Fully Distributed Leaderless Consensus of General Nonlinear Multi-agent Systems with Directed Topology and Bounded Disturbances

Tianhong Zhou, Shidong Zhai, Chaoyang Li, and Hao Peng

Abstract—**This paper focuses on addressing the leaderless consensus problem in general nonlinear multi-agent systems (MAS) with bounded disturbances and directed topology. Neural networks are utilized for estimating unknown nonlinear functions. By designing an adaptive observer, under the presence of bounded disturbances and unknown nonlinearities, a fully distributed leaderless consensus protocol is presented to make the closed-loop systems achieve consensus. By using sigmoid basis functions for disturbance compensation, the sensitivity of the controller to disturbances can be reduced, enabling the system to work more stably and robustly in the presence of bounded disturbances. Finally, the results are verified through a numerical example.**

Index Terms—**nonlinear systems, leaderless consensus, bounded disturbances, fully distributed consensus.**

I. Introduction

In recent decades, the problem of achieving consensus in multi-agent systems (MAS) has gained significant attention as a prominent research problem with wide practical applications. Research on MAS-related consensus is beneficial for people to perceive utility in various domains, including robot cooperation [1], satellite synchronization [2], multi-UAV cooperation [3], key-management scheme [4], and many others. The issue of consensus is central to MAS. When dealing with consensus problems, designing an adaptive protocol that uses local information effectively presented a challenging task, which needs to use various state information of multi-agent systems and design various intelligence through local information interaction to achieve consensus agreement, rather than relying on a centralized control structure.

Reaching consensus in MAS has been a widely researched topic, focusing on the collective behaviors that emerge in such systems [5], [6]. Consensus refers to the situation where multi-agents strive to reach consensus or alignment on a shared objective or task. To this end, we need to know the state of each agent and design an appropriate control strategy. The goal of solving the consensus problem is to enable all nodes to reach consensus on a particular value, sequence, or decision to maintain the consistency and reliability of the system. One calls without leadership consensus [5], [7], and the other is leadership to achieve consensus [8], [9], [10], [11]. The reference [12], [13] considered a leader in changing the topology of the consensus problem. The reference [14] suggested a holistic approach to solving the consensus problem in linear MAS. The reference [15] addressed the problem of cooperative tracking of networked systems with identical linear systems and also introduced the duality of cooperative controller design and cooperative observer design on inverse graphs. The reference [16] focused on the stability analysis of cluster synchronization manifold in networks consisting of diffusively coupled nonlinear systems with the directed topology. The reference [17] investigated the cluster synchronization problem in coupled nonlinear systems networks with the directed topology and competitive relationships. The reference [18] designed two distributed controllers with static and adaptive coupling gains, respectively, under which the follower's state will be close to that of the leader. The authors in [19], [20] examined the issue of output consensus in linear MAS that exhibits heterogeneity.

In practice, all physical systems exhibit intrinsic nonlinearity, such as robotic systems, aircraft systems, and induction motor systems [21]. The consensus problem of nonlinear MAS has not been extensively explored in the existing literature. In [22], it mainly addresses the synchronization problem of leaderless and leader-follower clusters in directed topologically interconnected nonlinear systems. [23] centered around the leaderless cluster agreement (CC) dilemma in second-order multi-agent frameworks (MAS) with inherent nonlinearity under the directed topology. [24] discussed a multi-agent system with second-order nonlinear dynamics to achieve consistency in the design. A continuous and robust consensus tracking scheme has been proposed for a MAS with an integrator [25], which takes into account disturbances and dynamics that are not explicitly modeled. Through the adaptive coupling gain, robust adaptive protocols were suggested in [18] to ensure that the consensus tracking error is ultimately bounded when the disturbances satisfy certain conditions. The result was then extended in [26] to address the problem consensus tracking problem for MAS, which has directed communication graphs and non-matching perturbations. To achieve consensus tracking while ensuring interference rejection performance.

We further study that the nonlinear multi-agent system does not include consensus problem leaders, the system contains a spanning tree of the general digraph. Innovative adaptive protocols are introduced and demonstrated to achieve

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leaderless consensus. We propose a novel fully distributed adaptive protocol that relies solely on the state information of each agent and its neighboring agents. The robust adaptive consensus protocols presented in this paper apply to scenarios where the disturbances are generally bounded. The contribution of this paper is mainly in two aspects. First, a fully distributed adaptive protocol is developed for the directed topology that contains a spanning tree to solve the problem of leaderless consensus. Second, neural networks are used to estimate and counteract nonlinearity for each agent subject to limited disturbances.

The remainder of this paper is organized as follows. Section II describes the fundamentals and the problem statement. In Section III, a fully distributed state observer is designed, and the issue of consensus with bounded disturbances is studied. Section IV uses a numerical simulation to prove the theory. Section V summarizes the paper by summarizing the key contributions.

II. Preliminaries

A. Graph theory

A complex network of N nodes can be described by a graph $\mathcal{G}$. The set of nodes V is the edges from node j to node i and A is the adjacency matrix. Assume that $a_{ij} = 0$ holds for any $I \in \mathcal{V}$. Let L be the Laplacian matrix of the graph $\mathcal{G}$

The graph of L is called an undirected graph, otherwise, it is called a directed graph. If a sequence of directed edges can be found such that a directed path exists from node j and node $i(i \neq j)$, it is called a directed path. Node j to node $i(i \neq j)$ are the start node and end node respectively. A graph is considered a directed spanning tree if there is a directed path between at least one node (the root node) and all other nodes. When each node can be considered as the root node, the graph is called a strongly connected graph [27], [28]. Graph $\mathcal{G}$ corresponding to matrix A can also be denoted by $\mathcal{G}$. Graph $\mathcal{G}$ can also be denoted by $\mathcal{G}$.

Assumption 1. *There is a directed spanning tree in the directed topology graph.*

Lemma 1. *[29] Assuming that the Assumption 1 holds, the graph $\mathcal{G}$ can be decomposed into two parts $\mathcal{V}_1$ are indexed as $1, \cdots, k$, and $\mathcal{V}_2$ are indexed as $k+1, \cdots, N, 1 \leq k \leq N$, where $\mathcal{V}_1$ node set is and $\mathcal{V}_2$ node set is. Then, Laplacian can decompose the form as $L_s = \begin{bmatrix} L_{s1} & 0 \\ L_{s3} & L_{s2} \end{bmatrix}$, where $L_{s1} \in R^{k \times k}$ is strongly connected subgraph, $L_{s2} \in R^{N-k \times N-k}$ in the following lemma is introduced [30], [28].*

Lemma 2. *[31], [32] Under Assumption 1, L_{s2} in Lemma 1 is a nonsingular M-matrix.*

Lemma 3. *[33] For a strongly connected graph $\mathcal{G}_1$ with Laplacian matrix L_{s1}, the matrix $\hat{L}_{s1} = \Xi_r L_{s1} + L_{s1}^T \Xi_r$ represents a weighted symmetric Laplacian matrix of an undirected connected graph, where $\Xi_r = \mathrm{diag}(r_1, \cdots, r_k) > 0$ with $r = [r_1, \cdots, r_k]$ being the left zero unit eigenvector of L_{s1}. Moreover, $min_{z^T x = 0} x^T \hat{L}_{s1} x \geq \frac{\lambda_2(\hat{L}_{s1})}{k} x^T x$, where $\lambda_2(\hat{L}_{s1})$ represents the minimum non-zero eigenvalue of $\lambda_2(\hat{L}_{s1})$ and z is a vector consisting of entirely positive elements.*

Lemma 4. *[34] For the non-singular M-matrix L_{s2}, there exists a diagonal positive definite matrix $G > 0$ such that $L_{s0} = GL_{s2} + L_{s2}^T G > 0$.*

B. Model description

In this subsection, taking into account a collection of N agents with unknown heterogeneous nonlinear dynamic, suppose that every agent is subjected to the subsequent general nonlinear dynamic:

$$\dot{x}_i(t) = Ax_i(t) + B[u_i(t) + f(x_i(t)) + d_i(t)], \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ is the state of the agent and $u_i \in \mathbb{R}^m$ is the control input to the system. $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ are constant matrices, $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $d_i(t) \in \mathbb{R}^m$ represent the heterogeneous smooth nonlinearity and the disturbance suffered by the ith agent, respectively. Consider the tracking issue of a cluster of N agents, where the disturbance $d_i(t)$ is bounded.

Assumption 2. *Each agent by bounded disturbances.*

$$\|d_i(t)\|_\infty \leq \omega_i, \ i = 1, 2, \cdots, N, \quad (2)$$

where $\omega_i > 0$ represents finite yet unidentified constants.

The undetermined nonlinearity can be expressed as follows

$$f_i(x_i) = W_i^T \phi_i(x_i) + \epsilon_i, \quad (3)$$

where $W_i \in \mathbb{R}^{s \times m}$ denotes a fixed real matrix that signifies the ideal weight matrix of the neural network, $\phi_i() : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ is a known activation function, and ϵ_i denotes the residual error of the neural network approximation, characterized as a bounded vector of approximation errors that satisfies the inequality $\|\epsilon_i\|_\infty \leq \bar{\epsilon}_i$. Usually, as $\bar{\epsilon}_i$ is not ascertainable, we shall exclude it from the subsequent controller design. Let us assume that $\phi_i()$ is bounded on closed and bounded sets.

Assumption 3. *The matrices (A, B) are controllable.*

Lemma 5. *[26] Assuming that Assumption 3 holds, there exists a positive definite matrix Q satisfying the following inequality $AQ + QA^T - 2BB^T < 0$.*

III. Main Results

In this section, the purpose of this section is to achieve consensus among multi-agent systems under static communication graphs, with the aim of $lim_{t \to \infty}(x_i(t) - x_j(t)) = 0, \forall i, j$ with random starting values $x_i(0)$. Let $\xi_i = \sum_{j=1}^N a_{ij}(v_i - v_j)$ represents the agreement of the deviation for each agent. Consequently, the consensus is achieved exclusively when ξ_i approaches zero. We design an internal state observer (4) where v_i is the internal state and η_i is the adaptive control gain, one has

$$\begin{aligned} \dot{v}_i &= Av_i + B(\eta_i + \xi_i^T P \xi_i) K \xi_i, \\ \dot{\eta}_i &= \xi_i^T P B B^T P \xi_i. \end{aligned} \quad (4)$$

By utilizing the adaptive amplification factor to approximate this shared constant amplification factor, fully distributed adaptive protocols were suggested by introducing an additional additive factor to address the asymmetrical Laplacian matrix [9].

For every individual agent, devise the controller in the following manner:

$$u_i = (\eta_i + \xi_i^T P\xi_i)K\xi_i + K\tilde{x}_i - \hat{W}_i^T\phi_i(x_i) - \gamma_i \text{sgn}(K\tilde{x}_i), \quad (5)$$

where $\hat{W}_i$ serves as an estimate of W_i and is employed to compensate for the nonlinearity in $f_i(x_i,t)$, where $K = -B^T P$ with $P = Q^{-1} > 0$. The adaptive control gain η_i is introduced to estimate the global eigenvalue information of the Laplacian matrix. Its purpose is to adjust and adapt to changes in the Laplacian matrix's eigenvalues. Let $\tilde{x}_i(t) = x_i(t) - v_i(t)$ represents the observer error, and γ_i is an adaptive gain designed in the following manner,

$$\dot{\gamma}_i = \left\| K\tilde{x}_i(t) \right\|_1. \quad (6)$$

Moreover, $\hat{W}_i$ is devised as

$$\begin{aligned} \dot{\hat{W}}_i &= m_i\phi_i(x_i)\tilde{x}_i^T PB - m_i c_i(\hat{W}_i - \bar{W}_i), \\ \dot{\bar{W}}_i &= n_i c_i(\hat{W}_i - \bar{W}_i), \end{aligned} \quad (7)$$

where m_i, n_i, and c_i are positive factors.

Noticing (1-5), one acquires the dynamics of $\tilde{x}_i(t)$:

$$\dot{\tilde{x}}_i = A\tilde{x}_i + B[\tilde{W}_i^T\phi_i(x_i) + \bar{d}_i + K\tilde{x}_i - \gamma_i \text{sgn}(K\tilde{x}_i)], \quad (8)$$

where $\tilde{W}_i = W_i - \hat{W}_i$ and $\bar{d}_i = d_i + \epsilon_i$. Then it follows $\left\|\bar{d}_i\right\|_\infty \le \omega_i + \bar{\epsilon}_i$.

We will demonstrate the achievement of consensus with G, which includes a directed spanning tree, as follows.

Let $\tilde{v}_1 = [v_1^T, \cdots, v_k^T]^T, \tilde{v}_2 = [v_{k+1}^T, \cdots, v_N^T]^T$, and $v = [\bar{v}_1^T, \bar{v}_2^T]^T$ and $\bar{\xi}_1 = [\xi_1^T, \cdots, \xi_k^T]^T, \bar{\xi}_2 = [\xi_{k+1}^T, \cdots, \xi_N^T]^T$ and $\xi = [\bar{\xi}_1^T, \bar{\xi}_2^T]^T$. It is evident that

$$\begin{aligned} \bar{\xi}_1 &= (L_{s1} \otimes I_n)\tilde{v}_1, \\ \bar{\xi}_2 &= (L_{s2} \otimes I_n)\tilde{v}_2 + (L_{s3} \otimes I_n)\tilde{v}_1. \end{aligned} \quad (9)$$

The dynamics of the closed-loop systems of $\bar{\xi}_1$ and $\bar{\xi}_2$ are depicted as follows:

$$\begin{aligned} \dot{\bar{\xi}}_1 =& [I_N \otimes A + L_{s1}(H_1 + \bar{\rho}_1) \otimes BK]\bar{\xi}_1, \\ \dot{\bar{\xi}}_2 =& [I_N \otimes A + L_{s2}(H_2 + \bar{\rho}_2) \otimes BK]\bar{\xi}_2 \\ &+ [L_{s3}(H_1 + \bar{\rho}_1) \otimes BK]\bar{\xi}_1, \\ \dot{\eta}_i =& \xi^T{}_i PBB^T P\xi_i. \end{aligned} \quad (10)$$

Let $H_1 = \text{diag}(\eta_1, \cdots, \eta_k)$, $H_2 = \text{diag}(\eta_{k+1}, \cdots, \eta_N)$, $\rho_i = \xi_i^T P\xi_i, i = 1, \cdots, N, \bar{\rho}_1 = \text{diag}(\rho_1, \cdots, \rho_k)$ and $\bar{\rho}_2 = \text{diag}(\rho_{k+1}, \cdots, \rho_N)$.

Theorem 1. *Assuming that Assumptions 1-3 are met, the consensus of the nonlinear MAS (1) can be attained by employing the controller (5).*

Proof: Consider the following Lyapunov function:

$$V = \alpha V_1 + V_2 + V_3, \quad (11)$$

with

$$\begin{aligned} V_1 =& \sum_{i=1}^{k} \frac{\Xi_r}{2}[(2\eta_i + \rho_i)\rho_i + (\eta_i - \beta_1)^2], \\ V_2 =& \sum_{i=k+1}^{N} \frac{\Xi_g}{2}[(2\eta_i + \rho_i)\rho_i + (\eta_i - \beta_2)^2], \\ V_3 =& \tilde{x}_i^T P\tilde{x}_i + \frac{1}{m_i}tr(\tilde{W}_i^T\tilde{W}_i) \\ &+ \frac{1}{n_i}tr(\breve{W}_i^T\breve{W}_i) + (\gamma_i - \gamma)^2, \end{aligned} \quad (12)$$

where α, β_1 and β_2 are positive constants, Ξ_r and Ξ_g is determined by the left zero unit eigenvector of L_{s1} and L_{s2}, respectively. Let $\breve{W}_i = W_i - \bar{W}_i$, where γ is a parameter to be determined. Differentiating V with respect to the relevant variables, one gets

$$\dot{V} = \alpha\dot{V}_1 + \dot{V}_2 + \dot{V}_3, \quad (13)$$

where

$$\begin{aligned} \dot{V}_1 =& \sum_{i=1}^{k} \Xi_r[(\eta_i + \rho_i)\dot{\rho}_i + (\eta_i + \rho_i - \beta_1)\dot{\eta}_i], \\ =& 2\bar{\xi}_1^T[(H_1 + \bar{\rho}_1)R \otimes P]\dot{\bar{\xi}}_1 \\ &+ \sum_{i=1}^{k} \Xi_r(\eta_i + \rho_i - \beta_1)\xi_i^T PBB^T P\xi_i, \\ =& \bar{\xi}_1^T[(H_1 + \bar{\rho}_1)R \otimes (PA + A^T P)]\bar{\xi}_1 \\ &- \bar{\xi}_1^T[(H_1 + \bar{\rho}_1)\hat{L}_{s1}(H_1 + \bar{\rho}_1) \otimes PBB^T P]\bar{\xi}_1 \\ &+ \bar{\xi}_1^T[(H_1 + \bar{\rho}_1 - \beta_1 I_k)R \otimes PBB^T P]\bar{\xi}_1, \end{aligned} \quad (14)$$

and

$$\begin{aligned} \dot{V}_2 =& \sum_{i=k+1}^{N} \Xi_g[(\eta_i + \rho_i)\dot{\rho}_i + (\eta_i + \rho_i - \beta_2)\dot{\eta}_i], \\ =& 2\bar{\xi}_2^T[(H_2 + \bar{\rho}_2)G \otimes P]\dot{\bar{\xi}}_2 \\ &+ \sum_{i=k+1}^{N} \Xi_g(\eta_i + \rho_i - \beta_2)\xi_i^T PBB^T P\xi_i, \\ =& \bar{\xi}_2^T[(H_2 + \bar{\rho}_2)R \otimes (PA + A^T P)]\bar{\xi}_2 \\ &- \bar{\xi}_2^T[(H_2 + \bar{\rho}_2)L_{s0}(H_2 + \bar{\rho}_2) \otimes PBB^T P]\bar{\xi}_2 \\ &- 2\bar{\xi}_2^T[(H_2 + \bar{\rho}_2)GL_{s3}(H_1 + \bar{\rho}_1) \otimes PBB^T P]\bar{\xi}_1 \\ &+ \bar{\xi}_2^T[(H_2 + \bar{\rho}_2 - \beta_2 I_{N-k})G \otimes PBB^T P]\bar{\xi}_2. \end{aligned} \quad (15)$$

Define $\psi = [(H_1 + \bar{\rho}_1)^{-1} \otimes I_n](\Xi_r^T \otimes 1_n)$, which has all positive elements. We have

$$\begin{aligned} \psi^T[(H_1 + \bar{\rho}_1) \otimes B^T P]\bar{\xi}_1 &= (\Xi_r \otimes 1_n^T)(I_k \otimes B^T P)\bar{\xi}_1 \\ &= (\Xi_r L_{s1} \otimes 1_n^T)(I_k \otimes B^T P)\tilde{v}_1 = 0. \end{aligned} \quad (16)$$

Based on the insights provided by Lemma 2, we can derive

$$\begin{aligned} &\bar{\xi}_1^T[(H_1 + \bar{\rho}_1)\hat{L}_{s1}(H_1 + \bar{\rho}_1) \otimes PBB^T P]\bar{\xi}_1 \\ &\le -\frac{\lambda_2(\hat{L}_{s1})}{k}\bar{\xi}_1^T[(H_1 + \bar{\rho}_1)^2 \otimes PBB^T P]\bar{\xi}_1. \end{aligned} \quad (17)$$

Denote λ_0 as the smallest eigenvalue of L_{s0}. Then we have

$$\begin{aligned} &\bar{\xi}_2^T[(H_2 + \bar{\rho}_2)L_{s0}(H_2 + \bar{\rho}_2) \otimes PBB^T P]\bar{\xi}_2 \\ &\le -\lambda_0\bar{\xi}_2^T[(H_2 + \bar{\rho}_2)^2 \otimes PBB^T P]\bar{\xi}_2. \end{aligned} \quad (18)$$

Through Young's Inequality, we can get

$$\begin{aligned}&2\bar{\xi}_2^T[(H_2+\bar{\rho}_2)GL_{s3}(H_1+\bar{\rho}_1)\otimes PBB^TP]\bar{\xi}_1\\ \leq&\frac{\lambda_0}{2}\bar{\xi}_2^T[(H_2+\bar{\rho}_2)^2\otimes PBB^TP]\bar{\xi}_2\\ &+\frac{2\sigma_{max}^2(GL_{s3})}{\lambda_0}\bar{\xi}_1^T[(H_1+\bar{\rho}_1)^2\otimes PBB^TP]\bar{\xi}_1.\end{aligned} \tag{19}$$

Substituting (14)-(19) into (13), define $\alpha\dot{V}_1+\dot{V}_2=\dot{\bar{V}}$ and choose $\alpha=\frac{\lambda_2(\hat{L}_{s1})}{k}(\alpha_1+\frac{2\sigma_{max}^2(GL_{s3})}{\lambda_0})$ yields

$$\begin{aligned}\dot{\bar{V}}\leq&\alpha\bar{\xi}_1^T(H_1+\bar{\rho}_1)R\otimes(PA+A^TP+PBB^TP)\bar{\xi}_1\\ &-\bar{\xi}_1^T[(\alpha_1(H_1+\bar{\rho}_1)^2+\alpha\beta_1R)\otimes PBB^TP]\bar{\xi}_1\\ &+\bar{\xi}_2^T[(H_2+\bar{\rho}_2)G\otimes(PA+A^TP+PBB^TP)]\bar{\xi}_2\\ &-\bar{\xi}_2^T[(\frac{\lambda_0}{2}(H_2+\bar{\rho}_2)^2-\beta_2G)\otimes PBB^TP]\bar{\xi}_2.\end{aligned} \tag{20}$$

By choosing $\beta_1\geq\frac{9\alpha\lambda_{max}(R)}{4\alpha_1}$ and in light of Young's Inequality, we have

$$\begin{aligned}&-\bar{\xi}_1^T[(\alpha_1(H_1+\bar{\rho}_1)^2+\alpha\beta_1R)\otimes PBB^TP]\bar{\xi}_1\\ \leq&-3\alpha\bar{\xi}_1^T[(H_1+\bar{\rho}_1)R\otimes PBB^TP]\bar{\xi}_1.\end{aligned} \tag{21}$$

Similarly, we can derive

$$\begin{aligned}&-\bar{\xi}_2^T[(\frac{\lambda_0}{2}(H_2+\bar{\rho}_2)^2-\beta_2G)\otimes PBB^TP]\bar{\xi}_2\\ \leq&-3\bar{\xi}_2^T[(H_2+\bar{\rho}_2)G\otimes PBB^TP]\bar{\xi}_2,\end{aligned} \tag{22}$$

where $\beta_2\geq\frac{9\lambda_{max}(G)}{4\lambda_0}$. Then, we can derive that

$$\begin{aligned}\alpha\dot{V}_1+\dot{V}_2\leq&-\alpha\bar{\xi}_1^T[(H_1+\bar{\rho}_1)R\otimes M]\bar{\xi}_1\\ &-3\bar{\xi}_2^T[(H_2+\bar{\rho}_2)G\otimes M]\bar{\xi}_2\\ \leq&0,\end{aligned} \tag{23}$$

where $M=-(PA+A^TP-2PBB^TP)$ is a positive definite matrix.

$$\begin{aligned}\dot{V}_3=&2\tilde{x}_i^TP\{A\tilde{x}_i+B[\tilde{W}_i^T\phi_i(x_i)+\bar{d}_i+K\tilde{x}_i\\ &-\gamma_i(t)\mathrm{sgn}(K\tilde{x}_i)]\}-\frac{2}{m_i}tr(\tilde{W}_i^T\dot{\hat{W}}_i)\\ &-\frac{2}{n_i}tr(\breve{W}_i^T\dot{\bar{W}}_i)+2(\gamma_i-\gamma)\dot{\gamma}_i.\end{aligned} \tag{24}$$

On the one hand,

$$\begin{aligned}\frac{1}{m_i}tr(\tilde{W}_i^T\dot{\hat{W}}_i)=&tr(\tilde{W}_i^T\phi_i(x_i)\tilde{x}_i^TPB-c_i\tilde{W}_i^T(\hat{W}_i-\bar{W}_i))\\ =&\tilde{x}_i^TPB\tilde{W}_i^T\phi_i(x_i)-c_itr(\tilde{W}_i^T(\hat{W}_i-\bar{W}_i)).\end{aligned} \tag{25}$$

On the other hand,

$$\frac{1}{n_i}tr(\breve{W}_i^T\dot{\bar{W}}_i)=c_itr(\breve{W}_i^T(\hat{W}_i-\bar{W}_i)). \tag{26}$$

It follows that

$$\begin{aligned}\dot{V}_3=&\tilde{x}_i^T(PA+A^TP-2PBB^TP)\tilde{x}_i+2\tilde{x}_i^TPB\bar{d}_i\\ &-2\gamma\left\|K\tilde{x}_i(t)\right\|_1\\ &-2c_itr((\hat{W}_i-\bar{W}_i)^T(\hat{W}_i-\bar{W}_i))\\ \leq&-\left\|\tilde{x}_i\right\|^2+2(\omega_i+\bar{\epsilon}_i-\gamma)\left\|K\tilde{x}_i(t)\right\|_1,\end{aligned} \tag{27}$$

where $\tilde{x}_i^TPB\mathrm{sgn}(K\tilde{x}_i)=\left\|K\tilde{x}_i(t)\right\|_1$ and $\tilde{W}_i-\breve{W}_i=-\hat{W}_i+\bar{W}_i$ are employed in the first equation. Choose $\gamma>\omega_i+\bar{\epsilon}_i$, we have $\dot{V}_3<0$.

Thus, we can derive that

$$\begin{aligned}\dot{V}\leq&-\alpha\bar{\xi}_1^T[(H_1+\bar{\rho}_1)R\otimes M]\bar{\xi}_1-3\bar{\xi}_2^T[(H_2+\bar{\rho}_2)G\otimes M]\\ &\bar{\xi}_2-\left\|\tilde{x}_i\right\|^2+2(\omega_i+\bar{\epsilon}_i-\gamma)\left\|K\tilde{x}_i(t)\right\|_1\\ \leq&0.\end{aligned} \tag{28}$$

Therefore, V is bounded, by invoking (10), $\bar{\xi}_1,\bar{\xi}_2,\eta_i,\tilde{x}_i,\gamma_i$ and $\tilde{W}_i$ are bounded, then $\tilde{x}_i^T\dot{\tilde{x}}_i$ is constrained by observing (8). Following Barbalat's Lemma, one has $\tilde{x}_i\to0$, as $t\to\infty$. The consensus problem is solved. ■

IV. Numerical Simulation

This section demonstrates a numerical simulation to illustrate the implementation of the controller (5). In this case, we consider that the followers are subject to bounded disturbances and possess general nonlinear dynamics:

$$\begin{aligned}f_i(x_i)&=x_{i2}\cdot\cos(x_{i3})+x_{i1}\cdot\cos(x_{i2}),\\ d_i(t)&=2\cos(1+x_{i2})+\sin(x_{i1}+x_{i3}).\end{aligned} \tag{29}$$

The multi-agent systems (MAS) comprises six agents, and the system matrices are represented as follows:

$$A=\begin{bmatrix}-2.5&8&0\\1&-1&1\\0&-20&0\end{bmatrix},\ B=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

According to Lemma 1, the positive definite solution of the Linear Matrix Inequality (LMI) is selected as

$$P=Q^{-1}=\begin{bmatrix}10.2796&7.3095&4.2746\\7.3095&62.2456&7.7236\\4.7245&7.7236&4.2584\end{bmatrix}.$$

It follows that

$$K=-B^TP=[-4.7245\ -7.7236\ -4.2584].$$

The sigmoid basis function $\sigma(z)=(1+e^{-z})^{-1}$ is utilized, and the neural network weights $\hat{W}_i$ are initialized to zero. Additionally, the parameters in (7) are selected as follows: $m_i=1000,c_i=0.1,n_i=20$.

Example 1. *In this example, the directed communication topology graph is illustrated in Fig. 1 and contains a directed spanning tree. It consists of six agents indexed as $i=1,\cdots,6$, which is not a leader-follower graph (yellow nodes represent strongly connected parts, and blue nodes represent the remaining nodes). Fig. 2- 4 consensus of state x_i for all $i=1,\cdots,6$ for six agents. The adaptive gain γ_i becomes stable after a certain value, as shown in Fig. 5.*

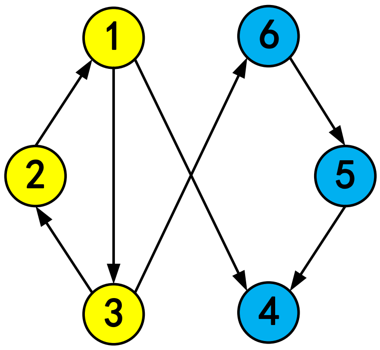

Fig. 1. The graph consists of a directed spanning tree with six agents indexed as $i = 1, \cdots, 6$, which is not a leader-follower graph.

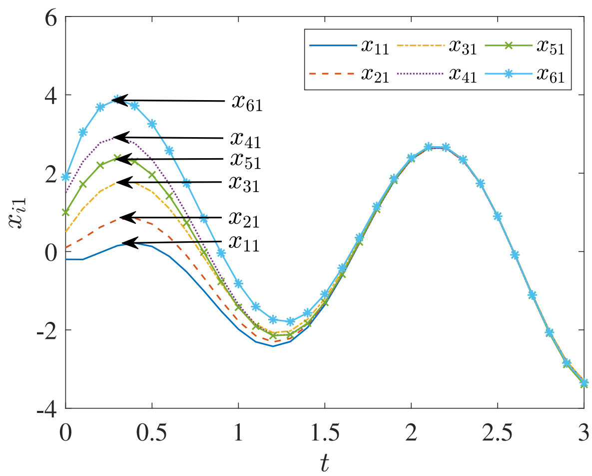

Fig. 2. The consensus of the state x_{i1} is achieved for all $i = 1, \cdots, 6$ among the six agents.

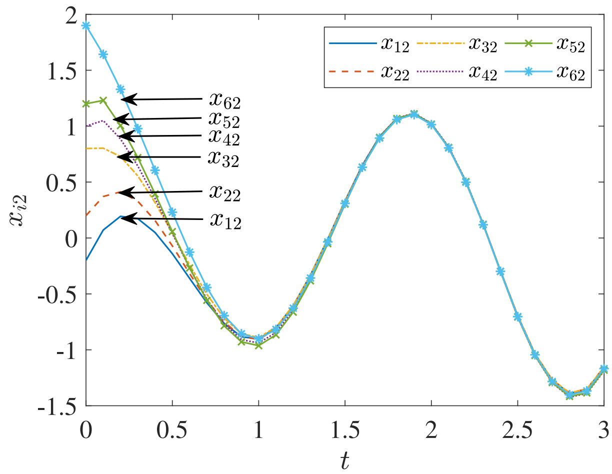

Fig. 3. The consensus of the state x_{i2} is achieved for all $i = 1, \cdots, 6$ among the six agents.

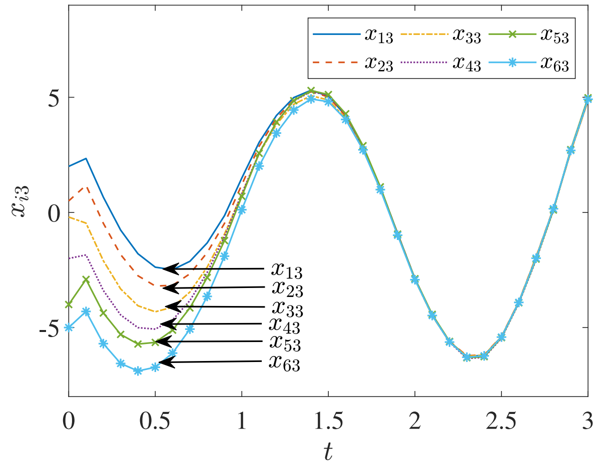

Fig. 4. The consensus of the state x_{i3} is achieved for all $i = 1, \cdots, 6$ among the six agents.

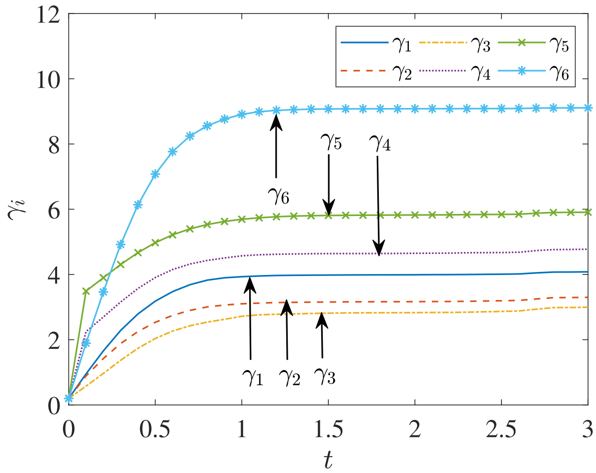

Fig. 5. The adaptive gain $\gamma_i, i = 1, \cdots, 6$ under the controller (5).

Example 2. *This example-directed communication topology is composed of eight agents indexed as $i = 1, \cdots, 8$ as shown in Fig. 6. This configuration satisfies Assumption 1 and can be partitioned into two strongly connected components, with nodes 1, 2, 3, and 4 acting as the roots(yellow nodes represent strongly connected parts and blue nodes represent the remaining nodes). Fig. 7- 9 achieve consensus of state x_i for all $i = 1, \cdots, 8$ for six agents. The adaptive gain γ_i becomes stable after a certain value, as shown in Fig. 10.*

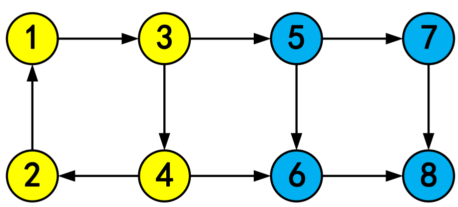

Fig. 6. The graph consists of a directed spanning tree with eight agents indexed as $i = 1, \cdots, 8$, which is not a leader-follower graph.

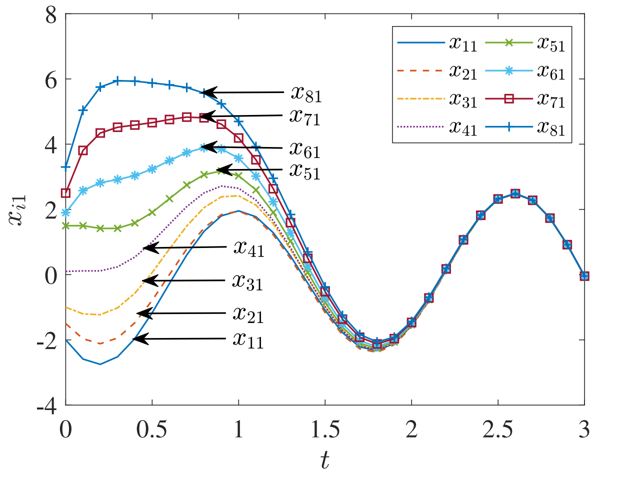

Fig. 7. The consensus of the state x_{i1} is achieved for all $i = 1, \cdots, 8$ among the six agents.

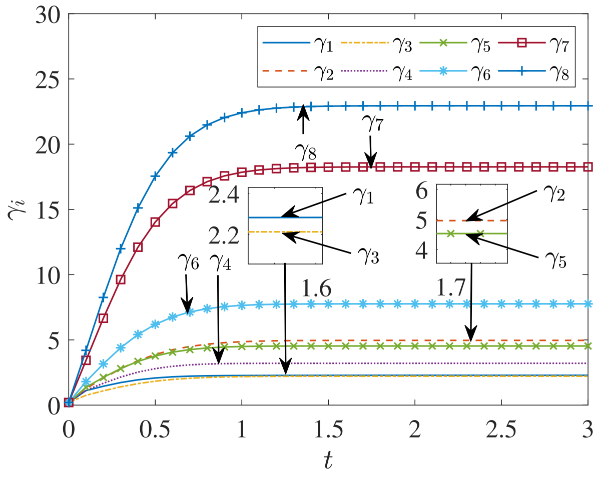

Fig. 10. The adaptive gain $\gamma_i, i = 1, \cdots, 8$ under the controller (5).

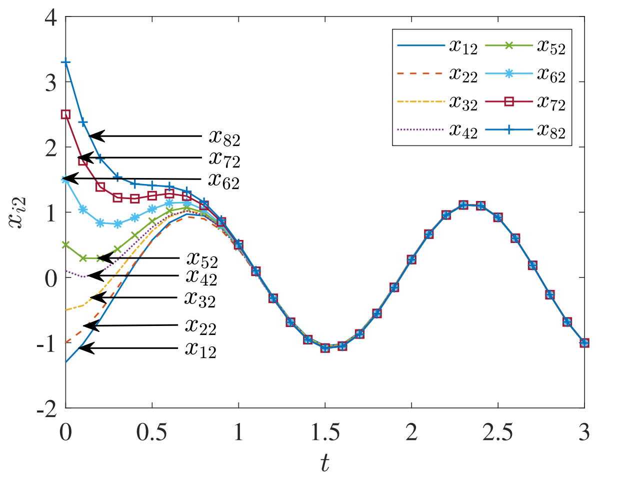

Fig. 8. The consensus of the state x_{i2} is achieved for all $i = 1, \cdots, 8$ among the six agents.

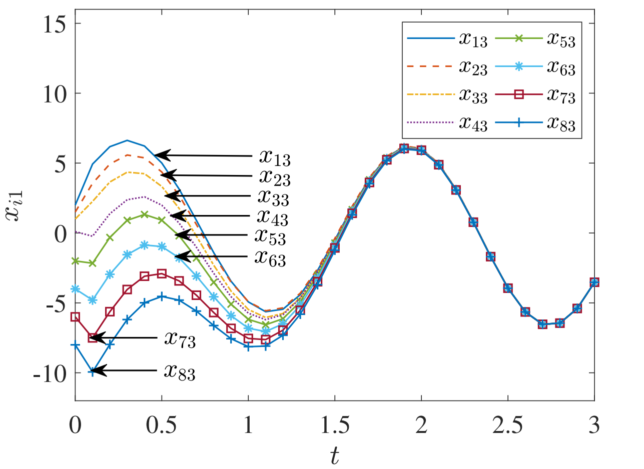

Fig. 9. The consensus of the state x_{i3} is achieved for all $i = 1, \cdots, 8$ among the six agents.

V. Conclusion

The main objective of this paper is to address the challenge of achieving leaderless consensus in general nonlinear multi-agent systems (MAS) with directed topology and bounded disturbances. To estimate unknown nonlinearities, the approach of neural network allocation approach is employed. An adaptive observer is developed to facilitate consensus among closed-loop systems. This fully distributed leaderless consensus protocol allows systems to reach mutual agreement and converge towards a consensus state. The design process incorporates adaptive σ-modification schemes, which result in reduced control gains and require a smaller amplitude for the control input, to ensure consensus convergence.

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