Maps on C^*-algebras Bi-skew Lie Derivable or Bi-skew Lie Triple Derivable at the Unit

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***Abstract*—Let B be a unital C^*-algebra, A a unital subalgebra of B. We prove that a continuous map φ from A to B is a $*$-derivation if it is continuous and bi-skew Lie derivable or bi-skew Lie triple derivable at the unit. Moreover, we prove that every bi-skew Lie or bi-skew Lie triple homomorphism at the unit is a Jordan $*$-homomorphism.**

***Index Terms*—C^*-algebra, bi-skew Lie derivation, Jordan derivation, Jordan homomorphism.**

I. INTRODUCTION

THE study of homomorphisms and derivations is one of the most important work in the area of operator algebras and attracts many mathematicians' attentions. One of the main problem is to find weaker conditions to characterize these maps. In the past decades, a line was to study maps which are derivable or are homomorphisms at a certain point, especially at the unit or zero. A linear map φ from an algebra A to an A-bimodule X is called to be derivable at a point $z \in A$, if

$$\varphi(xy) = \varphi(x)y + x\varphi(y)$$

for all $x, y \in A$ with $xy = z$. If φ is derivable at every point in A, then φ is called a derivation on A. Similarly, if for all $x, y \in A$ with $xy = z$,

$$\varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)],$$

or

$$\varphi(x \circ y) = \varphi(x) \circ y + x \circ \varphi(y)$$

where$[x, y] = xy - yx$ and $x \circ y = xy + yx$, then φ is called Lie derivable or Jordan derivable at z. If φ is Jordan or Lie derivable at every point in A, then φ is called a Jordan or Lie derivation. Let $\{\phi_n\}_{n=0}^{\infty}$ be a sequence of (linear) maps with $\phi_0 = id_A$ on the algebra A, if

$$\sum_{i+j=n} \phi_n(xy) = \phi_i(x)\phi_j(y)$$

for all $x, y \in A$ with $xy = z$ and all $n \in \mathbb{N}$, then $\{\phi_n\}$ is called to be higher derivable at z. If $\{\phi_n\}$ is higher derivable at every point in A, then $\{\phi_n\}$ is called a higher derivation. Fix a point z in A, if a linear map which is (Lie, Jordan) derivable at z is in fact a (Lie, Jordan) derivation on A, then z is called an all-(Lie, Jordan) derivable point in A. Which points are all-(Lie, Jordan) derivable points is an interesting problem and has been flowering out scores of achievements in the past decades.

J. Zhu, Ch. Xiong, and P. Li proved in [2], if H is a Hilbert space, then every nonzero point is an all-derivable point in $B(H)$. J. Zhu and Ch. Xiong proved in [1] that every strongly continuous map on a nest algebra which is derivable at the unit is an inner derivation. Jing proved in [3] that zero is a generalized Jordan all-derivable point in $B(H)$. Hou and Qi proved in [6] that zero is a generalized all-derivable point in $\mathcal{J}$-subspace lattice algebra. They also proved that an additive map on a $\mathcal{J}$-subspace lattice algebra which is derivable at zero is the sum of an additive derivation and a scalar multiplication. In addition, they proved the unit is an all-derivable point in $\mathcal{J}$-subspace lattice algebra when the base space X is a complex Banach algebra. Liu and Zhang proved in [7] that every sequence of linear maps higher derivable at commutative zero point is a higher derivation, where commutative zero point means $xy = yx = 0$ rather than $xy = 0$. Xue, An and Hou proved in [8] that a linear map from a Hilbert space nest algebra to the operator algebra of the Hilbert space, which is Jordan derivable at a nontrivial idempotent P, is a derivation. Furthermore, they proved a linear map from a Banach space nest algebra to itself, which is Jordan derivable at a nontrivial idempotent P, is a derivation. For more, we refer the readers to [4], [5], [9], [10] and the references there in. There are also many works about homomorphisms at one point. For example, Catalano and Julius proved in [11] every homomorphism at a fixed $n \times n$ matrix is in fact an inner automorphism on $n \times n$ matrix algebra. For more, we refer the readers to [12], [13] and the references there in. For derivations on lattice or logical algebras we refer the readers to [14], [15] and the references there in.

In this paper we continue to study the linear maps which are derivable or are homomorphisms at the unit. We will focus on the bi-skew Lie product on a C^*-algebra A, which is defined by $[x, y]_\diamond = xy^* - yx^*$.

Definition 1. Let A be a C^*-algebra. Fix an element z in A, a linear map from A into a Banach A-bimodule X is called bi-skew Lie derivable at z if

$$\varphi([x, y]_\diamond) = [\varphi(x), y]_\diamond + [x, \varphi(y)]_\diamond \tag{1}$$

for all $x, y \in A$ with $xy = z$.

Similarly, bi-skew Lie triple product of x, y, z is defined by $[[x, y]_\diamond, z]_\diamond$.

Definition 2. A linear map from a C^*-algebra A into a Banach A-bimodule X is called a bi-skew Lie triple derivable at z if

$$\begin{aligned}&\varphi([[x, y]_\diamond, x]_\diamond)\\ =&[[\varphi(x), y]_\diamond, x]_\diamond + [[x, \varphi(y)]_\diamond, x]_\diamond + [[x, y]_\diamond, \varphi(x)]_\diamond\end{aligned}$$

for all $x, y \in A$ with $xy = z$.

If φ is bi-skew Lie (triple) derivable at every point, then it is called a bi-skew Lie (triple) derivation on A. It is

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clear that every bi-skew Lie (triple) derivation is bi-skew Lie (triple) derivable at any one point. An interesting question is if the inverse is true. In [16], Kong and Zhang proved every nonlinear bi-skew Lie derivation on a factor von Neumann algebra whose dimension is greater than or equal to 2 is an additive $*$-derivation. Following the methods in [17], we will prove in section 2 that every continuous linear map bi-skew Lie (triple) derivable at the unit on a unital C^*-algebra is a $*$-derivation. And in section 3, we will prove every continuous linear bi-skew Lie (triple) homomorphism at the unit on a unital C^*-algebra is a Jordan $*$-homomorphism (see section 3 for the concrete definitions).

II. Maps bi-skew-Lie (triple) derivable at the unit

The works in this section will focus on the study of linear maps between C^*-algebras which are continuous and bi-skew-Lie (triple) derivable at the unit. Through out this section, A denote a unital C^*-subalgebra of a unital C^*-algebra B, and A contains the unit of B. For any self-adjoint element a in A, and any real number t, e^{ita} denote the functional calculation of a by the continuous function $\lambda \mapsto \mathrm{e}^{it\lambda}$ on $\mathbb{C}$. We begin this section with a lemma about a property of linear continuous map which is bi-skew-Lie derivable at the unit.

Lemma II.1. *If φ is a linear continuous map from A to B and is bi-skew-Lie derivable at the unit, then $\varphi(1_A) = 0$ and $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$.*

Proof: For any $a \in A_{sa}$ and any real number t, e^{ita} is a unitary in A, and $[\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond = \mathrm{e}^{2ita} - \mathrm{e}^{-2ita}$. Hence we get that

$$\begin{aligned}
&\varphi(\mathrm{e}^{2ita} - \mathrm{e}^{-2ita}) \\
=&[\varphi(\mathrm{e}^{ita}), \mathrm{e}^{-ita}]_\diamond + [\mathrm{e}^{ita}, \varphi(\mathrm{e}^{-ita})]_\diamond \\
=&\varphi(\mathrm{e}^{ita})\mathrm{e}^{ita} - \mathrm{e}^{-ita}\varphi(\mathrm{e}^{ita})^* + \mathrm{e}^{ita}\varphi(\mathrm{e}^{-ita})^* \\
&- \varphi(\mathrm{e}^{-ita})\mathrm{e}^{-ita}.
\end{aligned}$$

By taking derivative at t, we obtain that

$$\begin{aligned}
&2\varphi(a\mathrm{e}^{2ita}) + 2\varphi(a\mathrm{e}^{-2ita}) \\
=&\varphi(a\mathrm{e}^{ita})\mathrm{e}^{ita} + \varphi(\mathrm{e}^{ita})a\mathrm{e}^{ita} \\
&+ a\mathrm{e}^{-ita}\varphi(\mathrm{e}^{ita})^* + \mathrm{e}^{-ita}\varphi(a\mathrm{e}^{ita})^* \\
&+ a\mathrm{e}^{ita}\varphi(\mathrm{e}^{-ita})^* + \mathrm{e}^{ita}\varphi(a\mathrm{e}^{-ita})^* \\
&+ \varphi(a\mathrm{e}^{-ita})\mathrm{e}^{-ita} + \varphi(\mathrm{e}^{-ita})a\mathrm{e}^{-ita}.
\end{aligned} \tag{2}$$

Put $t = 0$ in (2), then

$$\varphi(a) = \varphi(a)^* + \varphi(1_A)a + a\varphi(1_A)^*,$$

from which we deduce that $\varphi(1_A) = \varphi(1_A) + 2\varphi(1_A)^*$ by putting $a = 1_A$. So $\varphi(1_A) = 0$ and so $\varphi(a) = \varphi(a)^*$ for all $a \in A_{sa}$. If x is an arbitrary element in A , then there are $a, b \in A_{sa}$ such that $x = a + ib$. Hence,

$$\varphi(x^*) = \varphi(a - ib) = \varphi(a) - i\varphi(b) = \varphi(x)^*.$$

■

With the above lemma, we are now ready for the proof of the following main theorem.

Theorem II.2. *Let φ from A to B be as in the Lemma II.1, then it is a $*$-derivation.*

Proof: By Lemma II.1, we get $\varphi(1_A) = 0$. Then by taking derivative of (2) at $t = 0$, we obtain

$$\varphi(a^2) = \varphi(a)a + a\varphi(a), \quad a \in A_{sa}.$$

By replacing a by $b + c$ for any $b, c \in A_{sa}$, we get

$$\varphi(bc + cb) = \varphi(b)c + b\varphi(c) + \varphi(c)b + c\varphi(b).$$

For any $x \in A$, there are $b, c \in A_{sa}$ such that $x = b + ic$. Hence

$$\begin{aligned}
\varphi(x^2) =&\varphi(b^2 - c^2 + i(bc + cb)) \\
=&\varphi(b^2) - \varphi(c^2) + i(\varphi(bc + cb)) \\
=&\varphi(b)b + b\varphi(b) - \varphi(c)c - c\varphi(c) \\
&+ i(\varphi(b)c + b\varphi(c) + \varphi(c)b + c\varphi(b)) \\
=&\varphi(x)x + x\varphi(x).
\end{aligned}$$

Namely, φ is a Jordan derivation. It follows from [18, Theorem 6.3] that φ is a $*$-derivation. ■

For bi-skew-Lie triple derivable maps, we can prove the following Lemma similar as the case of bi-skew-Lie derivable maps.

Lemma II.3. *If φ is a continuous linear map from A to B and is bi-skew-Lie triple derivable at the unit, then $\varphi(1_A) = 0$ and $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$.*

Proof:

For any $a \in A_{sa}$ and any real number t, e^{ita} is a unitary in A. Hence,

$$\begin{aligned}
&\varphi(\mathrm{e}^{ita} - \mathrm{e}^{-3ita} - \mathrm{e}^{-ita} + \mathrm{e}^{3ita}) \\
=&\varphi([[\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond, \mathrm{e}^{ita}]_\diamond) \\
=&[[\varphi(\mathrm{e}^{ita}), \mathrm{e}^{-ita}]_\diamond, \mathrm{e}^{ita}]_\diamond + [[\mathrm{e}^{ita}, \varphi(\mathrm{e}^{-ita})]_\diamond, \mathrm{e}^{ita}]_\diamond \\
&+ [[\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond.
\end{aligned}$$

By taking derivative at t, we get

$$\begin{aligned}
&\varphi(ia\mathrm{e}^{ita}) + 3\varphi(ia\mathrm{e}^{-3ita}) + \varphi(ia\mathrm{e}^{-ita}) + 3\varphi(ia\mathrm{e}^{3ita}) \\
=&[[\varphi(ia\mathrm{e}^{ita}), \mathrm{e}^{-ita}]_\diamond, \mathrm{e}^{ita}]_\diamond + [[\varphi(\mathrm{e}^{ita}), -ia\mathrm{e}^{-ita}]_\diamond, \mathrm{e}^{ita}]_\diamond \\
&+ [[\varphi(\mathrm{e}^{ita}), \mathrm{e}^{-ita}]_\diamond, ia\mathrm{e}^{ita}]_\diamond \\
&+ [[ia\mathrm{e}^{ita}, \varphi(\mathrm{e}^{-ita})]_\diamond, \mathrm{e}^{ita}]_\diamond + [[\mathrm{e}^{ita}, \varphi(-ia\mathrm{e}^{-ita})]_\diamond, \mathrm{e}^{ita}]_\diamond \\
&+ [[\mathrm{e}^{ita}, \varphi(\mathrm{e}^{-ita})]_\diamond, ia\mathrm{e}^{ita}]_\diamond \\
&+ [[ia\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond + [[\mathrm{e}^{ita}, -ia\mathrm{e}^{-ita}]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond \\
&+ [[\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond, \varphi(ia\mathrm{e}^{ita})]_\diamond.
\end{aligned} \tag{3}$$

Put $t = 0$ in (3), we get

$$4\varphi(a) + 4\varphi(a)^* + 4a\varphi(1_A)^* + 4\varphi(1_A)a = 8\varphi(a). \tag{4}$$

So $\varphi(1_A) = 0$ by putting $a = 1_A$, and so $\varphi(a) = \varphi(a)^*$. As in the proof of Lemma II.1, we can show that for all $x \in A$, $\varphi(x^*) = \varphi(x)^*$. ■

Theorem II.4. *If φ from A to B is as in the Lemma II.3, then it is a $*$-derivation.*

Proof: Taking the first and the second derivative of (3) at $t = 0$, we get

$$2\varphi(a^3) = \varphi(a^2)a + a\varphi(a^2) + \varphi(a)a^2 + a^2\varphi(a).$$

Replace a by $a + 1_A$, then a direct calculation shows that

$$\varphi(a^2) = \varphi(a)a + a\varphi(a), \quad a \in A_{sa}.$$

As what we have done in the proof of Theorem II.2, put $x \in A$, then $x = a + ib$ for some $a, b \in A_{sa}$. Hence

$$\begin{aligned}\varphi(x^2) =&\varphi(a^2 - b^2 + i(ab + ba))\\ =&\varphi(a^2) - \varphi(b^2) + i(\varphi(ab + ba))\\ =&\varphi(a)a + a\varphi(a) + \varphi(b)b + b\varphi(b)\\ &+ i(\varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a))\\ =&\varphi(x)x + x\varphi(x).\end{aligned}$$

Namely, φ is a Jordan derivation. It follows from [18, Theorem 6.3] that φ is a $*$-derivation. ■

III. Bi-skew-Lie (triple) homomorphisms at the unit

Every C^*-algebra becomes a Jordan algebra by introducing the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. If a linear map φ from a C^*-algebra A into another C^*-algebra B satisfies $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all x, y in A, then it is called a Jordan homomorphism. If furthermore, $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$, then it is called a Jordan $*$-homomorphism. Replace x by $x + y$, then a direct calculation shows that the linear map φ is a Jordan homomorphism if and only if $\varphi(x^2) = \varphi(x)^2$ for all x in A. In this section, we prove that every continuous linear bi-skew-Lie (triple) homomorphism at the unit is a Jordan $*$-homomorphism. Firstly, we give the definition of a bi-skew-Lie (triple) homomorphism at the unit.

Definition 3. Let φ be a linear map from a C^*-algebra A to another C^*-algebra B. If

$$\varphi([x, y]_\diamond) = [\varphi(x), \varphi(y)]_\diamond$$

or

$$\varphi([[x, y]_\diamond, x]_\diamond) = [[\varphi(x), \varphi(y)]_\diamond, \varphi(x)]_\diamond$$

for all $x, y \in A$ with $xy = 1$, then φ is called a bi-skew-Lie or bi-skew-Lie triple homomorphism at the unit.

Now we explore the behavior of the unit under a linear continuous bi-skew-Lie homomorphism at the unit. In the following, A, B will always be unital C^*-algebras, A need not be a C^*-subalgebra of B.

Lemma III.1. *If φ from A to B is a continuous linear bi-skew-Lie homomorphism at the unit, then $\varphi(1_A)$ is a projection.*

Proof: As in the proof of Lemma II.1, since for any $a \in A_{sa}$ and any real number t, e^{ita} is a unitary in A, and $[\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond = \mathrm{e}^{2ita} - \mathrm{e}^{-2ita}$, we get that

$$\begin{aligned}&\varphi(\mathrm{e}^{2ita}) - \varphi(\mathrm{e}^{-2ita})\\ =&[\varphi(\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond\\ =&\varphi(\mathrm{e}^{ita})\varphi(\mathrm{e}^{-ita})^* - \varphi(\mathrm{e}^{-ita})\varphi(\mathrm{e}^{ita})^*.\end{aligned}$$

By taking derivative at t, we obtain that

$$\begin{aligned}&2\varphi(a\mathrm{e}^{2ita}) + 2\varphi(a\mathrm{e}^{-2ita})\\ =&\varphi(a\mathrm{e}^{ita})\varphi(\mathrm{e}^{-ita})^* + \varphi(\mathrm{e}^{ita})\varphi(a\mathrm{e}^{-ita})^*\\ &+ \varphi(a\mathrm{e}^{-ita})\varphi(\mathrm{e}^{ita})^* + \varphi(\mathrm{e}^{-ita})\varphi(a\mathrm{e}^{ita})^*. \qquad (5)\end{aligned}$$

We get that $\varphi(1_A) = \varphi(1_A)\varphi(1_A)^*$ by putting $a = 1_A$ and $t = 0$ in (5). So $\varphi(1_A)^* = \varphi(1_A)\varphi(1_A)^* = \varphi(1_A)$ and $\varphi(1_A)^2 = \varphi(1_A)$. Namely, $\varphi(1_A)$ is a projection. ■

If furthermore, we assume $\varphi(1_A) = 1_B$, then we get the following main theorem.

Theorem III.2. *If $\varphi : A \to B$ is as in the Lemma III.1 and maps the unit of A to the unit of B, then it is a Jordan $*$-homomorphism.*

Proof: Since $\varphi(1_A) = 1_B$, we deduce from (5) by putting $t = 0$ that $\varphi(a) = \varphi(a)^*$ for all $a \in A_{sa}$. As in the proof of Lemma II.1, by putting $x = a + ib$ with $a, b \in A_{sa}$, we obtain that $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$.

Taking the first derivative of (5) at t we get

$$\begin{aligned}&4\varphi(a^2\mathrm{e}^{2ita}) - 4\varphi(a^2\mathrm{e}^{2ita})\\ =&\varphi(a^2\mathrm{e}^{ita})\varphi(\mathrm{e}^{-ita})^* + \varphi(a\mathrm{e}^{ita})\varphi(a\mathrm{e}^{-ita})^*\\ &+ \varphi(a\mathrm{e}^{ita})\varphi(a\mathrm{e}^{-ita})^* + \varphi(\mathrm{e}^{ita})\varphi(a^2\mathrm{e}^{-ita})^*\\ &- \varphi(a^2\mathrm{e}^{-ita})\varphi(\mathrm{e}^{ita})^* - \varphi(a\mathrm{e}^{-ita})\varphi(a\mathrm{e}^{ita})^*\\ &- \varphi(a\mathrm{e}^{-ita})\varphi(a\mathrm{e}^{ita})^* - \varphi(\mathrm{e}^{-ita})\varphi(a^2\mathrm{e}^{ita})^*. \qquad (6)\end{aligned}$$

Taking a new derivative of (6) at $t = 0$, noting that $\varphi(1_A) = 1_B$, we get

$$2\varphi(a^3) = \varphi(a^2)\varphi(a) + \varphi(a)\varphi(a^2).$$

Replacing a by $a + 1_A$, we get

$$\begin{aligned}&2\varphi(1_A) + 6\varphi(a) + 4\varphi(a)^2 + 2\varphi(a^2)\\ &+ \varphi(a^2)\varphi(a) + \varphi(a)\varphi(a^2)\\ =&2\varphi(1_A) + 6\varphi(a) + 6\varphi(a^2) + 2\varphi(a^3).\end{aligned}$$

Hence, $\varphi(a^2) = \varphi(a)^2$ for all $a \in A_{sa}$. Replacing a by $a+b$ for any a and b in A_{sa}, we get $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$. Now for any $x \in A$, we can written x as $x = a+ib$ for some $a, b \in A_{sa}$, so

$$\begin{aligned}\varphi(x^2) =&\varphi((a^2 - b^2) + i(ab + ba))\\ =&\varphi(a)^2 - \varphi(b)^2 + i(\varphi(a)\varphi(b) + \varphi(b)\varphi(a))\\ =&\varphi(x)^2.\end{aligned}$$

Therefore, φ is a Jordan $*$-homomorphism. ■

Generally, $\varphi(1_A)$ need not to be equal to 1_B. In the following, we explore more properties about φ, and drop the assumption $\varphi(1_A) = 1_B$ in the above theorem.

Theorem III.3. *If φ from A to B is a continuous linear bi-skew-Lie homomorphism at the unit, then $\varphi(1_A)$ is a projection such that $\varphi(A) \subset \varphi(1_A)B\varphi(1_A)$ and φ is a Jordan $*$-homomorphism.*

Proof: It follows from Lemma III.1 that $\varphi(1_A)$ is a projection. Put $t = 0$ in (5), we get

$$4\varphi(a) = 2\varphi(a)\varphi(1_A) + 2\varphi(1_A)\varphi(a)^*. \qquad (7)$$

Multiplying $\varphi(1_A)^\perp$ from both the left and right sides of (7), we get

$$\varphi(1_A)^\perp\varphi(a)\varphi(1_A)^\perp = 0.$$

Similarly, multiplying $\varphi(1_A)^\perp$ from the left side and $\varphi(1_A)$ from the right side of (7), we get

$$4\varphi(1_A)^\perp\varphi(a)\varphi(1_A) = 2\varphi(1_A)^\perp\varphi(a)\varphi(1_A).$$

So $\varphi(1_A)^\perp\varphi(a)\varphi(1_A) = 0$ and $\varphi(1_A)\varphi(a)^*\varphi(1_A)^\perp = 0$. Then multiplying $\varphi(1_A)^\perp$ from the right side and $\varphi(1_A)$ from the left side of (7), we get $\varphi(1_A)\varphi(a)\varphi(1_A)^\perp = 0$. Hence $\varphi(a) = \varphi(1_A)\varphi(a)\varphi(1_A)$ for all $a \in A_{sa}$. By putting

$x = a + ib$ with $a, b \in A_{sa}$, we deduce from the linearity of φ that $\varphi(A) \subset \varphi(1_A)B\varphi(1_A)$.

Since $\varphi(1_A)$ is the unit of $\varphi(1_A)B\varphi(1_A)$, we can regard φ as a map from A into $\varphi(1_A)B\varphi(1_A)$ which maps 1_A to the unit. Hence, by Theorem III.2, φ is a Jordan $*$-homomorphism. ■

A bi-skew-Lie triple homomorphism at the unit has similar properties as a bi-skew-Lie homomorphisms at the unit. In this case, $\varphi(1_A)$ is no longer a projection. However, it is a selfadjoint partial isometry. We will prove this in the following.

Recall that an element v is called a partial isometry if v^*v is a projection, see for example [19]. By a direct calculation, one can show v is a partial isometry if and only if $vv^*v = v$ and $v^*vv^* = v^*$.

Lemma III.4. *If φ from A to B is a continuous linear bi-skew-Lie triple homomorphism at the unit, then $\varphi(1_A)$ is a selfadjoint partial isometry.*

Proof: For any $a \in A_{sa}$ and any real number t, e^{ita} is a unitary in A. Hence,

$$\begin{aligned}&\varphi(\mathrm{e}^{ita} - \mathrm{e}^{-3ita} - \mathrm{e}^{-ita} + \mathrm{e}^{3ita})\\ =&\varphi([[\mathrm{e}^{ita}, \mathrm{e}^{-ita}]_\diamond, \mathrm{e}^{ita}]_\diamond)\\ =&[[\varphi(\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond.\end{aligned}$$

Taking derivative at t, we get

$$\begin{aligned}&\varphi(ia\mathrm{e}^{ita}) - \varphi(-3ia\mathrm{e}^{-3ita}) - \varphi(-ia\mathrm{e}^{-ita}) + \varphi(3ia\mathrm{e}^{ita})\\ =&[[\varphi(ia\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond\\ &+ [[\varphi(\mathrm{e}^{ita}), \varphi(-ia\mathrm{e}^{-ita})]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond\\ &+ [[\varphi(\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond, \varphi(ia\mathrm{e}^{ita})]_\diamond. \qquad (8)\end{aligned}$$

Put $t = 0$ in (8), then we get

$$\begin{aligned}&\varphi(1_A)\varphi(a)^*\varphi(1_A)^* + \varphi(a)(\varphi(1_A)^*)^2 + \varphi(1_A)\varphi(a)\varphi(1_A)^*\\ &+ \varphi(1_A)^2\varphi(a)^* = 4\varphi(a).\end{aligned}$$

Put $a = 1_A$, then

$$\varphi(1_A)(\varphi(1_A)^*)^2 + \varphi(1_A)^2\varphi(1_A)^* = 2\varphi(1_A).$$

So $\varphi(1_A)$ is selfadjoint and $\varphi(1_A)^3 = \varphi(1_A)$. ■

Theorem III.5. *If φ from A to B is a continuous linear bi-skew-Lie triple homomorphism at the unit and $\varphi(1_A) = 1_B$, then φ is a Jordan $*$-homomorphism.*

Proof: Since $\varphi(1_A) = 1_B$, it follows from (9) that $2\varphi(a) + 2\varphi(a)^* = 4\varphi(a)$. So $\varphi(a) = \varphi(a)^*$ and so as in the proof of Lemma II.1, $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$.

Taking derivative of (8) at t, we get

$$\begin{aligned}&\varphi(-a^2\mathrm{e}^{ita}) - \varphi(-9a^2\mathrm{e}^{-3ita})\\ &- \varphi(-a^2\mathrm{e}^{-ita}) + \varphi(-9a^2\mathrm{e}^{3ita})\\ =&[[\varphi(-a^2\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond\\ &+ [[\varphi(\mathrm{e}^{ita}), \varphi(-a^2\mathrm{e}^{-ita})]_\diamond, \varphi(\mathrm{e}^{ita})]_\diamond\\ &+ [[\varphi(\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond, \varphi(-a^2\mathrm{e}^{ita})]_\diamond\\ &+ 2[[\varphi(ia\mathrm{e}^{ita}), \varphi(-ia\mathrm{e}^{-ita})]_\diamond, \varphi(ia\mathrm{e}^{ita})]_\diamond\\ &+ 2[[\varphi(ia\mathrm{e}^{ita}), \varphi(\mathrm{e}^{-ita})]_\diamond, \varphi(ia\mathrm{e}^{ita})]_\diamond\\ &+ 2[[\varphi(\mathrm{e}^{ita}), \varphi(-ia\mathrm{e}^{-ita})]_\diamond, \varphi(ia\mathrm{e}^{ita})]_\diamond. \qquad (9)\end{aligned}$$

If we put $t = 0$ in (9), then we get an identical equation $0 = 0$. So we take a new derivative of (9) at $t = 0$, then it follows that

$$2\varphi(a^3) = \varphi(a^2)\varphi(a) + \varphi(a)\varphi(a^2).$$

Replace a by $1_A + a$, then a direct calculation shows that

$$\varphi(a^2) = \varphi(a)^2.$$

As in the proof of Theorem III.2, we can see that $\varphi(x^2) = \varphi(x)^2$ for all $x \in A$. So φ is a Jordan homomorphism. ■

By Lemma III.4, $\varphi(1_A)$ is a selfadjoint partial isometry, so $p := \varphi(1_A)^2$ is a projection. pBp is a subalgebra of B. In the following, we show that $\varphi(A) \subset pBp$.

Theorem III.6. *If $\varphi : A \to B$ is a linear continuous bi-skew-Lie triple homomorphism at the unit, then $\varphi(A) \subset pBp$. Furthermore, φ is a Jordan $*$-homomorphism if and only if $\varphi(1_A)$ is a projection.*

Proof: Since $\varphi(1_A)^* = \varphi(1_A)$, it follows from (9) that

$$\begin{aligned}&\varphi(1_A)\varphi(a)^*\varphi(1_A) + \varphi(a)(\varphi(1_A))^2 + \varphi(1_A)\varphi(a)\varphi(1_A)\\ &+ \varphi(1_A)^2\varphi(a)^* = 4\varphi(a), \qquad (10)\end{aligned}$$

which yields that $\varphi(a)^* = \varphi(a)$ for all $a \in A_{sa}$. As in the proof of Lemma II.1, we can show that $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$.

Notice that $(1_B - p)\varphi(1_A) = \varphi(1_A)(1_B - p) = 0$, by multiplying $p^\perp := 1_B - p$ from both the left and the right sides of (10), we get $p^\perp\varphi(a)p^\perp = 0$. Similarly, $p^\perp\varphi(a)p = p\varphi(a)p^\perp = 0$. So $\varphi(a) = p\varphi(a)p \in pBp$ and so $\varphi(A) \subset pBp$.

If φ is a Jordan $*$-homomorphism, then $\varphi(1_A)^2 = \varphi((1_A)^2) = \varphi(1_A)$. Hence $\varphi(1_A)$ is a projection. Conversely, if $\varphi(1_A)$ is a projection, then $\varphi(1_A) = p$ is the unit of pBp. We can regard φ as a map from A into pBp. By Theorem III.5, φ is a Jordan $*$-homomorphism.

■

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