Stability of Cubic n-dimensional Functional Equation in Non-Archimedean Banach Spaces

P. Elumalai, S. Sangeetha*

***Abstract*—The main goal of this article is to investigate the generalized Hyers-Ulam stability of cubic n-dimensional functional equation**

$$\mathfrak{g}\left(\mathfrak{t}\sum_{i=1}^{k-1}\nu_i+\nu_k\right)+\mathfrak{g}\left(\mathfrak{t}\sum_{i=1}^{k-1}\nu_i-\nu_k\right)+2\mathfrak{t}\sum_{i=1}^{k-1}\mathfrak{g}(\nu_i)$$
$$=2\mathfrak{t}^3\mathfrak{g}\left(\sum_{i=1}^{k-1}\nu_i\right)+\mathfrak{t}\sum_{i=1}^{k-1}\Big[\mathfrak{g}(\nu_i+\nu_k)+\mathfrak{g}(\nu_i-\nu_k)\Big]$$

in the setting of non-Archimedean Banach spaces(NABS) by using the direct method and fixed point method.

***Index Terms*—Cubic n-dimensional Functional Equation(FE), non-Archimedean Banach Spaces(NABS), Direct Method(DM) and Fixed Point Method(FPM).**

I. Introduction

THE Ulam-Hyers stability problem is concerned with establishing the conditions under which, given an approximate solution of a functional equation(FE), one can locate an exact key that is closer to it in some way. The exploration of the stability problem for functional equations(FEs) is described as a question by Ulam [31] regarding the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [16]. Several authors([1], [2], [4], [11], [29], [30]) generalized it an achieved intriguing results.

The method was developed by Hyers and gives the additive mapping generated from the approximation additive mapping, known as the direct method, which may be used to study the stability of various functional equations. This method is the most essential and powerful instrument for studying the stability of various functional equations. Aoki [4] and Bourgin [7] generalized Hyers' theorem for additive mappings by considering an unbounded Cauchy difference. Rassias [24] presented a generalization of Hyers' theorem for linear mapping in 1978, allowing the Cauchy difference to be unbounded. In 1991, Gajda [14] used the same methods as [24] and presented an affirmative answer to this question for $p > 1$. However, Gajda [14] and Rassias and Semrl [25] counterexamples that one cannot establish the Rassias' type theorem for $p = 1$ have motivated various mathematicians to propose new approximately additive or approximately linear mappings. One of the most famous FE is the additive FE

$$\mathfrak{g}(\nu+\upsilon)=\mathfrak{g}(\nu)+\mathfrak{g}(\upsilon) \tag{1}$$

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Cauchy solved it in the class of continuous real-valued functions for the first time in 1821. In honor of Cauchy [31], the additive functional equation is commonly referred to as a Cauchy additive functional equation. The theory of additive functional equations is commonly used to develop theories of other functional equations. Since the function $\mathfrak{g}(\nu)=\alpha\nu$ is the solution of the FE (1), every solution of the additive FE is called an additive function.

Rassias examined the H-U stability for the different FEs in various spaces [23], [26]. Czerwik [11] investigated the stability of a quadratic FE with several variables in normed spaces. Several authors investigated the different stability outcomes in ([11], [29], [30]).

Park et al. [21] presented an additive s-functional equation in 2019. He established the H-U stability for the aforementioned one in complex Banach spaces using the FPM and DM. In addition, he investigated the H-U stability of homomorphism and derivations in complex Banach algebras.

Almahalebi [3] explored the quadratic FE in Banach spaces in 2018. And using the FPM, he established the hyperstability result of the same equation. Radu [22] examined several stability results utilizing the FPM. He investigated the stability of the Cauchy FE and Jensen's FEs using the fixed point approach. Following his work, a number of authors investigated different FEs using the FPM [17], [29], [30]. Consider the functional equation

$$\mathfrak{g}(\nu+\upsilon)+\mathfrak{g}(\nu-\upsilon)=2\mathfrak{g}(\nu)+2\mathfrak{g}(\upsilon) \tag{2}$$

where $\mathfrak{g}(\nu)=\alpha\nu^2$ is a solution of this FE, so one can usually say that the above FE is quadratic. The H-U stability problem of the quadratic FE was first proved by F. Skof [27] for functions between a normed space and a Banach space. Later, the result was explored by S. Czerwisk [12]. A Stability problem of Ulam for the cubic FE

$$\mathfrak{g}(2\nu+\upsilon)+\mathfrak{g}(2\nu-\upsilon)=2\mathfrak{g}(\nu+\upsilon)+2\mathfrak{g}(\nu-\upsilon)+12\mathfrak{g}(\nu) \tag{3}$$

was introduced by Jun and Kim [19]. Additionally, they found a solution to Ulam's stability problem for the generalized Euler-Lagrange type cubic FE

$$\begin{aligned}\mathfrak{g}(a\nu+\upsilon)+\mathfrak{g}(\nu+a\upsilon)=&(a+1)(a-1)^2[\mathfrak{g}(\nu)+\mathfrak{g}(\upsilon)]\\&+a(a+1)\mathfrak{g}(\nu+\upsilon)\end{aligned} \tag{4}$$

for fixed integer a with $a\neq 0,1$, and

$$\begin{aligned}\mathfrak{g}(a\nu+b\upsilon)+\mathfrak{g}(b\nu+a\upsilon)=&(a+b)(a-b)^2[\mathfrak{g}(\nu)+\mathfrak{g}(\upsilon)]\\&+ab(a+b)\mathfrak{g}(\nu+\upsilon)\end{aligned} \tag{5}$$

for integers a,b with $a \neq 0, b \neq 0$, and $a \pm b \neq 0$, and the equations being equivalent to (1.3). In a later study, Chu et al. [9], [10] explored the H-U stability and extended the cubic FE to the generalized form:

$$\mathfrak{g}\left(\sum_{i=1}^{m-1} \nu_i + 2\nu_m\right) + \mathfrak{g}\left(\sum_{i=1}^{m-1} \nu_i - 2\nu_m\right) + \sum_{i=1}^{m-1} \mathfrak{g}(2\nu_i) = 2\mathfrak{g}\left(\sum_{i=1}^{m-1} \nu_i\right) + 4\sum_{i=1}^{m-1}\left[\mathfrak{g}(\nu_i + \nu_m) + \mathfrak{g}(\nu_i - \nu_m)\right] \quad (6)$$

where $n \geq 2$ is an integer. Furthermore, Jung and Chang the FPM in [18] to examine a generalized H-U-R stability for a cubic FE.

In this current work, we present a cubic n-dimensional FE

$$\mathfrak{g}\left(\mathfrak{t}\sum_{i=1}^{k-1} \nu_i + \nu_k\right) + \mathfrak{g}\left(\mathfrak{t}\sum_{i=1}^{k-1} \nu_i - \nu_k\right) + 2\mathfrak{t}\sum_{i=1}^{k-1} \mathfrak{g}(\nu_i) = 2\mathfrak{t}^3\mathfrak{g}\left(\sum_{i=1}^{k-1} \nu_i\right) + \mathfrak{t}\sum_{i=1}^{k-1}\left[\mathfrak{g}(\nu_i + \nu_k) + \mathfrak{g}(\nu_i - \nu_k)\right] \quad (7)$$

where $t \geq 2$ is a positive integer with $\mathbb{N} - \{0, 1\}$, and obtain its general solution. The objective of this work is to investigate the H-U stability of equation (7) by using the DM and FPM in NABS. It is clear that the mapping $\mathfrak{g}(\nu) = \alpha\nu^3$ is a solution of (7).

II. Preliminaries

In this section, we will present some basic definitions and theorems, which will be essential to prove our main results.

Definition 2.1. [15] Let $\mathbb{K}$ be a field. A NA absolute value on $\mathbb{K}$ is a function $|\ | : \mathbb{K} \to \mathbb{R}$ such that for every $\nu, \upsilon \in \mathbb{K}$ we have

(i) $|\nu| \geq 0$ iff $\nu = 0$;
(ii) $|\nu\upsilon| = |\nu||\upsilon|$
(iii) $|\nu + \upsilon| \leq max\{|\nu|, |\upsilon|\}$.

Definition 2.2. [15] Let $\mathcal{X}$ be a vector space over a scalar field k with a NA non-trivial valuation $|.|$. A function $\|.\| : \mathcal{X} \to \mathbb{R}^+$ is a NA norm if it satisfies the following conditions:

(i) $\|\nu\| = 0$ iff $\nu = 0$;
(ii) $\|r\nu\| = |r|\ \|\nu\|$ for every $r \in k$, $\nu \in \mathcal{X}$;
(iii) $\|\nu + \upsilon\| \leq \max\{\|\nu\|, \|\upsilon\|\}$ for every $\nu, \upsilon \in \mathcal{X}$.

Then $(\mathcal{X}\ \|.\|)$ is called a NA normed space.

Due to the fact that

$$\|\nu_n - \nu_m\| \leq \max\{\|\nu_{j+1} - \nu_j\| : m \leq j \leq n-1\}\ (n > m)$$

a sequence $\{\nu_n\}$ is Cauchy iff $\{\nu_{n+1} - \nu_n\}$ converges to zero in a NA space. By a complete NA normed space we mean one in which every Cauchy sequence is convergent.

Example 2.1. [15] Let p be a fixed prime number. For any non-zero rational number ν, there is a unique integer $n_\nu \in \mathbb{Z}$ such that

$$\nu = \frac{s}{t}p^{n_\nu},$$

where s and t are integers not divisible by p. Then, the function $|.|_p : \mathbb{Q} \to [0, +\infty)$ defined by

$$|\nu| = \begin{cases} 0, & \nu = 0, \\ p^{-n_\nu}, & \nu \neq 0 \end{cases}$$

is a NA valuation on $\mathbb{Q}$.

Example 2.2. [15] Let $\nu = \frac{60}{7}$. Suppose we want to find its 5-adic absolute value (hence $p = 5$). Expressed in the p-adic form, we have

$$\nu = \frac{60}{7} = 5^1.\frac{12}{7}$$

which mean $|\nu|_5 = \frac{1}{5} or 5^{-1}$.

7-adic absolute value for ν. It will be simple to $|\nu|_7 = 7$, because

$$\nu = 7^{-1}.60$$

$$|\nu|_7 = \frac{1}{7^{-1}} = 7.$$

which mean $|\nu|_7 = 7$.

Definition 2.4. [13] Let $\mathcal{X}$ be a set. A function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a generalized metric on $\mathcal{X}$ if ρ fulfills the following conditions:

(i) $\rho(\nu, \upsilon) = 0$ iff $\mathfrak{u} = \upsilon$;
(ii) $\rho(\nu, \upsilon) = \rho(\upsilon, \mathfrak{u})$ for every $\mathfrak{u}, \upsilon \in \mathcal{X}$;
(iii) $\rho(\nu, \omega) \leq \rho(\nu, \upsilon) + \rho(\upsilon, \omega)$ for every $\nu, \upsilon, \omega \in \mathcal{X}$.

We recall some of the following fundamental results.

Theorem 2.1. [8], [13] Let $(\mathcal{X}, \rho)$ be a complete generalized metric space(CGMS) and let $\Delta : \mathcal{X} \to \mathcal{X}$ be a strictly contractive with $\pounds < 1$. Then, for each given element $\mathfrak{u} \in \mathcal{X}$, either $\rho(\Delta^n\nu, \Delta^{n+1}\nu) = \infty$ for all $n \geq 0$ or there is a $n_0 \in \mathbb{N}$ satisfies

(i) $\rho(\Delta^n\nu, \Delta^{n+1}\nu) < \infty$, for every $n \geq n_0$;
(ii) the sequence $\{\Delta^n\nu\}$ converges to a fixed point υ^* of Δ;
(iii) υ^* is the unique fixed point of Δ in the set $\mathcal{Y} = \{\upsilon \in \mathcal{X}\ /\ \rho(\Delta^{n_0}\nu, \upsilon) < \infty\}$;
(iv) for every $\upsilon \in \mathcal{Y}$, we have

$$\rho(\upsilon, \upsilon^*) \leq \frac{1}{1 - \pounds}\ \rho(\upsilon, \Delta\upsilon).$$

The use of FEs to prove new fixed point theorems with applications was first made possible by Isac and Rassias in 1996 [17]. Several researchers have thoroughly explored the stability problems of several FEs using fixed point techniques (see [5], [6], [8], [13], [18], [28]).

For coding simplicity, we can define a function $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ by

$$D\mathfrak{g}(\nu_1, \nu_2, ..., \nu_k) = \mathfrak{g}\left(\mathfrak{t}\sum_{i=1}^{k-1} \nu_i + \nu_k\right) + \mathfrak{g}\left(\mathfrak{t}\sum_{i=1}^{k-1} \nu_i - \nu_k\right) + 2\mathfrak{t}\sum_{i=1}^{k-1} \mathfrak{g}(\nu_i) - 2\mathfrak{t}^3\mathfrak{g}\left(\sum_{i=1}^{k-1} \nu_i\right) - \mathfrak{t}\sum_{i=1}^{k-1}\left[\mathfrak{g}(\nu_i + \nu_k) + \mathfrak{g}(\nu_i - \nu_k)\right] \quad (8)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$, where $t \geq 2$ is a positive integer number.

III. General Solution

Lemma 3.1. Let $\mathcal{X}$ and $\mathcal{W}$ be linear space. A mapping $\mathfrak{g}:\mathcal{X}\to\mathcal{W}$ satisfies the FE (7) if and only if $\mathfrak{g}$ is cubic.

Proof. Assume that the function $\mathfrak{g}$ satisfies the FE (7). By taking $\nu_i=0,(i=1,2,...,k)$, we get $\mathfrak{g}(0)=0$, and by setting $2\nu_i=0,(i=1,2,...,k-1)$, and $\nu_m=\nu$, we may conclude that

$$\mathfrak{g}(-\nu)=-\mathfrak{g}(\nu)\qquad for\ every\ \nu\in\mathcal{X},$$

i.e., $\mathfrak{g}$ is a function. Now, we will show $\nu_m=\upsilon$ in the equation (7), we obtain

$$\mathfrak{g}(2\nu+\upsilon)+\mathfrak{g}(2\nu-\upsilon)+4\mathfrak{g}(\nu)=16\mathfrak{g}(\nu)+2[\mathfrak{g}(\nu+\upsilon)+\mathfrak{g}(\nu-\upsilon)]$$

for every $\nu,\upsilon\in\mathcal{X}$. Thus $\mathfrak{g}$ is cubic.

$$\mathfrak{g}(t\nu+\upsilon)+\mathfrak{g}(t\nu-\upsilon)+2t\mathfrak{g}(\nu)=2t^3\mathfrak{g}(\nu)+t[\mathfrak{g}(\nu+\upsilon)+\mathfrak{g}(\nu-\upsilon)]$$

for every $\nu,\upsilon\in\mathcal{X}$. Conversely, since $\mathfrak{g}$ is cubic, we have the FE

$$\mathfrak{g}(2\nu+\upsilon)+\mathfrak{g}(2\nu-\upsilon)=12\mathfrak{g}(\nu)+[\mathfrak{g}(\nu+\upsilon)+\mathfrak{g}(\nu-\upsilon)] \tag{9}$$

Then obtaining the following characteristics is simple:
1) $\mathfrak{g}(0)=0$
2) $\mathfrak{g}(-\upsilon)=-\mathfrak{g}(\upsilon)$
3) $\mathfrak{g}(t\nu)=t^3\mathfrak{g}(\nu)$
4) $\mathfrak{g}(\nu+2\upsilon)+\mathfrak{g}(\nu-2\upsilon)+6\mathfrak{g}(\nu)=4[\mathfrak{g}(\nu+\upsilon)+\mathfrak{g}(\nu-\upsilon)]$

this will be demonstrated through induction on $k\geq 2$. It holds on $k=2$: see [19]. Suppose it is true in the case when $k=m$; that is, we get

$$\mathfrak{g}\left(t\sum_{i=1}^{m-1}\nu_i+\nu_m\right)+\mathfrak{g}\left(t\sum_{i=1}^{m-1}\nu_i-\nu_m\right)+2t\sum_{i=1}^{m-1}\mathfrak{g}(\nu_i)$$
$$=2t^3\mathfrak{g}\left(\sum_{i=1}^{m-1}\nu_i\right)+t\sum_{i=1}^{m-1}\Big[\mathfrak{g}(\nu_i+\nu_m)+\mathfrak{g}(\nu_i-\nu_m)\Big]$$

for every $\nu_1,\nu_2,...,\nu_m\in\mathcal{X}$. Now, letting $\nu_1=\nu_1+\upsilon$ and $\nu_i=\nu,\ (i=2,3,...,m)$, we get

$$\mathfrak{g}\left(\sum_{i=1}^{m-1}t\nu_i+t\upsilon+\nu_m\right)+\mathfrak{g}\left(\sum_{i=1}^{m-1}t\nu_i+t\upsilon-\nu_m\right)+$$
$$2t\mathfrak{g}(\nu_1+\upsilon)+2t\sum_{i=2}^{m-1}\mathfrak{g}(\nu_1+\upsilon)=2t^3\mathfrak{g}\left(\sum_{i=1}^{m-1}\nu_i+\upsilon\right)$$
$$+t\mathfrak{g}(\nu_1+\upsilon+\nu_m)+t\mathfrak{g}(\nu_1+\upsilon-\nu_m)$$
$$+t\sum_{i=2}^{m-1}\Big[\mathfrak{g}(\nu_i+\upsilon_m)+\mathfrak{g}(\nu_i-\nu_m)\Big] \tag{10}$$

since $\mathfrak{g}$ is cubic, the following equation is possible:

$$\mathfrak{g}(\nu+\upsilon+2\omega)+\mathfrak{g}(\nu+\upsilon-2\omega)+\mathfrak{g}(2\nu)+\mathfrak{g}(2\upsilon)$$
$$=2[\mathfrak{g}(\nu+\upsilon)+2\mathfrak{g}(\nu+\omega)+2\mathfrak{g}(\nu-\omega)$$
$$+2\mathfrak{g}(\upsilon+\omega)+2\mathfrak{g}(\upsilon-\omega)] \tag{11}$$

for every $\nu,\upsilon,\omega\in\mathcal{X}$. By Putting $\nu=2\nu_1,\ \upsilon=2\upsilon$ and $\omega=\nu_m$ in the equation (11) and using the property (3), we get

$$8\mathfrak{g}(\nu_1+\upsilon+\nu_m)+8\mathfrak{g}(\nu_1+\upsilon-\nu_m)+8\mathfrak{g}(2\nu_1)+8\mathfrak{g}(2\upsilon)$$
$$=2[\mathfrak{g}(2\nu_1+2\upsilon)+2\mathfrak{g}(2\nu_1+\nu_m)+2\mathfrak{g}(2\nu_1-\nu_m)$$
$$+2\mathfrak{g}(2\upsilon+\nu_m)+2\mathfrak{g}(2\upsilon-\nu_m)]$$

for every $\nu,\upsilon,\nu_m\in\mathcal{X}$. Thus we get

$$2\mathfrak{g}(\nu_1+\upsilon+\nu_m)+2\mathfrak{g}(\nu_1+\upsilon-\nu_m)+2\mathfrak{g}(2\nu_1)+2\mathfrak{g}(2\upsilon)$$
$$=4\mathfrak{g}(2\nu_1+2\upsilon)+\mathfrak{g}(2\nu_1+\nu_m)+\mathfrak{g}(2\nu_1-\nu_m)$$
$$+\mathfrak{g}(2\upsilon+\nu_m)+\mathfrak{g}(2\upsilon-\nu_m)$$

for every $\nu,\upsilon,\nu_m\in\mathcal{X}$. From the equation (9), we may have

$$2\mathfrak{g}(\nu_1+\upsilon+\nu_m)+2\mathfrak{g}(\nu_1+\upsilon-\nu_m)+16\mathfrak{g}(\nu_1)+16\mathfrak{g}(\upsilon)$$
$$=4\mathfrak{g}\mathfrak{h}(\nu_1+\upsilon)+2[\mathfrak{g}(\nu_1+\nu_m)+\mathfrak{g}(\nu_1-\nu_m)]$$
$$+2[\mathfrak{g}(\upsilon+\nu_m)+\mathfrak{g}(\upsilon-\nu_m)]+12\mathfrak{g}(\nu_1)+12\mathfrak{g}(\upsilon)$$

for every $\nu_1,\upsilon,\nu_m\in\mathcal{X}$. Hence the equation (10) will be

$$\mathfrak{g}\left(\sum_{i=1}^{m-1}t\nu_i+t\upsilon+\nu_m\right)+\mathfrak{g}\left(\sum_{i=1}^{m-1}t\nu_i+t\upsilon-\nu_m\right)$$
$$+2t\mathfrak{g}(\nu_1+\upsilon)+2t\sum_{i=2}^{m-1}\mathfrak{g}(\nu_1+\upsilon)$$
$$=2t^3\mathfrak{g}\left(\sum_{i=1}^{m-1}\nu_i+\upsilon\right)+t\mathfrak{g}(\nu_1+\upsilon+\nu_m)+$$
$$t\mathfrak{g}(\nu_1+\upsilon-\nu_m)+t\sum_{i=2}^{m-1}\Big[\mathfrak{g}(\nu_i+\nu_m)+\mathfrak{g}(\nu_i-\nu_m)\Big]$$
$$=2t^3\mathfrak{g}\left(\sum_{i=1}^{m-1}\nu_i+\upsilon\right)-2t\mathfrak{g}(\nu_1)-2t\mathfrak{g}(\upsilon)$$
$$+4\mathfrak{g}(\nu_1+\upsilon)+t\mathfrak{g}(\upsilon+\nu_m)+\mathfrak{g}(\upsilon-\nu_m)$$
$$+t\sum_{i=1}^{m-1}\Big[\mathfrak{g}(\nu_i+\nu_m)+\mathfrak{g}(\nu_i-\nu_m)\Big]$$

for every $\nu_1,\nu_2,...,\nu_m\in\mathcal{X}$ and $\upsilon\in\mathcal{X}$. As a result, if $\mathfrak{g}$ is cubic, we get the required equation (7).

IV. H-U stability in Non-Archimedean Banach spaces

In this section, $\mathcal{X}$ and $\mathcal{W}$ are considered as NANS and NABS, respectively.

A. Stability Results:Direct method

In this part, we examine the H-U stability of the FE (8) by using the direct method.

Theorem 4.1. Let a mapping $\varphi:\mathcal{X}^k\to[0,\infty)$ and a mapping $\mathfrak{g}:\mathcal{X}\to\mathcal{W}$ be a mapping that satisfies $\mathfrak{g}(0)=0$ and

$$\|D\mathfrak{g}(\nu_1,\nu_2,...,\nu_k)\|\leq\varphi(\nu_1,\nu_2,...,\nu_k) \tag{12}$$

for every $\nu_1,\nu_2,...,\nu_k\in\mathcal{X}$, with

$$\lim_{l\to\infty}|\mathfrak{t}|^{3l}\varphi\left(\frac{\nu_1}{\mathfrak{t}^l},\frac{\nu_2}{\mathfrak{t}^l},...,\frac{\nu_k}{\mathfrak{t}^l}\right)=0 \tag{13}$$

Then, there is a unique cubic mapping $\mathcal{C}_3:\mathcal{X}\to\mathcal{W}$ that satisfies

$$\|\mathfrak{g}(\nu)-\mathcal{C}_3(\nu)\|\leq\sup_{l\in\mathbb{N}}\left\{\frac{1}{2}|t|^{3(l-1)}\varphi\left(\frac{\nu}{t^l},\frac{\nu}{t^l},0,...,0\right)\right\} \tag{14}$$

for every $\nu\in\mathcal{X}$.

Proof: Replacing $(\nu_1, \nu_2, ..., \nu_k)$ by $(\nu, \nu, 0, ..., 0)$ in (8), we get

$$\left\|2\mathfrak{g}(t\nu) - 2t^3\mathfrak{g}(\nu)\right\| \leq \varphi(\nu, \nu, 0, ..., 0) \tag{15}$$

for every $\nu \in \mathcal{X}$. This implies

$$\left\|\mathfrak{g}(\nu) - t^3\mathfrak{g}\left(\frac{\nu}{t}\right)\right\| \leq \frac{1}{2}\varphi\left(\frac{\nu}{t}, \frac{\nu}{t}, 0..., 0\right)$$

for every $\nu \in \mathcal{X}$. Hence

$$\begin{aligned}
&\left\|t^{3m}\mathfrak{g}\left(\frac{\nu}{t^m}\right) - t^{3\mathfrak{n}}\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{n}}}\right)\right\| \\
&\leq max\bigg\{\left\|t^{3m}\mathfrak{g}\left(\frac{\nu}{t^m}\right) - t^{3(m+1)}\mathfrak{g}\left(\frac{\nu}{t^{m+1}}\right)\right\| \\
&\qquad, ..., \left\|t^{3(\mathfrak{n}-1)}\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{n}-1}}\right) - t^{3\mathfrak{n}}\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{n}}}\right)\right\|\bigg\} \\
&\leq max\bigg\{|t|^{3m}\left\|\mathfrak{g}\left(\frac{\nu}{t^m}\right) - t^3\mathfrak{g}\left(\frac{\nu}{t^{m+1}}\right)\right\| \\
&\qquad, ..., |t|^{3(\mathfrak{n}-1)}\left\|\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{n}-1}}\right) - t^3\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{n}}}\right)\right\|\bigg\} \\
&\leq \sup_{l\in\{m,m+1,...\}}\left\{\frac{1}{2}|t|^{3l}\varphi\left(\frac{\nu}{t^{l+1}}, \frac{\nu}{t^{l+1}}, 0, ..., 0\right)\right\} \quad (16)
\end{aligned}$$

for any $\mathfrak{n} > m > 0$ and for any $\nu \in \mathcal{X}$. Therefore, we conclude from (16) and (13) that the sequence $\left\{t^{3l}\mathfrak{g}\left(\frac{\nu}{t^l}\right)\right\}$ is a Cauchy in $\mathcal{W}$ for every $\nu \in \mathcal{X}$. Since $\mathcal{W}$ is complete, the sequence $\left\{t^{3l}\mathfrak{g}\left(\frac{\nu}{t^l}\right)\right\}$ converges in $\mathcal{W}$ for every $\nu \in \mathcal{X}$.

Consequently, the mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ might be defined

$$\mathcal{C}_3(\nu) = \lim_{m\to\infty} t^{3m}\mathfrak{g}\left(\frac{\nu}{t^m}\right), \qquad \nu \in \mathcal{X}.$$

Taking $m = 0$ and the limit $\mathfrak{n} \to \infty$ in (16), we get (14). It follows from (12) and (13), we get

$$\begin{aligned}
\|\mathcal{D}\mathcal{C}_3(\nu_1, \nu_2, ..., \nu_k)\| &= \lim_{l\to\infty}|t|^{3l}\left\|\mathcal{D}\mathfrak{g}\left(\frac{\nu_1}{t^l}, \frac{\nu_2}{t^l}, ..., \frac{\nu_k}{t^l}\right)\right\| \\
&\leq \lim_{l\to\infty}|t|^{3l}\varphi\left(\frac{\nu_1}{t^l}, \frac{\nu_2}{t^l}, ..., \frac{\nu_k}{t^l}\right) = 0
\end{aligned}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$. Thus, we get

$$\mathcal{D}\mathcal{C}_3(\nu_1, \nu_2, ..., \nu_k) = 0.$$

From Lemma (1), the function $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ is cubic. If $\mathcal{C}_3^{'}$ is another cubic mapping satisfies (14), then

$$\begin{aligned}
&\|\mathcal{C}_3(\nu) - \mathcal{C}_3^{'}(\nu)\| \\
&= \left\|t^{3\mathfrak{s}}\mathcal{C}_3\left(\frac{\nu}{t^{\mathfrak{s}}}\right) - t^{3\mathfrak{s}}\mathcal{C}_3^{'}\left(\frac{\nu}{t^{\mathfrak{s}}}\right)\right\| \\
&\leq max\bigg\{\left\|t^{3\mathfrak{s}}\mathcal{C}_3\left(\frac{\nu}{t^{\mathfrak{s}}}\right) - t^{3\mathfrak{s}}\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{s}}}\right)\right\|, \\
&\qquad \left\|t^{3\mathfrak{s}}\mathfrak{g}\left(\frac{\nu}{t^{\mathfrak{s}}}\right) - t^{3\mathfrak{s}}\mathcal{C}_3^{'}\left(\frac{\nu}{t^{\mathfrak{s}}}\right)\right\|\bigg\} \\
&\leq \sup_{l\in\mathbb{N}}\left\{\frac{1}{2}|t|^{\mathfrak{s}+l-1}\varphi\left(\frac{\nu}{t^{\mathfrak{s}+l}}, \frac{\nu}{t^{\mathfrak{s}+l}}, 0, ..., 0\right)\right\}, \\
&\quad \to 0 \ as \ k \to \infty
\end{aligned}$$

for every $\nu \in \mathcal{X}$. So $\mathcal{C}_3(\nu) = \mathcal{C}_3^{'}(\nu)$. ■

Corollary 1. *If a function $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ with $\mathfrak{g}(0) = 0$ and satisfies*

$$\|\mathcal{D}\mathfrak{g}(\nu_1, \nu_2, ..., \nu_k)\| \leq \beta\left(\sum_{i=1}^{k}\|\nu_i\|^{\alpha\gamma} + \prod_{i=1}^{k}\|\nu_i\|^{\gamma}\right)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$, then there is a unique cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ satisfying

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \frac{\beta}{|t|^{\alpha\gamma}}\|\nu\|^{\alpha\gamma} \tag{17}$$

for every $\nu \in \mathcal{X}$, where $\alpha\gamma < 3$, $t \geq 2$ and β are in $\mathbb{R}^+$.

Theorem 4.2. Let a mapping $\varphi : \mathcal{X}^k \to [0, \infty)$ and a mapping $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ be a mapping that satisfies $f(0) = 0$ and

$$\|D\mathfrak{g}(\nu_1, \nu_2, ..., \nu_k)\| \leq \varphi(\nu_1, \nu_2, ..., \nu_k) \tag{18}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$, with

$$\lim_{l\to\infty}\frac{1}{|\mathrm{t}|^{3l}}\varphi\left(\mathrm{t}^{l-1}\nu_1, \mathrm{t}^{l-1}\nu_2, ..., \mathrm{t}^{l-1}\nu_k\right) = 0 \tag{19}$$

Then, there is a unique cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ that satisfies

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \sup_{l\in\mathbb{N}}\left\{\frac{1}{2}\frac{1}{|t|^{3l}}\varphi\left(\mathrm{t}^{l-1}\nu, \mathrm{t}^{l-1}\nu, 0, ..., 0\right)\right\} \tag{20}$$

for every $\nu \in \mathcal{X}$.

Proof: It follows from (15) that

$$\left\|\mathfrak{g}(\nu) - \frac{1}{t^3}\mathfrak{g}(t\nu)\right\| \leq \frac{1}{2}\frac{1}{|t|^3}\varphi(\nu, \nu, 0, ..., 0) \tag{21}$$

for every $\nu \in \mathcal{X}$. Hence

$$\begin{aligned}
&\left\|\frac{1}{t^{3m}}\mathfrak{g}\left(t^m\nu\right) - \frac{1}{t^{3\mathfrak{n}}}\mathfrak{g}\left(t^{\mathfrak{n}}\nu\right)\right\| \\
&\leq max\bigg\{\left\|\frac{1}{t^{3m}}\mathfrak{g}\left(t^m\nu\right) - \frac{1}{t^{3(m+1)}}\mathfrak{g}\left(t^{m+1}\nu\right)\right\| \\
&\qquad, ..., \left\|\frac{1}{t^{3(\mathfrak{n}-1)}}\mathfrak{g}\left(t^{\mathfrak{n}-1}x\right) - \frac{1}{t^{3\mathfrak{n}}}\mathfrak{g}\left(t^{\mathfrak{n}}\nu\right)\right\|\bigg\} \\
&\leq max\bigg\{\frac{1}{|t|^{3m}}\left\|\mathfrak{g}\left(t^m\nu\right) - \frac{1}{t^3}\mathfrak{g}\left(t^{m+1}\nu\right)\right\| \\
&\qquad, ..., \frac{1}{|t|^{3(\mathfrak{n}-1)}}\left\|\mathfrak{g}\left(t^{\mathfrak{n}-1}\nu\right) - \frac{1}{t^3}\mathfrak{g}\left(t^{\mathfrak{n}}\nu\right)\right\|\bigg\} \\
&\leq \sup_{l\in\{m,m+1,...\}}\left\{\frac{1}{2}\frac{1}{|t|^{3(l+1)}}\varphi\left(t^l\nu, t^l\nu, 0, ..., 0\right)\right\} \quad (22)
\end{aligned}$$

for any $\mathfrak{n} > m > 0$ and for any $\nu \in \mathcal{X}$. Therefore, we conclude from (22) and (19), that the sequence $\left\{\frac{1}{t^{3l}}\mathfrak{g}\left(t^l\nu\right)\right\}$ is Cauchy in $\mathcal{W}$ for every $\nu \in \mathcal{X}$. Since $\mathcal{W}$ is complete, the sequence $\left\{\frac{1}{t^{3l}}\mathfrak{g}\left(t^l\nu\right)\right\}$ converges in $\mathcal{W}$ for every $\nu \in \mathcal{X}$. Consequently, the mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ might be defined

$$\mathcal{C}_3(\nu) = \lim_{m\to\infty}\frac{1}{t^{3m}}\mathfrak{g}\left(t^m\nu\right), \qquad \nu \in \mathcal{X}.$$

Taking $m = 0$ and the limit $\mathfrak{n} \to \infty$ in (22), we get (20). As a result of (18) and (19), we get

$$\begin{aligned}
&\|\mathcal{D}\mathcal{C}_3(\nu_1, \nu_2, ..., \nu_k)\| \\
&= \lim_{l\to\infty}\frac{1}{|t|^{3l}}\left\|\mathcal{D}\mathfrak{g}\left(t^l\nu_1, t^l\nu_2, ..., t^l\nu_k\right)\right\| \\
&\leq \lim_{l\to\infty}\frac{1}{|t|^{3l}}\varphi\left(t^{l-1}\nu_1, t^{l-1}x_\nu 2, ..., t^{l-1}\nu_k\right) = 0
\end{aligned}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$. Thus, we get

$$\mathcal{D}\mathcal{C}_3(\nu_1, \nu_2, ..., \nu_k) = 0.$$

From Lemma (1), the function $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ is cubic. If $\mathcal{C}_3^{'}$ is another cubic mapping satisfies (20), then

$$\begin{aligned}
&\|\mathcal{C}_3(\nu) - \mathcal{C}_3^{'}(\nu)\| \\
&\quad = \left\| \frac{1}{t^{3\mathfrak{s}}}\mathcal{C}_3\left(t^{\mathfrak{s}}\nu\right) - \frac{1}{t^{3\mathfrak{s}}}\mathcal{C}_3^{'}\left(t^{\mathfrak{s}}\nu\right) \right\| \\
&\quad \le max\Bigg\{ \left\| \frac{1}{t^{3\mathfrak{s}}}\mathcal{C}_3\left(t^{\mathfrak{s}}\nu\right) - \frac{1}{t^{3\mathfrak{s}}}\mathfrak{g}\left(t^{\mathfrak{s}}\nu\right) \right\|, \\
&\qquad\qquad \left\| \frac{1}{t^{3\mathfrak{s}}}\mathfrak{g}\left(t^{\mathfrak{s}}\nu\right) - \frac{1}{t^{3\mathfrak{s}}}\mathcal{C}_3^{'}\left(t^{\mathfrak{s}}\nu\right) \right\| \Bigg\} \\
&\quad \le \sup_{l\in\mathbb{N}} \left\{ \frac{1}{2}\frac{1}{|t|^{3(\mathfrak{s}+l)}}\varphi\left(t^{\mathfrak{s}+l-1}\nu, t^{\mathfrak{s}+l-1}\nu, 0, ..., 0\right) \right\}, \\
&\qquad \to 0 \ as \ s \to \infty
\end{aligned}$$

for every $\nu \in \mathcal{X}$. So $\mathcal{C}_3(\nu) = \mathcal{C}_3^{'}(\nu)$. ■

Corollary 2. *If a function $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ with $\mathfrak{g}(0) = 0$ and satisfies*

$$\|\mathcal{D}\mathfrak{g}(\nu_1, \nu_2, ..., \nu_k)\| \le \beta \left(\sum_{i=1}^{k} \|\nu_i\|^{\alpha\gamma} + \prod_{i=1}^{k} \|\nu_i\|^{\gamma} \right)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$, then there is a unique cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ satisfying

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \le \frac{\beta}{|t|^3}\|\nu\|^{\alpha\gamma}, \tag{23}$$

for every $\nu \in \mathcal{X}$, where $\alpha\gamma > 3,\ t \ge 2$ and β are in $\mathbb{R}^+$.

Example 4.1. Let $p > 2$ be a prime number and $\mathcal{X} = \mathcal{W} = \mathbb{Q}_p$. Define $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ by $\mathfrak{g}(\nu) = \nu^3 + \nu$ for every $\nu \in \mathcal{X}$. Since $|2| = 1$, In particular $k = 2$ and $t = 2$

$$\begin{aligned}
&\|\mathcal{D}\mathfrak{g}(\nu_1, \nu_2)\| \\
&\quad = |12|.\|\nu_1\| \le \beta\left(\|\nu_1\|^{\alpha\gamma} + \|\nu_2\|^{\alpha\gamma} + \|\nu_1\|^{\gamma}\|\nu_2\|^{\gamma}\right)
\end{aligned}$$

for every $\nu_1, \nu_2 \in \mathcal{X}$, where β is a positive number and $\alpha\gamma > 0$.

(i) $\mathfrak{g}(\nu) = \nu^3 + \nu, \qquad \mathcal{C}_3(\nu) = \nu^3$

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \le \frac{\beta}{|2|^{\alpha\gamma}}\|\nu\|^{\alpha\gamma}$$
$$\|\nu\| \le \beta\|\nu\| \qquad (put\ \alpha\gamma = 1, |2| = 1)$$

(ii) $\mathfrak{g}(\nu) = \nu^3 + \nu, \qquad \mathcal{C}_3(\nu) = \nu^3$

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \le \frac{\beta}{|2|^{3}}\|\nu\|^{\alpha\gamma}$$
$$\|\nu\| \le \beta\|\nu\|^4 \qquad (put\ \alpha\gamma = 4, |2| = 1)$$

Therefore, equation (17) and (23) are satisfied.

B. Stability Results:Fixed point method

Theorem 4.3. Let $\varphi : \mathcal{X}^k \to [0, \infty)$ be a function such that there is a constant $\pounds < 1$ with

$$\varphi\left(\frac{\nu_1}{\mathfrak{t}}, \frac{\nu_2}{\mathfrak{t}}, ..., \frac{\nu_k}{\mathfrak{t}}\right) \le \frac{\pounds}{|\mathfrak{t}|^3}\varphi(\nu_1, \nu_2, ..., \nu_k) \tag{24}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$. Let $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ be a mapping satisfying

$$\|D\mathfrak{g}(\nu_1, \nu_2, ..., \nu_k)\| \le \varphi(\nu_1, \nu_2, ..., \nu_k) \tag{25}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$. Then there is a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{W}$ such that

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \le \frac{\pounds}{|t|^3(1 - \pounds)} \ \frac{1}{2}\ \varphi(\nu, \nu, 0, ...0) \tag{26}$$

for every $\nu \in \mathcal{X}$.

Proof: Replacing $(\nu_1, \nu_2, ..., \nu_k)$ by $(\nu, \nu, 0, ..., 0)$ in (25), we have

$$\begin{aligned}
\left\|2\mathfrak{g}(\mathfrak{t}\nu) - 2\mathfrak{t}^3\mathfrak{g}(\nu)\right\| &\le \Psi(\nu) \\
\left\|\mathfrak{g}(\mathfrak{t}\nu) - \mathfrak{t}^3\mathfrak{g}(\nu)\right\| &\le \frac{1}{2}\Psi(\nu)
\end{aligned} \tag{27}$$

for every $\nu \in \mathcal{X}$. Where

$$\Psi(\nu) = \frac{1}{2}\varphi(\nu, \nu, 0, ..., 0)$$

for every $\nu \in \mathcal{X}$. Consider the set

$$\Omega = \{\mathfrak{q} : \mathcal{X} \to \mathcal{W}, q(0) = 0\}$$

as well as the generalized metric ρ on Ω:

$$\rho(\mathfrak{p}, \mathfrak{q}) = inf\{\sigma \in \mathbb{R}^+ : \|\mathfrak{p}(\nu) - \mathfrak{q}(\nu)\| \le \sigma.\Psi(\nu),\ \ \forall\, \nu \in \mathcal{X}\},$$

where, as usual, $\inf \phi = +\infty$. It is simple to show that (Ω, ρ) is complete(see [20], Lemma 2.1).

Now, we take the linear mapping $\Delta : \Omega \to \Omega$ such that

$$\Delta\mathfrak{p}(\nu) = \mathfrak{t}^3\ \mathfrak{p}\left(\frac{\nu}{\mathfrak{t}}\right) \tag{28}$$

for every $\nu \in \mathcal{X}$. Let $\mathfrak{p}, \mathfrak{q} \in \Omega$ be given such that $\rho(\mathfrak{p}, \mathfrak{q}) = \varepsilon$. Then, we get

$$\|\mathfrak{p}(\nu) - \mathfrak{q}(\nu)\| \le \varepsilon\ \Psi(\nu) \tag{29}$$

for every $\nu \in \mathcal{X}$. Hence

$$\begin{aligned}
\|\Delta\mathfrak{p}(\nu) - \Delta\mathfrak{q}(\nu)\| &= \left\|t^3\mathfrak{p}\left(\frac{\nu}{\mathfrak{t}}\right) - t^3\mathfrak{q}\left(\frac{\nu}{\mathfrak{t}}\right)\right\| \\
&\le |t|^3\ \varepsilon\ \Psi\left(\frac{\nu}{\mathfrak{t}}, \frac{\nu}{\mathfrak{t}}, ..., 0\right) \\
&\le |\mathfrak{t}|^3\ \varepsilon\ \frac{\pounds}{|\mathfrak{t}|^3}\ \ \Psi(\nu) \\
&\le \varepsilon\ \pounds\ \Psi(\nu)
\end{aligned}$$

for every $\nu \in \mathcal{X}$.

By definition $\rho(\Delta(\mathfrak{p}), \Delta(\mathfrak{q})) \le \pounds\ \varepsilon$. Therefore

$$\rho(\Delta(\mathfrak{p}), \Delta(\mathfrak{q})) \le \pounds\rho(\mathfrak{p}, \mathfrak{q}) \qquad \text{for every } \mathfrak{p}, \mathfrak{q} \in \Omega$$

This means that Δ is a Lipschits constant $\pounds$ strictly contractive self-mapping of $\mathcal{V}$.

It follows form (27) that

$$\begin{aligned}
\left\|\mathfrak{g}(\mathfrak{t}\nu) - \mathfrak{t}^3\mathfrak{g}(\nu)\right\| &\le \frac{1}{2}\varphi\left(\frac{\nu}{t}, \frac{\nu}{t}, 0, ..., 0\right) \\
&\le \frac{1}{2}\frac{\pounds}{|t|^3}\varphi(\nu, \nu, 0, ..., 0)
\end{aligned}$$

for every $\nu \in \mathcal{X}$. So $\rho(\mathfrak{g}, \Delta\mathfrak{g}) \le \frac{1}{2}\frac{\pounds}{|\mathfrak{t}|^3}$.

By Theorem 2.1, there is a cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ satisfying the following:

(i) $\mathcal{C}_3$ is a fixed point of Δ in the set

$$\delta = \{\mathfrak{p} \in \Omega\ :\ \rho(\mathfrak{p}, \mathfrak{q}) < \infty\}$$

$$\mathcal{C}_3(\nu) = t^3 \ \mathcal{C}_3\left(\frac{\nu}{\mathfrak{t}}\right) \tag{30}$$

This yields that $\mathcal{C}_3$ is a unique mapping satisfying (30) such that there is a $\sigma \in (0, \infty)$ satisfying

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \sigma \ . \ \Psi(\nu)$$

for every $\nu \in \mathcal{X}$.

(ii) $\rho(\Delta^n \mathfrak{g}, \mathcal{C}_3) \to 0$ as $n \to \infty$. This indicates inequality

$$\lim_{n\to\infty} t^{3n} \mathfrak{g}\left(\frac{\nu}{\mathfrak{t}^n}\right) = \mathcal{C}_3(\nu) \qquad \nu \in \mathcal{X}$$

(iii) Moreover, $\rho(\mathfrak{g}, \mathcal{C}_3) \leq \dfrac{1}{1-\pounds}\rho(\mathfrak{g}, \Delta\mathfrak{g})$, and it implies the following:

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \frac{\pounds}{|t|^3(1-\pounds)}\Psi(\nu)$$

for every $\nu \in \mathcal{X}$.

As a result (24) and (25) that

$$\begin{aligned}\|\mathcal{D}\mathcal{C}_3(\nu_1,\nu_2, ..., \nu_k)\| &= \lim_{n\to\infty} |\mathfrak{t}|^{3n} \left\|\mathcal{D}\ \mathfrak{g}\left(\frac{\nu_1}{\mathfrak{t}^n}, \frac{\nu_2}{\mathfrak{t}^n}, ..., \frac{\nu_k}{\mathfrak{t}^n}\right)\right\| \\ &\leq \lim_{n\to\infty} |\mathfrak{t}|^{3n}\ \varphi\left(\frac{\nu_1}{\mathfrak{t}^n}, \frac{\nu_2}{\mathfrak{t}^n}, ..., \frac{\nu_k}{\mathfrak{t}^n}\right) = 0.\end{aligned}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$.

Thus $\|D\mathcal{C}_3(\nu_1, \nu_2, ..., \nu_k)\| = 0$. The mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ is cubic. ■

Corollary 3. *If a mapping $\varphi : \mathcal{X} \to \mathcal{W}$ such that $\varphi(0) = 0$ and*

$$\left\|D\mathfrak{g}\left(\frac{\nu_1}{\mathfrak{t}}, \frac{\nu_2}{\mathfrak{t}}, ..., \frac{\nu_k}{\mathfrak{t}}\right)\right\| \leq \ \lambda\left(\sum_{i=1}^{k}\|\nu_i\|^{\alpha\beta} + \prod_{i=1}^{k}\|\nu_i\|^{\beta}\right)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$, where $\alpha\beta < 3, \lambda$ are in $\mathbb{R}^+$ and $t \geq 2$, then there is a unique cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ satisfying

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \frac{\lambda.\|\nu\|^{\alpha\beta}}{2.|t|^{\alpha\beta} - |t|^3} \tag{31}$$

for every $\nu \in \mathcal{X}$.

Proof: Letting

$$\varphi(\nu_1, \nu_2, ..., \nu_k) = \lambda\left(\sum_{i=1}^{k}\|\nu_i\|^{\alpha\beta} + \prod_{i=1}^{k}\|\nu_i\|^{\beta}\right)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$.

The proof follows from Theorem 4.3. After that, we may use $\pounds = \dfrac{1}{2}|t|^{3-\alpha\beta}$ to obtain our required result. ■

Theorem 4.4. Let $\varphi : \mathcal{X}^k \to [0, \infty)$ be a function such that there is a constant $\pounds < 1$ with

$$\varphi(\mathfrak{t}\nu_1, \mathfrak{t}\nu_2, ..., \mathfrak{t}\nu_k) \leq |\mathfrak{t}|^3 \pounds\varphi(\nu_1, \nu_2, ..., \nu_k) \tag{32}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$. Let $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ be a mapping satisfying

$$\|D\mathfrak{g}(\nu_1, \nu_2, ..., \nu_k)\| \leq \varphi(\nu_1, \nu_2, ..., \nu_k) \tag{33}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$. Then there is a unique cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ such that

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \frac{1}{|t|^3(1-\pounds)} \ \ \frac{1}{2}\varphi(\nu, \nu, 0, ...0) \tag{34}$$

for every $\nu \in \mathcal{X}$.

Proof: It follows from (27) that

$$\left\|\mathfrak{g}(\nu) - \frac{1}{\mathfrak{t}^3}\mathfrak{g}(\mathfrak{t}\nu)\right\| \leq \frac{1}{|t|^3}\frac{1}{2}\varphi(\nu, \nu, 0, ..., 0) \tag{35}$$

for every $\nu \in \mathcal{X}$. Let us denoted by

$$\Psi(\nu) = \frac{1}{2}\varphi(\nu, \nu, 0, ..., 0)$$

for every $\nu \in \mathcal{X}$. Consider the linear mapping $\Delta : \Omega \to \Omega$ which has the following attribute

$$\Delta\mathfrak{p}(\nu) = \frac{1}{\mathfrak{t}^3}\ \mathfrak{p}(\mathfrak{t}\nu) \tag{36}$$

for every $\nu \in \mathcal{X}$. Let $\mathfrak{p}, \mathfrak{q} \in \Omega$ be given such that $\rho(\mathfrak{p}, \mathfrak{q}) = \varepsilon$. Then, we get

$$\|\mathfrak{p}(\nu) - \mathfrak{q}(\nu)\| \leq \varepsilon\ \Psi(\nu) \tag{37}$$

for every $\nu \in \mathcal{X}$. Hence

$$\begin{aligned}\|\Delta\mathfrak{p}(\nu) - \Delta\mathfrak{q}(\nu)\| &= \left\|\frac{1}{t^3}\mathfrak{p}(\mathfrak{t}\nu) - \frac{1}{t^3}\mathfrak{q}(\mathfrak{t}\nu)\right\| \\ &\leq \frac{1}{|t|^3}\ \varepsilon\ \Psi(\mathfrak{t}\nu) \\ &\leq \frac{1}{|t|^3}\ \varepsilon\ |t|^3\Psi(\nu) \\ &\leq \varepsilon\ \pounds\ \Psi(\nu)\end{aligned}$$

for every $\nu \in \mathcal{X}$.

By definition $\rho(\Delta\mathfrak{p}, \Delta\mathfrak{q}) \leq \pounds\ \varepsilon$. Therefore

$$\rho(\Delta\mathfrak{p}, \Delta\mathfrak{q}) \leq \pounds\rho(\mathfrak{p}, \mathfrak{q}) \qquad \forall\ \mathfrak{p}, \mathfrak{q} \in \Omega$$

This means that Δ is a strictly contractive self mapping of $\mathcal{V}$ with a Lipschits constant $\pounds$.

It follows from (35) that

$$\begin{aligned}\left\|\mathfrak{g}(\mathfrak{t}\nu) - \frac{1}{\mathfrak{t}^3}\mathfrak{g}(\mathfrak{t}\nu)\right\| &\leq \frac{1}{2}\varphi(t\nu, t\nu, 0, ..., 0) \\ &\leq \frac{1}{2}\frac{1}{|t|^3}\varphi(\nu, \nu, 0, ..., 0)\end{aligned}$$

for every $\nu \in \mathcal{X}$. So

$$\rho(\mathfrak{g}, \Delta\mathfrak{g}) \leq \frac{1}{2}\frac{1}{|\mathfrak{t}|^3}.$$

By Theorem 2.1, there is a cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ satisfying the following:

(i) $\mathcal{C}_3$ is a fixed point of Δ in the set

$$\delta = \{\mathfrak{p} \in \Omega\ :\ \rho(\mathfrak{p}, \mathfrak{q}) < \infty\}$$

$$\mathcal{C}_3(\mathfrak{t}\nu) = t^3\ \mathcal{C}_3(\nu) \tag{38}$$

This result that $\mathcal{C}_3$ being a unique mapping satisfying (38) such that there is a $\sigma \in (0, \infty)$ satisfying

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \sigma\ .\ \Psi(\nu)$$

for every $\nu \in \mathcal{X}$.

(ii) $\rho(\Delta^n \mathfrak{g}, \mathcal{C}_3) \to 0$ as $n \to \infty$. This indicates inequality

$$\lim_{n\to\infty} \frac{1}{t^{3n}} \mathfrak{g}\left(\mathfrak{t}^n \nu\right) = \mathcal{C}_3(\nu) \qquad \nu \in \mathcal{X}$$

(iii) Moreover, $\rho(\mathfrak{g}, \mathcal{C}_3) \leq \dfrac{1}{1-\pounds}\rho(\mathfrak{g}, \Delta\mathfrak{g})$, and it implies the following

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \frac{1}{|t|^3(1-\pounds)}\Psi(\nu)$$

for every $\nu \in \mathcal{X}$.

As a results (32) and (33) that

$$\begin{aligned}
&\|\mathcal{D}\mathcal{C}_3(\nu_1,\nu_2, ..., \nu_k)\| \\
&\quad= \lim_{n\to\infty} \frac{1}{|\mathfrak{t}|^{3n}} \left\|\mathcal{D}\ \mathfrak{g}(\mathfrak{t}^n\nu_1, \mathfrak{t}^n\nu_2, ..., \mathfrak{t}^n\nu_k)\right\| \\
&\quad\leq \lim_{n\to\infty} \frac{1}{|\mathfrak{t}|^{3n}}\ |\mathfrak{t}|^{3n}\pounds^n\varphi(\mathfrak{t}^n\nu_1, \mathfrak{t}^n\nu_2, ..., \mathfrak{t}^n\nu_k) = 0.
\end{aligned}$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$.

Thus $\|D\mathcal{C}_3(\nu_1, \nu_2, ..., \nu_k)\| = 0$. The mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ is cubic. ■

Corollary 4. *If a mapping $\varphi : \mathcal{X} \to \mathcal{W}$ such that $\varphi(0) = 0$ and*

$$\|D\mathfrak{g}(\mathfrak{t}\nu_1, \mathfrak{t}\nu_2, ..., \mathfrak{t}\nu_k)\| \leq\ \lambda\left(\sum_{i=1}^{k}\|\nu_i\|^{\alpha\beta} + \prod_{i=1}^{k}\|\nu_i\|^{\beta}\right)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$, where $\alpha\beta > 3$, λ are in $\mathbb{R}^+$ and $t \geq 2$, then there is a unique cubic mapping $\mathcal{C}_3 : \mathcal{X} \to \mathcal{W}$ satisfying

$$\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| \leq \frac{\lambda.2.\|\nu\|^{\alpha\beta}}{2|t|^3 - |t|^{\alpha\beta}} \tag{39}$$

for every $\nu \in \mathcal{X}$.

Proof: Letting

$$\varphi(\nu_1, \nu_2, ..., \nu_k) = \lambda\left(\sum_{i=1}^{k}\|\nu_i\|^{\alpha\beta} + \prod_{i=1}^{k}\|\nu_i\|^{\beta}\right)$$

for every $\nu_1, \nu_2, ..., \nu_k \in \mathcal{X}$.

The proof is based on Theorem 4.4. After that, we may use $\pounds = \frac{1}{2}|t|^{\alpha\beta-3}$ to obtain our required result. ■

Example 4.2. Let $p > 2$ be a prime number and $\mathcal{X} = \mathcal{W} = \mathbb{Q}_p$. Define $\mathfrak{g} : \mathcal{X} \to \mathcal{W}$ by $\mathfrak{g}(\nu) = \nu^3 + \nu$ for every $\nu \in \mathcal{X}$. Since $|2| = 1$, In particular $k = 2$ and $t = 2$

$$\begin{aligned}
&\|\mathcal{D}\mathfrak{g}(\nu_1, \nu_2)\| \\
&\quad= |12|.\|\nu_1\| \leq \lambda\left(\|\nu_1\|^{\alpha\beta} + \|\nu_2\|^{\alpha\beta} + \|\nu_1\|^{\beta}\|\nu_2\|^{\beta}\right)
\end{aligned}$$

for every $\nu_1, \nu_2 \in \mathcal{X}$, where λ is a positive number and $\alpha\beta > 0$.

(i) $\mathfrak{g}(\nu) = \nu^3 + \nu$, $\qquad \mathcal{C}_3(\nu) = \nu^2$, we get

$$\begin{aligned}
\|\mathfrak{g}(\upsilon) - \mathcal{C}_3(\nu)\| &\leq \frac{\lambda.\|\nu\|^{\alpha\beta}}{2.|2|^{\alpha\beta} - |2|^3} \\
\|\nu\| &\leq \lambda.\|\nu\| \quad (put\ \alpha\beta = 1, |2| = 1)
\end{aligned}$$

(ii) $\mathfrak{g}(\nu) = \nu^3 + \nu$, $\qquad \mathcal{C}_3(\nu) = \nu^3$, we get

$$\begin{aligned}
\|\mathfrak{g}(\nu) - \mathcal{C}_3(\nu)\| &\leq \frac{\lambda.2.\|\nu\|^{\alpha\beta}}{2.|2|^3 - |2|^{\alpha\beta}} \\
\|\nu\| &\leq \lambda.\|\nu\|^4 \qquad (put\ \alpha\beta = 4, |2| = 1)
\end{aligned}$$

Therefore, equation (31) and (39) are satisfied.

V. Conclusion

Many authors discussed the generalized Hyers-Ulam(H-U) stability of cubic n-dimensional functional equation in non-Archimedean Banach space(NABS) in recent years. In this article, we have studied cubic n-dimensional functional equation(FE) (8) in non-Archimedean Banach space(NABS) by using the direct method(DM) and fixed point method(FPM).

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