Analytical Solutions of Time-Caputo-type and Space Riesz-type Distributed Order Diffusion Equation In Three Dimensional Space

Dayi Zheng, Jiangbin Chen

Abstract The difficulty in solving distributed order differential equations lies in the fact that the order of the derivative is distributed within a finite interval. This paper discusses the initial boundary value problem of three-dimensional diffusion equations of time Caputo type distribution order and the initial boundary value problem of three-dimensional diffusion equations of time Caputo type space Riesz type distribution order. The analytical solution of the initial boundary value problem of three-dimensional diffusion equations of time Caputo type distribution order is obtained using the separation of variables method, and the analytical solution of the initial boundary value problem of three-dimensional diffusion equations of time Caputo type space Riesz type distribution order is obtained using spectral method and Laplace transform.

Index Terms—Distributed order derivative, Caputo-type derivative, Riesz-type derivative, Spectral method

I. Introduction

After the concept of variable order integral and variable order derivative was proposed by Samko in 1993, the variable order derivative model was applied to the modeling of viscoelastic materials and viscous fluids [2] and the distributed order derivative of the order distribution of derivatives in a finite interval was also more and more widely used. For example, ultra-low speed diffusion or strong abnormal diffusion phenomena in polymer physics are usually described by the distributed order diffusion equation. The distribution order diffusion equation can also be used to describe the sub diffusion stochastic process that belongs to the Wiener process. Its diffusion index decreases with time. Many complex diffusion processes whose diffusion index changes with time, such as decelerated hyper-diffusion and accelerated slow diffusion, decelerated slow diffusion and accelerated hyperdiffusion, can be described by the distribution order diffusion equation. At present, distributed order differential equations have been widely used to describe the rheological properties of composite materials, signal control and processing, dielectric induction and diffusion, and stress-strain behavior of viscoelastic materials [3]. The research on distributed order differential equations has received attention in the past decade or two. Authors of [5, 6, 1] obtained a fundamental solution for the one-dimensional time-fractional diffusion equation and multi-dimensional diffusion-wave equation of distributed order. Authors of [8] studied the distributed order time-fractional diffusion equations characterized by multifractal memory kernels, in contrast to the simple power-law kernel of common time-fractional diffusion equations. An explicit strong solution and stochastic analogues for distributed order time-fractional diffusion equations are proposed in [7]. An improved meshless method for solving two-dimensional distributed order time-fractional diffusion-wave equation with error estimate are proposed in [4]. This paper discusses the initial boundary value problem of three-dimensional diffusion equations of time Caputo type distribution order and the initial boundary value problem of three-dimensional diffusion equations of time Caputo type space Riesz type distribution order. The analytical solution of the initial boundary value problem of three-dimensional diffusion equations of time Caputo type distribution order is obtained by the separation of variables method, and the analytical solution of the initial boundary value problem of three-dimensional diffusion equations of time Caputo type space Riesz type distribution order is obtained byspectral method and

Manuscript received May 25, 2023; revised October 28, 2023; This work was supported by Natural Science Foundation of Fujian Province (2019J01783) and Young and Middle-Aged Teachers Education Scientific Research Projects of Fujian Province Education Department (JAT201525), New Century Excellent Talents Support Program of Higher Education in Fujian Province(2017).

D. Y. Zheng is an associate professor of Mathematics Research Department, Fujian Institute of Education, Fuzhou Fujian 350025, China (e-mail: 3454231513@qq.com).

J. B. Chen is an associate professor of Zhicheng College, FuZhou University, Fuzhou, Fujian 350001, China (corresponding author to provide phone: 0591-83769360; fax: 0591-83769360; e-mail: jbchen@fzu.edu.cn).

Laplace transform.

II. Preliminaries

Definition 2.1. ***(Caputo type derivative fractional derivative)***

$$ {}^C_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-a)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (n-1 < \alpha \le n, t > a). $$

Definition 2.2. ***(Riesz fractional derivative on bounded intervals [10])***

$$ \frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) = -c_\alpha ({}_0D_x^\alpha + {}_xD_L^\alpha) u(x,t), \quad (n-1 < \alpha \le n, 0 \le x \le L), $$

where

$$ c_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})}, \ \alpha \ne 1, $$

$$ {}_0D_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}} d\xi, $$

$$ {}_xD_L^\alpha u(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_x^L \frac{u(\xi,t)}{(\xi-x)^{\alpha+1-n}} d\xi. $$

Lemma 2.1. *([10]) For the function $u(x)$ defined on infinite intervals, when $n-1 < \alpha < n$, we have*

$$ -(-\Delta_x)^{\frac{\alpha}{2}} u(x) = -\frac{1}{2\cos(\frac{\pi\alpha}{2})} [{}_{-\infty}D_x^\alpha u(x) + {}_xD_\infty^\alpha u(x)] = \frac{\partial^\alpha}{\partial |x|^\alpha} u(x). $$

After defining $u^*(x)$ as follow

$$ u^*(x) = \begin{cases} u(x), & x \in (0,L) \\ 0, & x \notin (0,L) \end{cases} $$

we get the following Corollary by Lemma 2.1

Corollary 2.1. *([10])*

$$ -(-\Delta)^{\frac{\alpha}{2}} u^*(x) = -\frac{1}{2\cos(\frac{\pi\alpha}{2})} [{}_0D_x^\alpha u(x) + {}_xD_L^\alpha u(x)] = \frac{\partial^\alpha}{\partial |x|^\alpha} u(x). $$

Lemma 2.2. *([10]) Let $\{\phi_n\}$ be a sequence of orthogonal eigenfunctions of Laplace operator $-\Delta$ defined on closed and bounded region D and satisfy*

$$ (-\Delta)\phi_n = \lambda_n^2 \phi_n \ \ and \ \ B(\phi) = 0 \ \ on \ \ \partial D $$

where $B(\phi)$ represents any one of three Boundary Conditions.

Denote

$$ \mathfrak{R}_\gamma = \{ f = \sum_{n=1}^\infty c_n \phi_n, \ c_n = \langle f, \phi_n \rangle, \ \sum_{n=1}^\infty |c_n|^2 |\lambda|_n^\gamma < \infty \} $$

where $\gamma = \max(\alpha, 0)$. Then

$$ (-\Delta)^{\frac{\alpha}{2}} f = \sum_{n=1}^\infty c_n (\lambda_n^2)^{\frac{\alpha}{2}} \phi_n, \quad f \in \mathfrak{R}_\gamma. $$

Proposition 2.1. *([9]) Laplace transform for Caputo type fractional order derivative*

$$ L\{ {}^C_0 D_t^\mu f(t); s\} = s^\mu F(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} f^{(k)}(0) $$

where $n-1 < \mu \le n$.

III. The analytical solutions of the initial value problem (IBVP) of three-dimensional diffusion equations of time Caputo type distribution order

Consider the following IBVP of 3D diffusion equation of time Caputo type distribution order

$$ \begin{cases} \int_0^1 {}_0D_t^\mu u(x,y,z,t) d\mu = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} \\ \qquad + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}, \\ 0 < x < L_1, 0 < y < L_2, 0 < z < L_3, \end{cases} \tag{1} $$

subject to

$$ \begin{cases} u(0,y,z,t) = u(L_1,y,z,t) = 0, \\ u(x,0,z,t) = u(x,L_2,z,t) = 0, \\ u(x,y,0,t) = u(x,y,L_3,t) = 0, \\ u(x,y,z,0) = g(x,y,z). \end{cases} \tag{2} $$

We set

$$ u(x,y,z,t) = T(t)Y(x)H(y)P(z), \tag{3} $$

substitute (3) to (1), then it follows

$$ Y(x)H(y)P(z) \int_0^1 {}_0D_t^\mu T(t) d\mu = T(t)H(y)P(z) \frac{d^2Y(x)}{dx^2} + T(t)Y(x)P(z) \frac{d^2H(y)}{dy^2} + T(t)Y(x)H(y) \frac{d^2P(z)}{dz^2}. $$

Divided on both sides of above equation by $T(t)Y(x)H(y)P(z)$, the equation is changed into

$$ \frac{\int_0^1 {}_0D_t^\mu T(t) d\mu}{T(t)} = \frac{\frac{d^2Y(x)}{dx^2}}{Y(x)} + \frac{\frac{d^2H(y)}{dy^2}}{H(y)} + \frac{\frac{d^2P(z)}{dz^2}}{P(z)}. \tag{4} $$

Substitute (3) to (2), then we have

$$ \begin{cases} T(t)Y(0)H(y)P(z) = T(t)Y(L_1)H(y)P(z) = 0, \\ T(t)Y(x)H(0)P(z) = T(t)Y(x)H(L_2)P(z) = 0, \\ T(t)Y(x)H(y)P(0) = T(t)Y(x)H(y)P(L_3) = 0, \\ T(0)Y(x)H(y)P(z) = g(x,y,z). \end{cases} \tag{5} $$

By observing equation (4), obviously we can see that the left side is a function only with respect to variable t while there are three functions about x, y and z respectively on the right side. Therefore (4) holds on whole domain only if three functions on right side are all constants as follows

$$\frac{\frac{d^2Y(x)}{dx^2}}{Y(x)} = -\lambda_1,\ \frac{\frac{d^2H(y)}{dy^2}}{H(y)} = -\lambda_2,\ \frac{\frac{d^2P(z)}{dz^2}}{P(z)} = -\lambda_3.$$

Thus

$$\frac{d^2Y(x)}{dx^2} + \lambda_1 Y(x) = 0, \quad (6)$$
$$\frac{d^2H(y)}{dy^2} + \lambda_2 H(y) = 0, \quad (7)$$
$$\frac{d^2P(z)}{dz^2} + \lambda_3 P(z) = 0, \quad (8)$$
$$\int_0^1 {}_0D_t^{\mu}T(t)d\mu + (\lambda_1+\lambda_2+\lambda_3)T(t) = 0. \quad (9)$$

We discuss the general solutions of eqation (6) in there different cases:

Case 1: when $\lambda_1 < 0$, the general solutions to (6) are $Y(x) = c_1 e^{\sqrt{-\lambda_1}x} + c_2 e^{-\sqrt{-\lambda_1}x}$.

Formula (5) imply

$$Y(0)T(t)H(y)P(z) = Y(L_1)T(t)H(y)P(z) = 0$$

and $T(t)$, $P(z)$ and $H(y)$ are all not zero functions, which indicate

$$Y(0) = Y(L_1) = 0.$$

Namely

$$\begin{cases} Y(0) = c_1 + c_2 = 0, \\ Y(L_1) = c_1 e^{\sqrt{-\lambda_1}L_1} + c_2 e^{-\sqrt{-\lambda_1}L_1} = 0, \end{cases}$$

by the fact

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda_1}L_1} & e^{-\sqrt{-\lambda_1}L_2} \end{vmatrix} = e^{-\sqrt{-\lambda_1}L_1} - e^{\sqrt{-\lambda_1}L_1} \neq 0,$$

we get $c_1 = c_2 = 0$, and the solution to (6) is trivial.

Case 2: when $\lambda_1 = 0$, the general solution to (6) are $Y(x) = c_1 + c_2 x$.

Thus

$$\begin{cases} Y(0) = c_1 + 0 = 0, \\ Y(L_1) = c_1 + c_2 L_1 = 0, \end{cases}$$

hence $c_1 = c_2 = 0$. So the general solution is trivial too in this case.

Case 3: when $\lambda_1 > 0$, the general solution (6) are $Y(x) = c_1\cos(\sqrt{\lambda_1}x) + c_2\sin(\sqrt{\lambda_1}x)$.

Boundary conditions indicate

$$\begin{cases} Y(0) = c_1 = 0, \\ Y(L_1) = c_1\cos(\sqrt{\lambda_1}L_1) + c_2\sin(\sqrt{\lambda_1}L_1) = 0. \end{cases}$$

In order to get nontrivial solutions, it is necessary to set $\sin(\sqrt{\lambda_1}L_1) = 0$, which imply

$$\sqrt{\lambda_1}L_1 = n\pi,$$

and further more

$$\lambda_1 = \frac{n^2\pi^2}{L_1^2},\ (n = 1, 2, \cdots),$$

therefore

$$Y_n(x) = c_n \sin\frac{n\pi x}{L_1},\ n = 1, 2, \cdots.$$

As to equation (7) and (8), in the same way, there are trivial solutions on case 1 and case 2, and in case 3, we obtain

$$\lambda_2 = \frac{m^2\pi^2}{L_2^2},\ (m = 1, 2, \cdots),\ \lambda_3 = \frac{k^2\pi^2}{L_2^2},\ (k = 1, 2, \cdots),$$

thus

$$H_m(y) = c_m \sin\frac{m\pi y}{L_2},\ m = 1, 2, \cdots$$
$$P_k(z) = c_k \sin\frac{k\pi z}{L_3}, k = 1, 2, \cdots.$$

Then we consider equation (9), assume $\lambda_1 + \lambda_2 + \lambda_3 = \lambda$, we have

$$\int_0^1 {}_0D^{\mu}T(t)d\mu + \lambda T(t) = 0, \quad (10)$$

Taking Laplace transform with respect to t on both sides in (10), we get

$$\int_0^1 [s^{\mu}\widehat{T}(s) - s^{\mu-1}T(0)]d\mu + \lambda\widehat{T}(s) = 0.$$

By initial condition $u(x, y, z, 0) = g(x, y, z)$, we have $T(0) = d$ is a constant.

Furthermore, we get

$$\widehat{T}(s) = \frac{d(s-1)}{s(s-1+\lambda\ln s)}. \quad (11)$$

Taking inverse Laplace transform with respect to s on both sides of (11), we get

$$T(t) = \frac{d}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{s-1}{s(s-1+\lambda\ln s)} e^{st} ds, \quad (12)$$

substituting $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = \frac{n^2\pi^2}{L_1^2} + \frac{m^2\pi^2}{L_2^2} + \frac{k^2\pi^2}{L_3^2} = \lambda_{nmk}$ into (12), we have

$$T_{nmk}(t) = \frac{d}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{s-1}{s(s-1+\lambda_{nmk}\ln s)}e^{st}ds.$$

According to

$$Y_n(x) = c_n \sin\frac{n\pi x}{L_1}$$
$$H_m(y) = c_m \sin\frac{m\pi y}{L_2}$$
$$P_k(z) = c_k \sin\frac{k\pi z}{L_3},$$

we get

$$\begin{aligned}
&u_{nmk}(x,y,z,t)\\
=&T_{nmk}(t)Y_n(x)H_m(y)P_k(z)\\
=&f_{nmk}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3}\\
&\times\frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{s-1}{s(s-1+\lambda_{nmk}\ln s)}e^{st}ds
\end{aligned}$$

where $f_{nmk} = dc_nc_mc_k$ is any constant.

Superposition of $u_{nmk}(x,y,z,t)$ and satisfy the initial condition:

$$\begin{aligned}
&u(x,y,z,t)\\
=&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}f_{nmk}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3}\\
&\times\frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{s-1}{s(s-1+\lambda_{nmk}\ln s)}e^{st}ds.
\end{aligned}\tag{13}$$

From the initial condition $u(x,y,z,0) = g(x,y,z)$ and $T(0) = d$ as a constant, we obtain

$$\begin{aligned}
&u(x,y,z,0)\\
=&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}f_{nmk}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3}\\
=&g(x,y,z),
\end{aligned}$$

furthermore, we get

$$\begin{aligned}
f_{nmk} =&\frac{8}{L_1L_2L_3}\int_0^{L_3}\{\int_0^{L_2}[\int_0^{L_1}g(x,y,z)\\
&\sin\frac{n\pi x}{L_1}dx]\sin\frac{m\pi y}{L_2}dy\}\sin\frac{k\pi z}{L_3}dz.
\end{aligned}\tag{14}$$

Therefore, the solution of equation (1) and (2) is (13), where f_{nmk} is given by (14).

IV. The analytical solution of IBVP of 3-dimensional diffusion equations of time Caputo type space Riesz type distribution order

Consider the following IBVP of 3D diffusion equations of time Caputo type space Riesz type distribution order

$$\begin{cases}
\int_0^1 {}_0D_t^{\mu}u(x,y,z,t)d\mu = \int_1^2 \frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,y,z,t)d\alpha\\
+\int_1^2 \frac{\partial^{\beta}}{\partial|y|^{\beta}}u(x,y,z,t)d\beta + \int_1^2 \frac{\partial^{\gamma}}{\partial|z|^{\gamma}}u(x,y,z,t)d\gamma,\\
t\geq 0,\ 0<x<L_1,\ 0<y<L_2,\ 0<z<L_3,
\end{cases}\tag{15}$$

subject to

$$\begin{cases}
u(0,y,z,t) = u(L_1,y,z,t) = 0,\\
u(x,0,z,t) = u(x,L_2,z,t) = 0,\\
u(x,y,0,t) = u(x,y,L_3,t) = 0,\\
u(x,y,z,0) = g(x,y,z),
\end{cases}\tag{16}$$

where $u(x,y,z,t)$, $g(x,y,z)$ are real value functions and sufficiently smooth.

By separation of variables method used in sections 3, we set

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3},\tag{17}$$

from Corollary 2.1 and Lemma 2.2, we get

$$\begin{aligned}
&\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,y,z,t) = -(-\Delta_x)^{\frac{\alpha}{2}}u(x,y,z,t)\\
=&-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)(\frac{n^2\pi^2}{L_1^2})^{\frac{\alpha}{2}}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3},
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\frac{\partial^{\beta}}{\partial|y|^{\beta}}u(x,y,z,t) = -(-\Delta_y)^{\frac{\beta}{2}}u(x,y,z,t)\\
=&-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)(\frac{m^2\pi^2}{L_2^2})^{\frac{\beta}{2}}\sin\frac{n\pi x}{L_1}\\
&\times\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3},
\end{aligned}\tag{19}$$

$$\begin{aligned}
&\frac{\partial^{\gamma}}{\partial|z|^{\gamma}}u(x,y,z,t) = -(-\Delta_z)^{\frac{\gamma}{2}}u(x,y,z,t)\\
=&-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)(\frac{k^2\pi^2}{L_3^2})^{\frac{\gamma}{2}}\sin\frac{n\pi x}{L_1}\\
&\times\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3},
\end{aligned}\tag{20}$$

substituting (17), (18), (19) and (20) into (15), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3} \times\int_0^1 {}_0D_t^{\mu}T_{nmk}(t)d\mu$$
$$=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3} \times\int_1^2 (\frac{n^2\pi^2}{L_1^2})^{\frac{\alpha}{2}}d\alpha$$
$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3} \times\int_1^2 (\frac{m^2\pi^2}{L_2^2})^{\frac{\beta}{2}}d\beta$$
$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(t)\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3} \times\int_1^2 (\frac{k^2\pi^2}{L_3^2})^{\frac{\gamma}{2}}d\gamma. \quad (21)$$

(21) lead to

$$\int_0^1 {}_0D_t^{\mu}T_{nmk}(t)d\mu+[\int_1^2(\frac{n^2\pi^2}{L_1^2})^{\frac{\alpha}{2}}d\alpha +\int_1^2(\frac{m^2\pi^2}{L_2^2})^{\frac{\beta}{2}}d\beta+\int_1^2(\frac{k^2\pi^2}{L_3^2})^{\frac{\gamma}{2}}d\gamma]T_{nmk}(t)=0, \quad (22)$$

combining (16) and (17) indicates

$$u(x,y,z,0) =\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}T_{nmk}(0)\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3} =g(x,y,z)$$

and the Fourier coefficients is

$$T_{nmk}(0) =\frac{8}{L_1L_2L_3}\int_0^{L_3}\{\int_0^{L_2}[\int_0^{L_1}g(x,y,z)\sin\frac{n\pi x}{L_1}dx] \times\sin\frac{m\pi y}{L_2}dy\}\sin\frac{k\pi z}{L_3}dz.$$

Let

$$c_{nmk} =\int_1^2(\frac{n^2\pi^2}{L_1^2})^{\frac{\alpha}{2}}d\alpha+\int_1^2(\frac{m^2\pi^2}{L_2^2})^{\frac{\beta}{2}}d\beta+\int_1^2(\frac{k^2\pi^2}{L_3^2})^{\frac{\gamma}{2}}d\gamma$$

and take Laplace transform with respect to t on both side, we get

$$\int_0^1[s^{\mu}\widehat{T}_{nmk}(s)-s^{\mu-1}T_{nmk}(0)]d\mu+c_{nmk}\widehat{T}_{nmk}(s)=0,$$

and further more

$$\widehat{T}_{nmk}(s)=\frac{s-1}{s(s-1+c_{nmk}\ln s)}T_{nmk}(0). \quad (23)$$

Taking inverse Laplace transform with respect to s on both sides of (23), we get

$$T_{nmk}(t) =T_{nmk}(0)\frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{s-1}{s(s-1+c_{nmk}\ln s)}e^{st}ds =\frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{s-1}{s(s-1+c_{nmk}\ln s)}e^{st}ds\frac{8}{L_1L_2L_3} \times\int_0^{L_3}\{\int_0^{L_2}[\int_0^{L_1}g(x,y,z)\sin\frac{n\pi x}{L_1}dx]\sin\frac{m\pi y}{L_2}dy\}\sin\frac{k\pi z}{L_3}dz.$$

Substituting above formula to (17), we obtain solution to (15), (16)

$$u(x,y,z,t) =\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{4}{\pi L_1L_2L_3j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{(s-1)e^{st}}{s(s-1+c_{nmk}\ln s)}ds \times\int_0^{L_3}\{\int_0^{L_2}[\int_0^{L_1}g(x,y,z)\sin\frac{n\pi x}{L_1}dx]\sin\frac{m\pi y}{L_2}dy\}\sin\frac{k\pi z}{L_3}dz \times\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{k\pi z}{L_3}. \quad (24)$$

V. The analytical solution to IBVP of 3-dimensional nonhomogeneous diffusion equations of time Caputo type space Riesz type distribution order

Consider the following IBVP of 3D nonhomogeneous diffusion equations of time Caputo type space Riesz type distribution order

$$\begin{cases} \int_0^1 {}_0D_t^{\mu}u(x,y,z,t)d\mu \\ =\int_1^2\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,y,z,t)d\alpha+\int_1^2\frac{\partial^{\beta}}{\partial|y|^{\beta}}u(x,y,z,t)d\beta \\ \quad+\int_1^2\frac{\partial^{\gamma}}{\partial|z|^{\gamma}}u(x,y,z,t)d\gamma+f(x,y,z,t), \\ t\ge 0,\ 0<x<L_1,\ 0<y<L_2,\ 0<z<L_3, \end{cases} \quad (25)$$

subject to

$$\begin{cases} u(0,y,z,t)=u(L_1,y,z,t)=0, \\ u(x,0,z,t)=u(x,L_2,z,t)=0, \\ u(x,y,0,t)=u(x,y,L_3,t)=0, \\ u(x,y,z,0)=g(x,y,z), \end{cases} \quad (26)$$

where both $u(x,y,z,t)$ and $g(x,y,z)$ are real value functions which are sufficiently smooth.

By principle of superposition, the above problem can be equivalently transformed into following two problems:

(A)

$$\begin{cases} \int_0^1 {}_0D_t^{\mu} u(x,y,z,t)d\mu = \int_1^2 \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x,y,z,t)d\alpha \\ + \int_1^2 \frac{\partial^{\beta}}{\partial |y|^{\beta}} u(x,y,z,t)d\beta + \int_1^2 \frac{\partial^{\gamma}}{\partial |z|^{\gamma}} u(x,y,z,t)d\gamma, \\ 0 \le t \le T,\ 0 < x < L_1,\ 0 < y < L_2,\ 0 < z < L_3, \\ u(0,y,z,t) = u(L_1,y,z,t) = 0, \\ u(x,0,z,t) = u(x,L_2,z,t) = 0, \\ u(x,y,0,t) = u(x,y,L_3,t) = 0, \\ u(x,y,z,0) = g(x,y,z), \end{cases}$$

(B)

$$\begin{cases} \int_0^1 {}_0D_t^{\mu} u(x,y,z,t)d\mu = \int_1^2 \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x,y,z,t)d\alpha \\ + \int_1^2 \frac{\partial^{\beta}}{\partial |y|^{\beta}} u(x,y,z,t)d\beta + \int_1^2 \frac{\partial^{\gamma}}{\partial |z|^{\gamma}} u(x,y,z,t)d\gamma \\ + f(x,y,z,t), \\ 0 \le t \le T,\ 0 < x < L_1,\ 0 < y < L_2,\ 0 < z < L_3, \\ u(0,y,z,t) = u(L_1,y,z,t) = 0, \\ u(x,0,z,t) = u(x,L_2,z,t) = 0, \\ u(x,y,0,t) = u(x,y,L_3,t) = 0, \\ u(x,y,z,0) = 0. \end{cases}$$

Suppose $u(x,y,z,t)$ is solution to (25), (26), and $u^1(x,y,z,t)$, $u^2(x,y,z,t)$ are solutions to (A), (B) respectively, then $u(x,y,z,t) = u^1(x,y,z,t) + u^2(x,y,z,t)$, by the result of section IV, we know $u^1(x,y,z,t)$ is expressed as (24).
Assume the solution to (B) is the form

$$u^2(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} W_{nmk}(t) \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \sin\frac{k\pi z}{L_3}. \tag{27}$$

Expanding $f(x,y,z,t)$ in Fourier series as follow

$$f(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} p_{nmk}(t) \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \sin\frac{k\pi z}{L_3} \tag{28}$$

where

$$p_{nmk}(t) = \frac{8}{L_1L_2L_3}\int_0^{L_3}\{\int_0^{L_2}[\int_0^{L_1} f(x,y,z,t) \times \sin\frac{n\pi x}{L_1}dx]\sin\frac{m\pi y}{L_2}dy\}\sin\frac{k\pi z}{L_3}dz.$$

Substituting (27) and (28) to (25), and by Lemma 2.1 and Lemma 2.2, we have

$$\int_0^1 {}_0D_t^{\mu} W_{nmk}(t)d\mu + [\int_1^2 (\frac{n^2\pi^2}{L_1^2})^{\frac{\alpha}{2}}d\alpha + \int_1^2 (\frac{m^2\pi^2}{L_2^2})^{\frac{\beta}{2}}d\beta + \int_1^2 (\frac{k^2\pi^2}{L_3^2})^{\frac{\gamma}{2}}d\gamma]W_{nmk}(t) - P_{nmk}(t) = 0. \tag{29}$$

Let

$$c_{nmk} = \int_1^2 (\frac{n^2\pi^2}{L_1^2})^{\frac{\alpha}{2}}d\alpha + \int_1^2 (\frac{m^2\pi^2}{L_2^2})^{\frac{\beta}{2}}d\beta + \int_1^2 (\frac{k^2\pi^2}{L_3^2})^{\frac{\gamma}{2}}d\gamma,$$

taking Laplace transform w.r.t t on both sides of (29), then by condition $u(x,y,z,0) = 0$ of (B), we get

$$\int_0^1 [s^{\mu}\widehat{W}_{nmk}(s)]d\mu + c_{nmk}\widehat{W}_{nmk}(s) - \widehat{p}_{nmk}(s) = 0,$$

thus

$$\widehat{W}_{nmk}(s) = \frac{\ln s}{s - 1 + c_{nmk}\ln s}\widehat{p}_{nmk}(s),$$

taking inverse Laplace transform w.r.t s on both sides of above equation, we have

$$W_{nmk}(t) = p_{nmk}(t) * \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{\ln s}{s - 1 + c_{nmk}\ln s} e^{st}ds$$

where $*'$ represent convolution. Therefore

$$u^2(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\{p_{nmk}(t) * \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{\ln s}{s - 1 + c_{nmk}\ln s} \times e^{st}ds\} \sin\frac{n\pi x}{L_1} \times \sin\frac{m\pi y}{L_2} \sin\frac{k\pi z}{L_3},$$

combining above several results, we obtain

$$\begin{aligned} u(x,y,z,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{4}{\pi L_1L_2L_3}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{(s-1)e^{st}}{s - 1 + c_{nmk}\ln s}ds \\ &\times \int_0^{L_3}\{\int_0^{L_2}[\int_0^{L_1} g(x,y,z)\sin\frac{n\pi x}{L_1}dx]\sin\frac{m\pi y}{L_2}dy\} \\ &\times \sin\frac{k\pi z}{L_3}dz \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \sin\frac{k\pi z}{L_3} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \\ &\{p_{nmk}(t) * \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{\ln s}{s - 1 + c_{nmk}\ln s}e^{st}ds\} \sin\frac{n\pi x}{L_1} \\ &\times \sin\frac{m\pi y}{L_2} \sin\frac{k\pi z}{L_3}. \end{aligned}$$

VI. Conclusion

The difficulty in solving distributed order differential equations lies in the fact that the order of the derivative is distributed within a finite interval. We overcome this difficulty by separating variables and using Laplace transform as well as Lemma 2.1 and spectral methods, and then obtain their analytical solutions.

References

[1] T. M. Atanackovic, S. Pilipovic, D. Zorica, "Time distributed-order diffusion-wave equation. II.Applications of Laplace and Fourier transformations," Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, vol. 465, no. 2016, pp.1893 - 1917, 2009.

[2] W. Chen, H. Sun, X. Li, Fractional derivative modeling of mechanical and engineering problems, Beijing: Science Press, 2010.

[3] X. Liu, Research on fast algorithm of time distribution order diffusion equation(Chinese), Jinan: Shandong University, May 2018, page 2.

[4] A. Mostafa, D. Mehdi, "An improved meshless method for solving two-dimensional distributed order time-fractional diffusion-wave equation with error estimate," Numer Algor, vol 75, no. 1, pp. 173 - 211, 2017.

[5] F. Mainardi, G. Pagnini, R. Gorenflo, "Some aspects of fractional diffusion equations of single and distributed order," Appl. Math. Comput, vol. 187, no. 1, pp. 295 - 305, 2007.

[6] F. Mainardi, A. Mura, G. Pagnini, R. Gorenflo, "Time-fractional diffusion of distributed order," J.Vib. Control, vol. 14, no. 9-10, pp. 1267 - 1290, 2008.

[7] M .M. Meerschaert, E. Nane, P. Vellaisamy, "Distributed-order fractional diffusions on bounded domains," J. Math. Anal. Appl, vol. 379, no. 1, pp. 216 - 228, 2011.

[8] T. Sandev, A. V. Chechkin, N. Korabel, et al, "Distributed-order diffusion equations and multifractality: models and solutions," Phys. Rev. E, vol. 92, no. 4 pp. 042117, 2015.

[9] I. Podlubny, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999. xxiv+340 pp.

[10] Q. Yang, F. Liu, I. Turner, "Numerical methods for fractional partial differential equations with Riesz space fractional derivatives," Applied Mathematical Modelling, vol. 34, no. 1, pp. 200-218, 2010.