. W_c is the customer's waiting time in the classical M/M/1 queue. W_d is the waiting under the effect of vacations. W_d has the following LST:

$$\begin{aligned} W_d^*(s) &= \frac{\left[\mu+\eta+q^2(s-\lambda)\right]\left[G_2(z)+G_3(z)\right]}{F} \\ &\quad\times\frac{\mu-\lambda q+s}{F} \\ &\quad+\frac{\eta\left[\mu+\eta+q^2(s-\lambda)\right](\theta+s)G_2(z)}{F} \\ &\quad+\frac{\left[\mu+\eta+q^2(s-\lambda)\right](\lambda q+\theta)}{H} \\ &\quad\times\frac{(\mu-\gamma)(s+\xi+\eta)}{H}P_2(0), \end{aligned} \tag{61}$$

where

$$F = \left(\mu+\eta-\lambda q^2\right)(\mu-\lambda q+s)(s+\xi)- \eta(\lambda q-s)\left(\mu+\eta-\lambda q^2\right),$$

$$H = \gamma\left(\mu+\eta-\lambda q^2\right)(\mu-\lambda q+s)(s-\gamma)- \gamma\eta(\lambda q-s)\left(\mu+\eta-\lambda q^2\right).$$

Proof.

From the Little's Law,

$$N(z) = W^*\left[\lambda(1-z)\right].$$

Let $s=\lambda(1-z)$, then we get $z=1-\frac{s}{\lambda}$. Bringing them into (60), we get the desired expression.

Subsequently, we consider the strategy of customer in individual equilibrium. The customer's benefit is 0, if he balks. Conversely, if he decides to enter the system to be served, the average net is

$$S(q) = R - CE[W],$$

where $E[W]$ is given by (55).

We define q_e is the equilibrium joining probability.

If $S(1) > 0$, this means their expected individual is positive, the customers choose to join the system, so $q_e = 1$.

If $S(0) < 0$, this means their expected individual is negative, the customers choose not to join the system, so $q_e = 0$.

In addition, when $S(1) < 0$ and $S(0) > 0$, q_e is an equilibrium joining probability, which means the customer's expected individual benefit is 0.

Then we establish the social benefit function per unit of time:

$$SW_{uo} = \lambda q (R - CE[W]).$$

Due to the complexity of the expressions for individual and social benefit, it is difficult to derive specific results through traditional calculations. In the following, we directly perform numerical analysis.

V. Numerical analysis

We delved into the sensitivity analysis of $n_e(i)$ for the first model and q_e for the second model in this part. Subsequently, we proceed to compare the equilibrium social gain of the two cases.

As intuitively expected, customers are preferring to enter the system when the server has the ability to serve more people in Figure 3. The impact of the broken rate η on the equilibrium threshold is examined in Figure 4. It is observed that equilibrium threshold is decrease with η. It is because when the server frequently transfers from a normal state to a broken state, the waiting time of customers will increase. In Figure 5, customers are perfer to join the line with higher repair rate. In the next figure, we get the relationship between the revenue R and the equilibrium threshold. As expected, with the revenue increases, customers prefer to join the queue. Also from Figure 3-6 we can see that the equilibrium threshold for the broken state is always smaller than the equilibrium threshold in the other states, this is because the server stopping working. This can increase the waiting time. So when people arrive and find the server in the broken state they will be more reluctant to join the queue than in the other states.

The impact of various parameters in second situation is depicted in Figures 7-11. From figure 7, we can know the effect of R and μ on the equilibrium joining probability. It can be seen that the probability is increasing with μ and becoming 1 after a certain point. The increased willingness of customers to join the queue is a result of the server's higher rate of service, which effectively decreases the waiting time. In addition, we learn that the system can add R to attract customers to join the queue. Figure 8 shows that the equilibrium joining probability starts at 1 and does not change with λ, then decreases with the increase of λ after a certain point. The reason of this phenomenon is that as the

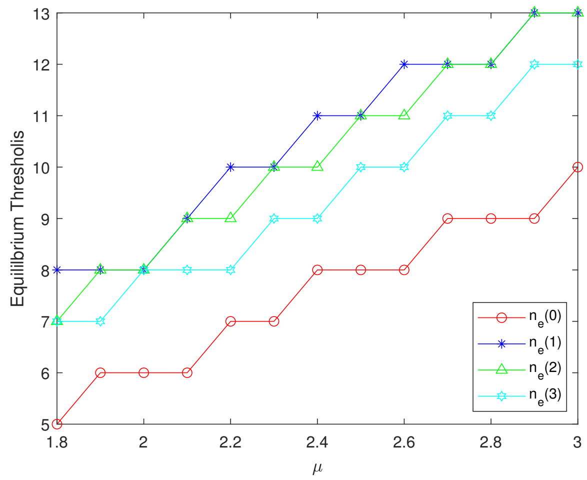

Fig. 3: Equilibrium thresholds for the fully observable model versus μ for $R = 30, C = 3, \xi = 0.4, \eta = 0.4, \mu_v = 0.5, \theta = 2, \theta_v = 1.$

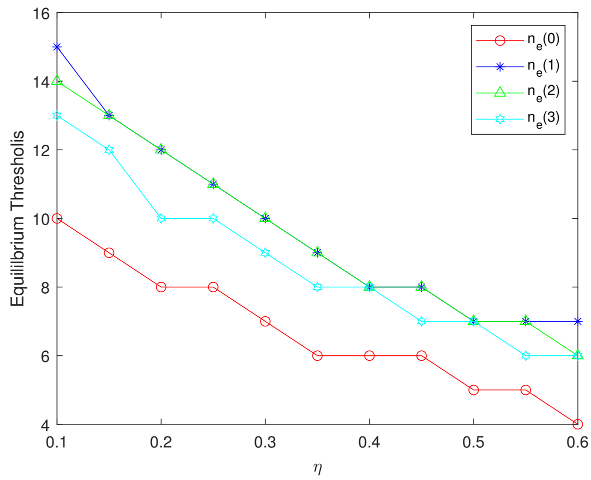

Fig. 4: Equilibrium thresholds for the fully observable model versus η for $R = 30, C = 3, \xi = 0.4, \mu = 2, \mu_v = 0.5, \theta = 2, \theta_v = 1.$

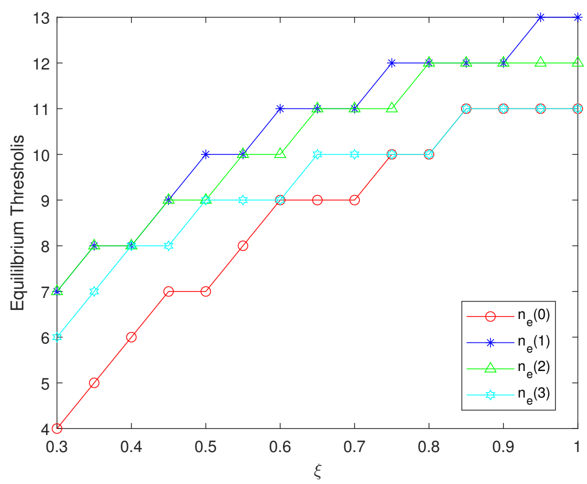

Fig. 5: Equilibrium thresholds for the fully observable model versus ξ for $R = 30, C = 3, \eta = 0.4, \mu = 2, \mu_v = 0.5, \theta = 2, \theta_v = 1.$

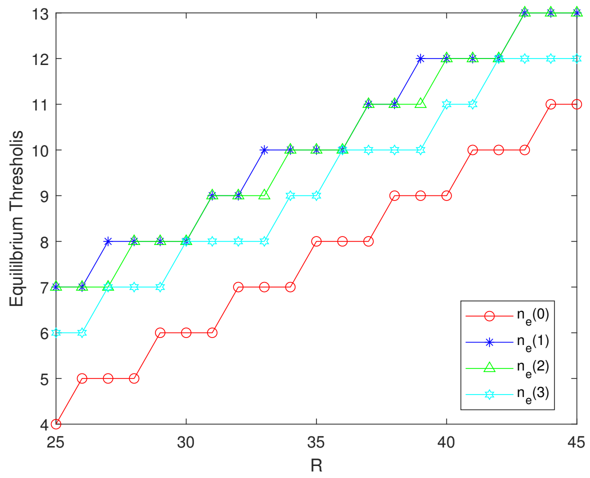

Fig. 6: Equilibrium thresholds for the fully observable model versus R for $C = 3, \xi = 0.4, \eta = 0.4, \mu = 2, \mu_v = 0.5, \theta = 2, \theta_v = 1$.

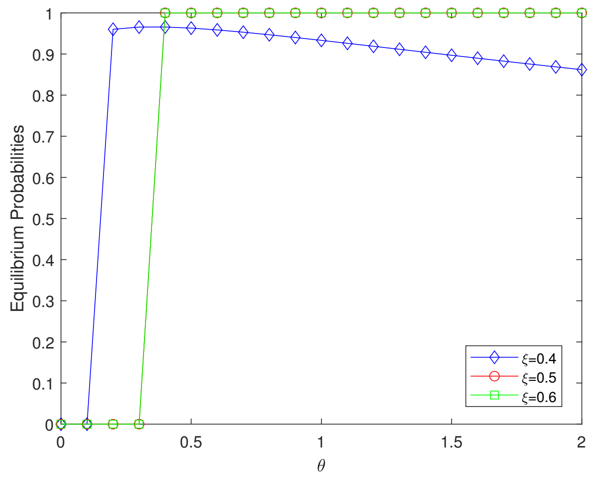

Fig. 9: Variation of the equilibrium probability of the fully unobservable model with ξ and θ for $R = 25, C = 3, \gamma = 1, \eta = 0.4, \mu = 2, \lambda = 0.8, \theta_v = 1, \mu_v = 0.5$.

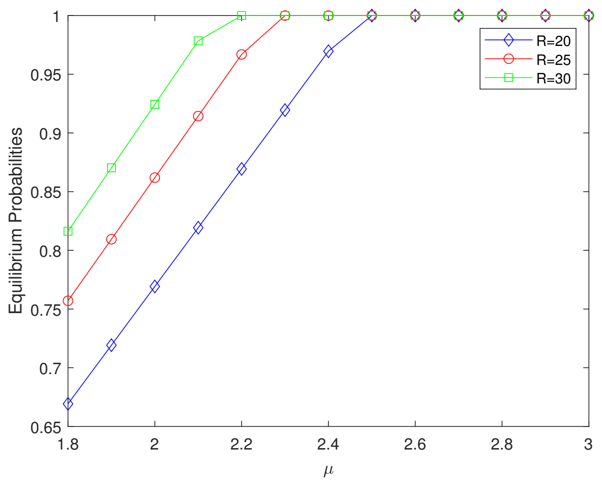

Fig. 7: Variation of the equilibrium probability of the fully unobservable model with R and μ for $C = 3, \gamma = 1, \lambda = 0.8, \eta = 0.4, \xi = 0.4, \theta_v = 1, \theta = 2, \mu_v = 0.5$.

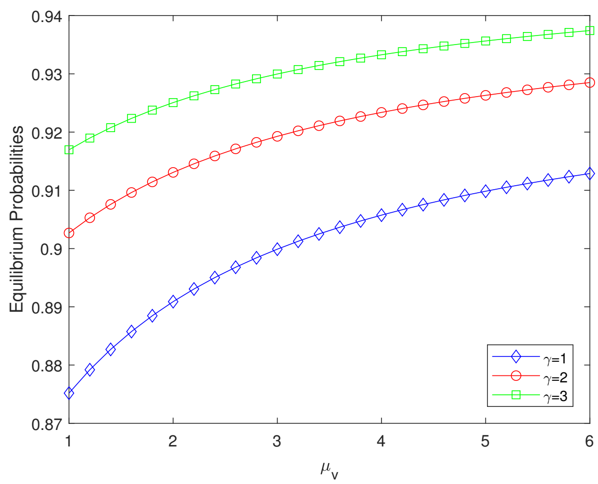

Fig. 10: Variation of equilibrium probability of fully unobservable models with γ and μ_v for $R = 25, C = 3, \eta = 0.4, \xi = 0.4, \mu = 2, \lambda = 0.8, \theta_v = 1, \theta = 2$.

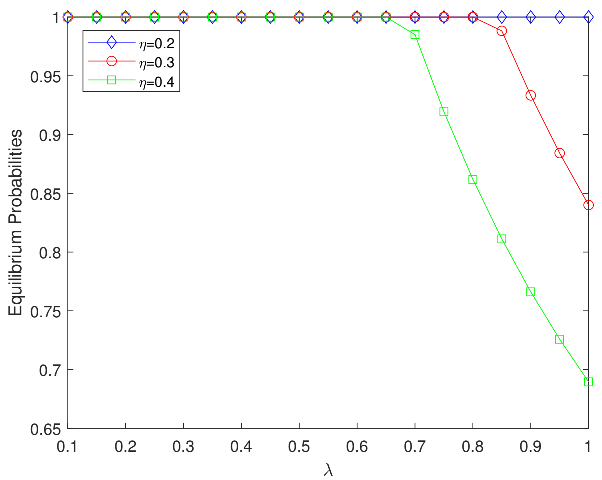

Fig. 8: Variation of the equilibrium probability of the fully unobservable model with η and λ for $R = 25, C = 3, \gamma = 1, \xi = 0.4, \mu = 2, \theta_v = 1, \theta = 2, \mu_v = 0.5$.

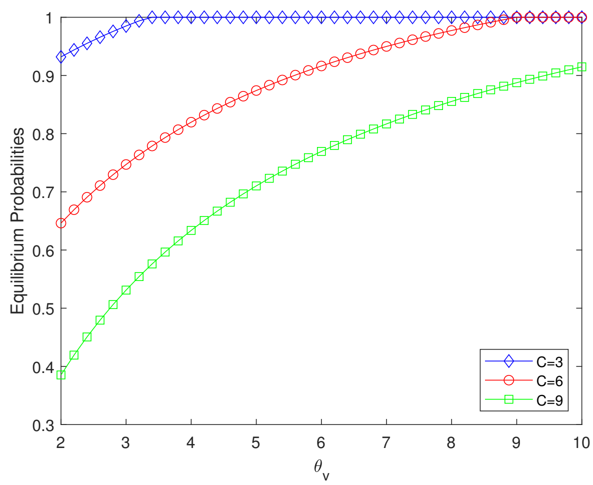

Fig. 11: Variation of equilibrium probability of fully unobservable models with C and θ_v for $R = 25, C = 3, \gamma = 1, \eta = 0.4, \xi = 0.4, \mu = 2, \lambda = 0.8, \theta = 2, \mu_v = 0.5$.

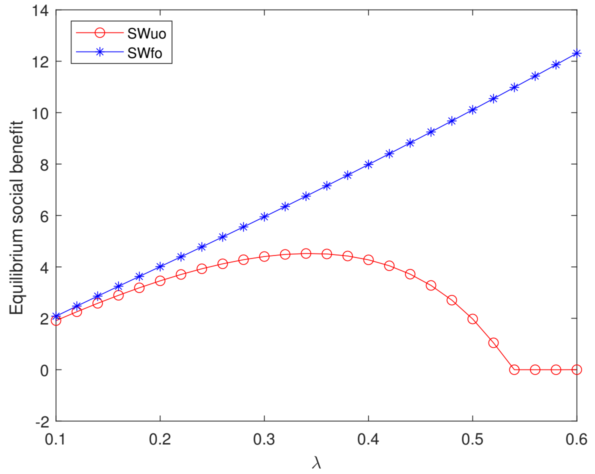

Fig. 12: Variation of equilibrium social benefit with λ under the two cases for $R = 30, C = 3, \gamma = 1, \mu = 2, \theta_v = 1, \theta = 2, \mu_v = 0.5, \eta = 0.6, \xi = 0.4$.

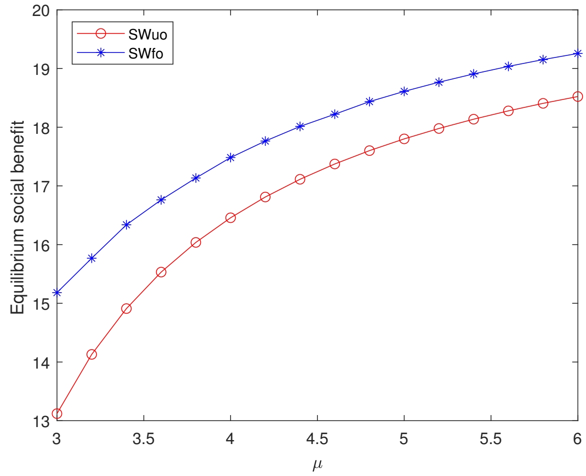

Fig. 14: Variation of equilibrium social benefit with μ under the two cases for $R = 30, C = 3, \lambda = 0.8, \gamma = 1, \theta_v = 1, \theta = 2, \mu_v = 0.5, \eta = 0.4, \xi = 0.4$.

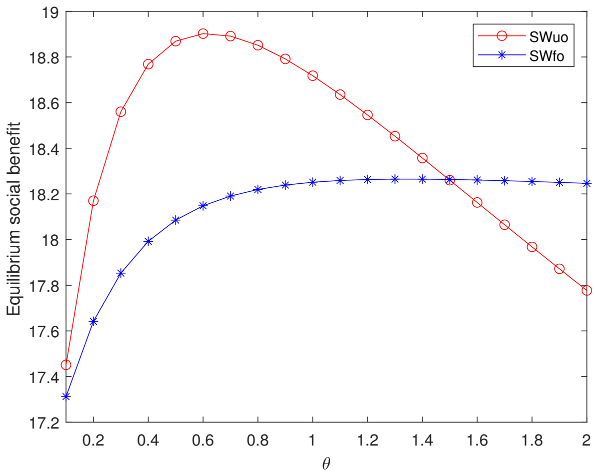

Fig. 13: Variation of equilibrium social benefit with θ under the two cases for $R = 30, C = 3, \lambda = 0.8, \gamma = 1, \mu = 2, \theta_v = 1, \mu_v = 0.5, \eta = 0.4, \xi = 1$.

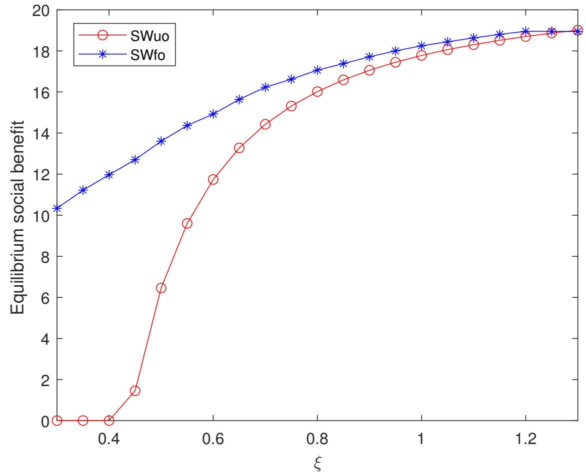

Fig. 15: Variation of equilibrium social benefit with ξ under the two cases for $R = 30, C = 3, \lambda = 0.8, \gamma = 1, \mu = 2, \theta_v = 1, \theta = 2, \mu_v = 0.5, \eta = 0.4$.

number of expected customers increases with λ, the waiting time of customers increases and customers are more inclined not to join the queue. For the same λ, as η increases, the queue fails to attract customers. Because the service will not be provided service during server broken. From Figure 9, it can be seen that at $\xi = 0.4$, the probability of equilibrium joining starts at 0 and does not change with θ, then increases suddenly, and decreases slowly after that point with the increase of θ. At $\xi = 0.5$ and $\xi = 0.6$, the equilibrium joining probability starts at 0 and does not change with the increase of θ, after a certain point the equilibrium probability becomes 1 with the increase of θ and then does not change with θ. In Figure 10, it is clear that the probability of joining the line increases with μ_v in the working vacation state, and increase with γ for a certain level of μ_v. In Figure 11, the equilibrium probability of customer joining system increases with the vacation rate for vacation in the case of $C = 3$ and $C = 6$ and does not change with the vacation time after increasing to 1. In the case of $C = 9$, it always increases with θ_v.

Figures 12-15 examine the equilibrium social benefit across various levels of information. The data presented in Figure 12 indicates that the social benefit of equilibrium is invariably greater in the fully visible case when compared to the other case. In Figure 13, the fully visible case demonstrates a noticeable trend where the social benefit initially grows with θ, but eventually reaches a point where it no longer increases. In the fully invisible case, the social benefit initially increases, reaches a peak, then decreases as θ. The data from Figure 14 clearly indicates that there is a positive correlation between μ and social welfare. And the social benefit is higher in the fully visible case compared to the other case. With the increase of μ, more customers are served, so the social benefits also increase accordingly. As we can see in Figure 15, with an increase in ξ, the social welfare rises in both cases. But the equilibrium joining probability of customers is 0, when ξ changes between 0 and 0.4. As we can see in Figures 12-15, the social benefit in the fully visible model is not always greater than it in the fully invisible model. This suggests that in equilibrium case, increasing in systemic information disclosure is not always conducive to increased social benefits. In some cases, increasing the accuracy of information can increase social gains, but in some cases, it can also hurt customers.

VI. Conclusion

We analyze customers' strategic behavior in the model with breakdowns, vacation interruptions, and two-stage vacations at different information levels. Based on their expected net benefits, arriving customers make a choice on whether or not to join the queue. In the first case, we get the customers' sojourn time and the strategies of equilibrium threshold. And in the other case, the derivation includes the stationary probability, performance measures, and equilibrium joining probability. Equilibrium social benefits are also calculated and compared for the two cases, and it is found that an increase in the degree of system information disclosure does not always benefit to an increase in social benefits.

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