Study of a Superlinear Problem for a Time Fractional Parabolic Equation Under Integral Over-determination Condition

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Abstract—**The main purpose of this paper is to examine the inverse problem associated with determining the right-hand side of a nonlinear fractional parabolic equation. This equation is accompanied by an integral over-determination supplementary condition. With the use of the functional analysis method, we establish the continuity, existence and uniqueness based on the construction of the direct problem. Such a method relies on the density of the range of the operator established for the problem at hand coupled with the energy inequality scheme. This scheme, also referred to as the method of a priori estimates, allows us to derive the existence theorem from the solution of the given problem, starting with the uniqueness theorem. For the solvability of the inverse problem and its uniqueness, we establish certain suitable conditions, and to demonstrate the existence and uniqueness of its solution, we utilize the fixed point theorem.**

Index Terms—**Inverse nonlinear problem, Fixed point theorem, Nonlocal integral condition.**

I. Introduction

THIS manuscript is dedicated to investigate the solvability of determining a pair of functions $[u(x,t), f(t)]$ that satisfies the Fractional Parabolic Problem (FPP):

$$^{C}D_t^{\alpha}u - \Delta u + \beta u + u^p = f(t)g(x,t), \quad x \in \Omega, t \in (0,T), \tag{1}$$

with the initial condition

$$u(x,0) = 0, x \in \Omega, \tag{2}$$

the boundary condition

$$u(x,t) = 0, \ \ (x,t) \in \partial\Omega \times [0,T], \tag{3}$$

and the nonlocal condition

$$\int_{\Omega} v(x)u(x,t)dx = E(t), \qquad t \in [0,T], \tag{4}$$

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where p is a given positive odd number, β is a positive constant, $g, \ E$ are known functions, and Ω is a bounded domain in $\mathbb{R}^n$ with a regular boundary $\partial\Omega$.

Inverse problems related to the heat equation (refer to [1]–[3], and relevant references) arises commonly in engineering and physics when modeling various phenomena, including seismology, medicine, fusion welding, gas and oil production and well operation [4]–[8]. In these problems, additional or supplementary information regarding the solution is provided in the form of condition (4). This condition serves as a significant modeling tool in the partial differential equations (PDEs) theory within engineering and physics [9]–[14]. It is important to highlight that the integral over-determination is often connected to nonlocal issues [15]–[18].

Many researches have certainly shown that a lot of traditional approaches may not be entirely effective when dealing with nonlocal problems [19], [20]. As a result, various strategies have been recently proposed to address problems arising from nonlocal phenomena. The choice of strategy relies on the specific type of nonlocal boundary values. Numerous researchers have explored the inverse parabolic problem with condition (4) and its particular solvability. Examples of such investigations can be found in works by authors such as [2], [3], [21]–[24].

There are several research papers dedicated to studying the existence and uniqueness of solutions to inverse problems for different parabolic equations in accordance with certain source functions [25], [26]. Such problems involve outlining the term of a parabolic equation with an over-determination condition, see [27], [28] and the references therein to get examples on these problems.

Fractional differential equations (FDEs) are generated by extending ordinary differential equations to non-integer orders [29]–[31]. FDEs hold significant importance in applied mathematics, physics and engineering as they serve as models for complex phenomena [32], [33]. Consequently, engineers and scientists have shown a growing interest in FDEs in recent years [34]–[36]. FDEs are particularly useful for modeling complex phenomena due to their ability to incorporate memory and nonlocal relationships in both space and time. References such as [37]–[41] have contributed to the understanding and application of FDEs in various fields.

II. Materials and Methods

This part is devoted first to call to mind some essentials of fractional calculus. Then it recalls several necessary definitions connected with the energy inequality scheme, fixed point theorem and the issues of existence, uniqueness and continuity of the inverse FPPs.

Definition 1: The Rieman-Liouville and Caputo differentiators are outlined in the following manner:

(i) The left Riemann-Liouville differentiator:

$$^{R}D_t^{\alpha}u(x,t) := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha}}d\tau, \quad (5)$$

(ii) The right Riemann-Liouville differentiator:

$$_{t}^{R}D^{\alpha}u(x,t) := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{u(x,\tau)}{(t-\tau)^{\alpha}}d\tau, \quad (6)$$

(iii) The left Caputo differentiator:

$$^{C}D_t^{\alpha}u(x,t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,\tau)}{\partial \tau}\frac{1}{(t-\tau)^{\alpha}}d\tau, \quad (7)$$

for all $0<\alpha<1$.

Numerous authors assert that the Caputo version of fractional differentiation is less accidental and manageable, particularly when handling homogeneous initial states. The relationship between the two notions expressed in equations (6) and (7), can be verified through direct calculation:

$$^{r}D_t^{\alpha}u(x,t) = \ ^{c}D_t^{\alpha}u(x,t) + \frac{u(x,0)}{\Gamma(1-\alpha)t^{\alpha}}. \quad (8)$$

The scheme of the energy inequality can be summarized by first writing the problem in the form of an operational equation:

$$Lu = F; \quad u \in D(L),$$

where L is an operator defined from a Banach space E to a suitably chosen Hilbert space F, then by establishing a priori estimate for the operator L, and finally by proving the density of the set of values of this operator in the space F.

Definition 2: [42] The space $^{l}H_0^{\alpha}(I)$ is determined as the closure of $C_0^{\infty}(I)$ in relation to the following norm:

$$\|u\|_{^{l}H^{\alpha}(I)} := (\|u\|^2_{L^2(I)} + |u|^2_{^{l}H_0^{\alpha}(I)})^{\frac{1}{2}}, \quad (9)$$

where

$$|u|^2_{^{l}H^{\alpha}(I)} = \|_0^R D_t^{\alpha}u\|_{L^2(I)},$$

for any real $\alpha>0$.

Definition 3: The space $^{r}H_0^{\alpha}(I)$ is determined as the closure of $C_0^{\infty}(I)$ in relation to the following norm:

$$\|u\|_{^{r}H_0^{\alpha}(I)} := (\|u\|^2_{L^2(I)} + |u|^2_{^{r}H_0^{\alpha}(I)})^{\frac{1}{2}}, \quad (10)$$

where

$$|u|^2_{^{r}H_0^{\alpha}(I)} = \|_t^R \partial_T^{\alpha}u\|^2_{L^2(I)},$$

for any real $\alpha>0$.

Lemma 1: [13], [42] If $u \in \ ^{l}H^{\alpha}(I)$ and $v \in C_0^{\infty}(I)$, then

$$(^{R}D_t^{\alpha}u(t), v(t))_{L^2(I)} = (u(t), \ _t^R D^{\alpha}v(t))_{L^2(I)},$$

for any real $\alpha \in \mathbb{R}_+$.

Lemma 2: [13], [42] For $u \in H_0^{\frac{\alpha}{2}}(I)$ and $0<\alpha<2$ such that $\alpha \neq 1$, we have:

$$^{R}D_t^{\alpha}u(t) = \ ^{R}D_t^{\frac{\alpha}{2}} \ ^{R}D_t^{\frac{\alpha}{2}}u(t).$$

Lemma 3: [13], [42] The semi-norms $|.|_{^{l}H^{\alpha}(I)}$, $|.|_{^{r}H^{\alpha}(I)}$ and $|.|_{^{c}H^{\alpha}(I)}$ are equivalent for $\alpha \in \mathbb{R}_+$, $\alpha \neq n+\frac{1}{2}$. Then we pose

$$|u|_{^{l}H^{\alpha}(I)} \cong |.|_{^{r}H^{\alpha}(I)} \cong |.|_{^{c}H^{\alpha}(I)}.$$

Lemma 4: [42] For each real $\alpha>0$, the space $^{l}H_0^{\alpha}(I)$ in relation to the norm (9) is complete.

Definition 4: The scalar product can be integrated in the Bochner sense with the space of square functions $L_2(0,T,L_2(0,d))$ (denoted by $L_2(Q)$) as

$$(u,w)_{L_2(0,T,L_2(0,d))} = \int_0^T (u,w)_{L_2(0,d)}dt. \quad (11)$$

III. Solvability of direct problem

The aim of this part is to discuss the issue of the solvability of the direct FPP. This will be done by first setting out the aimed problem, and then by investigating the existence of its solution.

A. Setting of the problem

Our objective here is to address, within the rectangular domain $Q=(0,d)\times(0,T)$, the existence and uniqueness of the solution $u=u(x,t)$ of the FPP such that d and T are finite values.

$$\begin{cases} ^{c}D_t^{\alpha}u - (\frac{\partial^2 u(x,t)}{\partial x^2}) + \beta u + u^p = \tilde{f}(x,t) \ \text{ in } Q, \\ u(x,0)=0, \ \ \forall x \in (0,d), \\ u(0,t)=u(d,t)=0, \ \ \forall t \in (0.T), \end{cases} \quad (P)$$

where $0<\alpha<1$. This problem enjoys nonlinear FPP of the form

$$\mathcal{L}u =^{c} D_t^{\alpha}u - \frac{\partial^2 u}{\partial x^2} + \beta u + u^p = \tilde{f},$$

with initial condition

$$lu = u(x,0) = 0, \ \ \forall x \in (0,d),$$

$$u(0,t)=u(d,t)=0, \ \ \forall t \in (0,T),$$

where $\beta \in \mathbb{R}_*^+$, $\tilde{f}$ is known function. Herein, we demonstrate the existence and uniqueness of the solution of problem (1)-(3) as a solution of the equation

$$Lu = \mathcal{F}, \quad (12)$$

where $L=(\mathcal{L},l)$ is an operator defined to the Hilbert space F from the Banach space B over the domain $D(L)=B$, which can be determined by

$$D(L) = \left\{u : u \in L^2(Q)\cap L^{p+1}(Q), \ ^{c}D_t^{\alpha}u, \frac{\partial u}{\partial x} \in L^2(Q)\right\}$$

Here, B represents the Banach space that comprises of all $u(x,t)$ with the following finite norm:

$$\|u\|^2_B = \|^{c}D_t^{\frac{\alpha}{2}}u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)}$$

Furthermore, F refers to the Hilbert space that comprises all Fourier elements $(\tilde{f},0)$, satisfying the condition that their norm in $L^2(Q)$ is finite.

Theorem 1: For each $u \in B$, the inequality

$$\|u\|_B \leq C\|Lu\|_{L^2(Q)} \quad (13)$$

holds, where $C \in \mathbb{R}_+$ so that it is not dependent on the function u.

Proof: To prove this result, we first use the scalar product in $L^2(Q)$, where $Q=(0,d)\times(0,T)$. Also, we apply the following function:

$$Mu = u(x,t),$$

on (1) to obtain the following assertion:

$$\int_Q \mathcal{L}u.Mudxdt = \int_Q^c D_t^\alpha u(x,t).u(x,t)dxdt$$
$$-\int_Q (\frac{\partial^2 u(x,t)}{\partial x^2}).u(x,t)dxdt + b\int_Q u^2(x,t)dxdt \quad (14)$$
$$+\int u^{p+1}(x,t)dxdt = \int_Q \tilde{f}(x,t)u(x,t)dxdt.$$

Now as $u(x,0) = 0$ and by applying Lemmas 1, 2 and 3, we get

$$\int_Q^c D_t^\alpha u(x,t).u(x,t)dxdt = ({}^c D_t^\alpha u(x,t).u(x,t)dxdt)_{L^2(Q)}$$
$$= ({}^R D_t^{\frac{\alpha}{2}}\ {}^R D_t^{\frac{\alpha}{2}} u(x,t))_{L^2(Q)},\ \text{(By Lemma 2)}$$
$$= ({}^R D_t^{\frac{\alpha}{2}} u(x,t),\ {}_t^R D_t^{\frac{\alpha}{2}} u(x,t))_{L^2(Q)},\ \text{(By Lemma 1)}$$
$$= |u|^2_{{}^c H^\alpha(Q)} \cong |u|^2_{{}^l H^\alpha(Q)}$$
$$= \|{}^c D_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)},\ \text{(By Lemma 3)}.$$

By applying on ($|ab| \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$) coupled with using the integral by parts of the previous equality, we find

$$\|{}^c D_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + (\beta - \frac{\varepsilon}{2})\|u\|^2_{L^2(Q)}$$
$$+\|u\|^{p+1}_{L^{p+1}(Q)} \leq \frac{1}{2\varepsilon}\|\tilde{f}\|^2_{L^2(Q)}.$$

Thus, for ($\varepsilon \leq 2\beta$), we get

$$\|{}^c D_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)}$$
$$\leq c\|\tilde{f}\|^2_{L^2(Q)}$$

with

$$c = \frac{1}{2\varepsilon min(1, \beta - \frac{\varepsilon}{2})}.$$

As a result, we obtain

$$\|u\|_B \leq C\|Lu\|_{L^2(Q)}$$

with

$$C = \sqrt{c},$$

which completes the proof of this result. ■

Proposition 1: There is a closure for the operator L that maps from B to F.

Proof: Let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence in which

$$u_{n\to} 0 \quad \text{in } B \tag{15}$$

and

$$Lu_{n\to}\mathcal{F} \quad \text{in } F. \tag{16}$$

Now, we have to show

$$\tilde{f} \equiv 0.$$

To this end, it should be noted that the convergence of u_n to 0 in B causes

$$u_{n\to} 0 \quad \text{in } (C_0^\infty(Q))'. \tag{17}$$

Due to the continuity of the fractional derivative coupled with the derivation of the first-order of $(C_0^\infty(Q))'$ in $(C_0^\infty(Q))'$ and the continuity distribution of the function u^p, then (17) would involve

$$\mathcal{L}u_{n\to} 0 \quad \text{in } (C_0^\infty(Q))'. \tag{18}$$

In addition, the convergence of Lu_n to f in $L^2(Q)$ yields

$$\mathcal{L}u_{n\to}\tilde{f} \quad \text{in } (C_0^\infty(Q))'. \tag{19}$$

Since we know that the limit in $(C_0^\infty(Q))'$ is unique, then from (18) and (19), one might infer

$$f \equiv 0.$$

As a result, the operator L is closeable. This could let us to denote $\bar{L}$ and $D(\bar{L})$ as the closure of L and its domain respectively. ■

It is necessary here to draw our attention to the fact that the solution to the following operator equation:

$$\bar{L}u = \mathcal{F}$$

is known as a strong solution of problem (1)-(3). Hence, one might expand the prior estimate to this solution by proposing the following estimate:

$$\|u\|_B \leq C\|\bar{L}u\|_{F'}, \quad \forall u \in D(\bar{L}). \tag{20}$$

Proposition 2: The strong solution of problem (1)-(3) is unique and depends continuously on $\tilde{f} \in F$.

Proposition 3: $R(\bar{L})$ is closed in F and

$$R(\bar{L}) = \overline{R(L)}.$$

Proof: Initially, we intend here to examine the solution's uniqueness of problem (1)-(3) provided that the existence issue. For this purpose, we let u_1 and u_2 be two solutions of (1)-(3). Assume $\eta = u_1 - u_2$, then η satisfies:

$$\begin{aligned} &{}^c D_t^\alpha \eta(x,t) - (\tfrac{\partial^2 \eta(x,t)}{\partial x^2}) + \beta\eta(x,t) + u_1^p - u_2^p = 0 \text{ in } Q, \\ &\eta(x,0) = 0,\ \forall x \in (0,d), \\ &\eta(x,t) = 0,\ \forall (x,t) \in \partial\Omega \times (0.T). \end{aligned} \tag{P'}$$

If one uses η as a scalar product in $L^2(\Omega)$ of the equation:

$${}^c D_t^\alpha \eta(x,t) - (\frac{\partial^2 \eta(x,t)}{\partial x^2}) + \beta\eta(x,t) + u_1^p - u_2^p = 0 \text{ in } Q, \tag{21}$$

and takes the integral over Ω, we obtain

$$\int_\Omega^c D_t^\alpha \eta(x,t).\eta(x,t)dx - \int_\Omega (\frac{\partial^2 \eta(x,t)}{\partial x^2}).\eta(x,t)dx$$
$$+\beta\int_\Omega \eta^2(x,t)dx + \int_\Omega (u_1^p - u_2^p)(u_1 - u_2)dx = 0.$$

Due to $\eta(x,0) = 0$ and by applying Lemmas 1, 2 and 3 coupled with integrating by parts, we get

$$\begin{aligned} &\|{}^c D_t^{\frac{\alpha}{2}} \eta\|^2_{L^2(\Omega)} + \|\frac{d\eta}{dx}\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)} \\ &+ \int_\Omega (u_1^p - u_2^p)(u_1 - u_2)dx = 0. \end{aligned} \tag{22}$$

Since λ^p is a monotone function in λ (on $\Omega = (0,d)$) and based on some ours analysis, we can conclude that $\int_\Omega (u_1^p - u_2^p)(u_1 - u_2)dx$ of (22) is positive. Consequently, it follows from equation (21) that

$$\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

which gives

$$u_1 = u_2.$$

for all $t \in (0, T)$. Now, we will return to demonstrate the assertions declared in this result. For this purpose, we let $z \in \overline{R(l)}$. Then $\exists (z_n)_{n\in\mathbb{N}}$ a sequence in $R(L)$ such that

$$\lim_n z_n = z.$$

Similarly, $\exists (u_n)_{n\in\mathbb{N}}$ a sequence in $D(L)$ such that

$$Lu_n = z_n.$$

Now, let $\varepsilon, n \geq n_0$ and $m, m' \in \mathbb{N}$, $m \geq m'$ such that u_m and $u_{m'}$ are two solutions, i.e.,

$$Lu_m = \tilde{f} \quad Lu_{m'} = \tilde{f}.$$

Consequently, putting $y = u_m - u_{m'}$ makes y satisfying

$$\begin{gathered} {}^cD_t^\alpha y(x,t) - (\tfrac{\partial^2 y(x,t)}{\partial x^2}) + \beta y(x,t) + u_m^p - u_{m'}^p = 0 \text{ in } Q, \\ y(x,0) = 0,\ \forall x \in (0,d), \\ y(x,t) = 0,\ \forall (x,t) \in \partial\Omega \times (0.T). \qquad (P'') \end{gathered}$$

By employing the same approach used to examine the solution's uniqueness, we can deduce that $y = 0$, which yields

$$0 \leq \|u_m - u_{m'}\| \leq 0,$$

for all $t \in (0,T)$. In other words, we have $\forall \varepsilon \geq 0$, $\exists n_0 \in \mathbb{N}$ such that $\|u_m - u_{m'}\| \leq \varepsilon$, $\forall m, m' \geq n_0$. Thus, one might conclude that $(u_n)_{n\in\mathbb{N}}$ is a cauchy sequence in E, and hence $\exists u \in E$ such that

$$\lim_n u_n = u$$

By using the definition of $\bar{L}$ ($\lim_n u_n = u$ in E, if $\lim_n u_n = \lim_n z_n = u$, then $\lim_n \bar{L}u_n = z$. Also, as $\bar{L}$ is closed, then $\bar{L}u = z$), we can assert that the function u satisfies

$$\bar{L}u = z \text{ so that } u \in D(\bar{L}).$$

Therefore, $z \in R(\bar{L})$, and hence $\overline{R(L)} \subset R(\bar{L})$. Furthermore, due to $R(\bar{L})$ is a Banach space, then we conclude that it is closed. Now, it is still necessary to demonstrate the opposing inclusion. For achieving this aim, we let $z \in R(\bar{L})$. Then there exists a sequence of $(z_n)_n$ in F consisting of the elements of $R(\bar{L})$ in which

$$\lim_n z_n = z.$$

As a consequence, $\exists (u_n)_{n\in\mathbb{N}}$, a corresponding sequence in which

$$\lim_n \bar{L}u_n = z_n.$$

Alternatively, we have a cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in F. So, $\exists u \in E$ such that

$$\lim_n u_n = u \quad \text{ in } E.$$

Thus, we have

$$\lim_n \bar{L}u_n = z.$$

As a consequence $z \in \overline{R(L)}$, which asserts that $\overline{R(L)} = R(\bar{L})$. ■

B. *Existence of solution*

To establish the existence of solution of (1)-(3), we need to demonstrate that $R(L)$ is dense in F, $\forall u \in B$ and for any arbitrary $\mathcal{F} = (\tilde{f}, 0) \in F$. To this end, we list the following result.

Theorem 2: Problem (1)-(3) has a solution.
In order to show this result, we should first notice that the scalar product of F is defined by

$$(Lu, W)_F = \int_Q \mathcal{L}u.wdxdt, \tag{23}$$

where $W = (w, 0) \in D(L)$. If one puts $w \in (R(L))^\perp$, we get

$$\int_Q^c D_t^\alpha u(x,t)w(x,t)dxdt - \int_Q (\frac{\partial^2 u(x,t)}{\partial x^2}).w(x,t)dxdt + \beta \int_Q u(x,t)w(x,t)dxdt + \int_Q u^p(x,t).w(x,t)dxdt = 0.$$

Setting $w = u$ yields

$$\int_Q^c D_t^\alpha u(x,t).u(x,t)dxdt - \int_Q (\frac{\partial^2 u(x,t)}{\partial x^2}).u(x,t)dxdt + \beta \int_Q u^2(x,t)dxdt + \int_Q u^{p+1}(x,t)dxdt = 0. \tag{24}$$

By performing integration by parts on each term of equation (24) and considering the given condition of u, we obtain

$$\|{}^cD_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + \beta\|u\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)} = 0.$$

Consequently, we get

$$\|{}^cD_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \beta\|u\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)} = -\|\frac{du}{dx}\|^2_{L^2(Q)} \leq 0$$

which implies $\|u\|^2_{L^2(Q)} \leq 0$. This means that $u = 0$ in Q, and hence $w = 0$ in Q.

IV. Existence and uniqueness of the Solution of problem (1)-(4)

In this part, for necessary reasons, we need to define the following function:

$$g^*(t) = \int_\Omega g(x,t).v(x)dx, \qquad Q = \Omega \times (0,T). \tag{25}$$

In addition, we suppose that all functions involved in the problem at hand meet the conditions

$$\begin{gathered} g \in C((0,T), L^2(\Omega)), v \in W_2^1(\Omega) \cap L^{p+1}(\Omega), \\ \|g(x,t)\| \leq m, \quad |g^*(t)| \geq r > 0, \qquad (H) \end{gathered}$$

for $E \in W_2^2(0,T)$, $r \in \mathbb{R}$ and $(x,t) \in Q$. Besides, we also suppose that these functions are generally measurable. Now, one might notice that the relationship between f and u can be described by

$$A : L^2(0,T) \to L^2(0,T), \tag{26}$$

where A is a linear operator defined as

$$(Af(t)) = \frac{1}{g^*}\left\{\int_\Omega \frac{du}{dx}\frac{dv}{dx}dx + \int_\Omega u^p(x,t).v(x)dx\right\}. \tag{27}$$

Consequently, the aforementioned relationship between f and u might be outlined as a 2^{nd}-linear equation for the function f defined over the interval $L^2(0,T)$ as

$$f(t) = Af(t) + W, \tag{28}$$

where

$$W = \frac{{}^cD_t^\alpha E + \beta E}{g^*} \tag{29}$$

in which $E(0) = 0$.

Theorem 3: Assume that the inverse problem's data functions (1)-(4) satisfy condition (H). Then we have the following equivalent statements:

(i) If there is a solution of the inverse problem (1)-(4), then equation (28) is solvable.

(ii) If there is a solution of (28) the compatibility condition $E(0) = 0$ is satisfied, then there is also a solution of the inverse problem (1)-(4).

Proof:

(i) Suppose that problem (1)-(4) is solved with designating its solution by $\{u, f\}$. Multiplying both sides of (1) by v and integrating the result over Ω yield

$$\begin{aligned} &{}^cD_t^\alpha \int_\Omega u(x,t).v(x)dx + \int_\Omega \frac{du}{dx}\frac{dv}{dx}dx \\ &\quad + \beta \int_\Omega u(x,t)v(x)dx + \int_\Omega u^p(x,t).v(x)dx \\ &\quad = f(t)g^*(t). \end{aligned} \tag{30}$$

Now, using (4) and (27) implies

$$f = Af + \frac{\beta E +^c D_t^\alpha E}{g^*}.$$

This confirms that f solves (28).

(ii) By assuming that equation (28) has a solution, say f, and then substituting f into equation (1), then the resulting relation (1)-(3) will be viewed as a direct problem with a unique solution. We still need to demonstrate that u meets the condition of integral over-determination (4) as well. To this end, we should note that according to equation (30), the function u is contingent on the following assertion:

$$\begin{aligned} &{}^cD_t^\alpha E + \int_\Omega \frac{du}{dx}\frac{dv}{dx}dx + \beta E + \int_\Omega u^p(x,t).v(x)dx \\ &\quad = f(t)g^*(t). \end{aligned} \tag{31}$$

Consequently, subtracting equation (30) from equation (31) yields

$$\begin{aligned} &{}^cD_t^\alpha \int_\Omega u(x,t).v(x)dx + \beta \int_\Omega u(x,t).v(x)dx \\ &\quad =^c D_t^\alpha E + \beta E. \end{aligned} \tag{32}$$

By integrating both sides of (32) and considering $E(0) = 0$, we deduce that u meets the integral condition (4). Therefore, we can immediately infer that $\{u, f\}$ is the solution of the inverse problem (1)-(4), which concludes the proof. ■

In the following content, we intend to discuss some properties of the operator A by proposing the next result. This result would pave the way to establish a further result connected with the existence and uniqueness of solution of (1)-(4).

Lemma 5: Assuming condition (H) holds, then there exists a positive δ such that the operator A is a contraction mapping in $L^2(0,T)$.

Proof: Observe that the following estimate can be inferred from (27) easily:

$$|Af(t)|^2 \leq \frac{2}{r^2} \times \left[\| \frac{du}{dx} \|^2_{L^2(\Omega)} \| \frac{dv}{dx} \|^2_{L^2(\Omega)} + \|u\|^{2p}_{L^{p+1}(\Omega)} \|v\|^2_{L^{p+1}(\Omega)} \right].$$

Now, we assume that

$$\|u\|^p_{L^\infty(0,T,L^{p+1}(\Omega))} = \gamma \geq 0.$$

Then we can have

$$|Af(t)|^2 \leq \frac{2}{r^2} \times \left[\| \frac{du}{dx} \|^2_{L^2(\Omega)} \| \frac{dv}{dx} \|^2_{L^2(\Omega)} + \gamma \|u\|^{p+1}_{L^{p+1}(\Omega)} \|v\|^2_{L^{p+1}(\Omega)} \right]$$

By integrating the above inequality over $(0,T)$, we obtain

$$\begin{aligned} \int_0^T |Af(t)|^2 dt \leq \frac{2}{r^2} max \left(\| \frac{dv}{dx} \|^2_{L^2(\Omega)}, \gamma \|v\|^2_{L^{p+1}(\Omega)} \right) \\ \times \left[\int_0^T \| \frac{du}{dx} \|^2_{L^2(\Omega)} dt + \int_0^T \|u\|^{p+1}_{L^{p+1}(\Omega)} dt \right], \end{aligned} \tag{33}$$

which immediately yields

$$\|Af\|_{L^2(0,T)} \leq K \left[\int_0^T \| \frac{du}{dx} \|^2_{L^2(\Omega)} dt + \int_0^T \|u\|^{p+1}_{L^{p+1}(\Omega)} dt \right]^{\frac{1}{2}}$$

where

$$K = \sqrt{\frac{2}{r^2} max \left(\| \frac{dv}{dx} \|^2_{L^2(\Omega)}, \gamma \|v\|^2_{L^{p+1}(\Omega)} \right)}$$

Then, using the A priori estimate and removing some terms lead us to infer

$$\| \frac{du}{dx} \|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)} \leq C \|f\|^2_{L^2(Q)}.$$

This consequently gives

$$\|Af\|_{L^2(0,T)} \leq \delta \|f\|_{L^2(0,T)}, \tag{34}$$

where $\delta = K\sqrt{C}$. Obviously, it can be noticed from the previous discussion that there is a positive δ in which $\delta \leq 1$. Thus, inequality (34) demonstrates that A is a contracting operator on $L^2(0,T)$. ■

Theorem 4: Assume condition (H) and the compatibility condition are satisfied, then there is a unique solution $\{u, f\}$ for inverse problem (1)-(4).

Proof: It follows that equation (28) possesses a unique solution f in $L^2(0,T)$. Also, based on Lemma 3, the existence of a solution of (1)-(4) can be established. However, the uniqueness of such a solution has not been yet shown. For this purpose, we assume that there are two different solutions for the inverse problem at hand, denoted as $\{u_1, f_1\}$ and $\{u_2, f_2\}$. In this regard, it should be noted that if the linear operator A contracts on $L(0,T)$ leading to $f_1 = f_2$, then according to Lemma 5, the uniqueness theorem of the solution to the main direct problem (1)-(3) implies $z_1 = z_2$, and this completes the proof. ■

Corollary 1: If the same assumptions of Theorem 4 hold, then the solution f of (28) exhibits continuous dependence on the data W.

Proof: Consider ω and v are two sets of W that meet all assumptions of Theorem 4. Suppose f and g are two solutions of (28) for the data ω and v, respectively. Now, based on (28), we obtain

$$f = Af + v \quad \text{and} \quad g = Ag + \omega.$$

So, with the help of (34), we can calculate the difference $f - g$ as follows

$$\begin{aligned}\|f-g\|_{L^2(0,T)} &= \|(Af+v)-(Ag+\omega)\|_{L^2(0,T)} \\ &\leq \delta\|f-g\|_{L^2(0,T)} + \|v-\omega\|_{L^2(0,T)}.\end{aligned}$$

As a result, we obtain

$$\|f-g\|_{L^2(0,T)} \leq \frac{1}{1-\delta}\|v-\omega\|_{L^2(0,T)},$$

which finishes the proof. ■

V. Discussion and conclusion

In this study, we have examined the inverse problem of outlining the right-hand side of a nonlinear fractional parabolic problem, incorporating an integral overdetermination condition, with a homogeneous initial condition. The primary focus has been on the theoretical aspects of this inverse problem. We have established certain conditions for the existence, uniqueness, and continuity depending on the data of the problem. Thus, many numerical analysis perspectives can be further studied and based on the results of this work, especially in the aspect of developing effective numerical methods that are compatible with the non local conditions of the integrative type, and also to think about solving the same problem by considering certain incompatible initial conditions.

References

[1] M. Ivanchov, "Inverse Problems for Equations of Parabolic Type," Lviv, Ukraine: VNTL Publishers, 2003

[2] T.E. Oussaeif and A. Bouziani, "An inverse coefficient problem for a parabolic equation under nonlocal boundary and integral overdetermination conditions," International Journal of Differential Equations and Applications, vol. 2, no. 3, pp38-43, 2014

[3] T.E. Oussaeif and A. Bouziani, "Inverse problem of a hyperbolic equation with an integral overdetermination condition," Electronic Journal of Differential Equations, vol. 2016, no. 138, pp1-7, 2016

[4] I.M. Batiha, I. Rezzoug, T.E. Oussaeif, A. Ouannas and I.H. Jebril, "Pollution detection for the singular linear parabolic equation," Journal of Applied Mathematics & Informatics, vol. 41, no. 3, pp647-656, 2023

[5] I.M. Batiha, L. Ben Aoua, T.E. Oussaeif, A. Ouannas, S. Alshorman, I.H. Jebril and S. Momani, "Common fixed point theorem in non-Archimedean menger PM-spaces using CLR property with application to functional equations," IAENG International Journal of Applied Mathematics, vol. 53, no. 1, pp360-368, 2023

[6] H. Qawaqneh, M.S.M. Noorani, H. Aydi, A. Zraiqat and A.H. Ansari, "On fixed point results in partial b-metric spaces," Journal of Function Spaces, vol. 2021, Article ID 8769190, 2021

[7] S. Doley, A.V. Kumar, K.R. Singh and L. Jino, "Study of time fractional Burgers' equation using Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives," Engineering Letters, vol. 30, no. 3, pp1017-1024, 2022

[8] T. Loganathan and K. Ganesan, "Solution of fully fuzzy multi-objective linear fractional programming problems- a Gauss elimination approach," Engineering Letters, vol. 30, no. 3, pp1085-1091, 2022

[9] J.R. Cannon, Y. Lin and S. Wang, "Determination of a control parameter in a parabolic partial differential equation," The ANZIAM Journal, vol. 33, no. 2, pp149-163, 1991

[10] J.R. Cannon, Y. Lin and S. Wang, "Determination of source parameter in a parabolic equations," Meccanica, vol. 27, no. 2, pp85-94, 1992

[11] M.I. Ivanchov and N.V. Pabyrivska, "Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions," Ukrainian Mathematical Journal, vol. 53, no. 5, pp674-684, 2001

[12] A.G. Fatullayev, N. Gasilov and I. Yusubov, "Simultaneous determination of unknown coefficients in a parabolic equation," Applicable Analysis, vol. 87, no. 10-11, pp1167-1177, 2008

[13] F. Kanca and M. Ismailov, "Inverse problem of finding the time-dependent coefficient of heat equation from integral overdetermination condition data," Inverse Problems in Science and Engineering, vol. 20, no. 4, pp463-476, 2012

[14] M.I. Ismailov and F. Kanca, "An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions," Mathematical Methods in the Applied Sciences, vol. 34, no. 6, pp692-702, 2011

[15] J.R Cannon and Y. Lin, "An inverse problem of finding a parameter in a semi-linear heat equation," Journal of Mathematical Analysis and Applications, vol. 145, no. 2, pp470-484, 1990

[16] V.L. Kamynin, "Unique solvability of the inverse problem of determination of the leading coefficient in a parabolic equation," Differential Equations, vol. 47, no. 1, pp91-101, 2011

[17] I.M. Batiha, Z. Chebana, T.E. Oussaeif, A. Ouannas, I.H. Jebril and M. Shatnawi, "Solvability of nonlinear wave equation with nonlinear integral Neumann conditions," International Journal of Analysis and Applications, vol. 21, Article ID 34, 2023

[18] I.M. Batiha, Z. Chebana, T.E. Oussaeif, A. Ouannas, S. Alshorm and A. Zraiqat, "Solvability and dynamics of superlinear reaction diffusion problem with integral condition," IAENG International Journal of Applied Mathematics, vol. 53, no. 1, pp360-368, 2023

[19] N.I. Ionkin, "The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition," Differentsial'nye Uravneniya, vol. 13, no. 2, pp294-304, 1977

[20] V.L. Kamynin and I.A. Vasin, "Asymptotic behaviour of the solutions of inverse problems for parabolic equations with irregular coefficient," Sbornik Mathematics, vol. 188, no. 3, pp371-387, 1997

[21] A.I. Prilepko and D.G. Orlovskii, "Determination of the parameter of an evolution equation and inverse problems of mathematical physics," Differentsial'nye Uravneniya, vol. 21, no. 4, pp694-701, 1985

[22] T.E. Oussaeif and A. Bouziani, "Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions," Electronic Journal of Differential Equations, vol. 2014, no. 179, pp1-10, 2014

[23] A. Lopushans'kyi, H. Lopushans'ka and V. Rapita, "Inverse problem in the space of generalized functions," Ukrainian Mathematical Journal, vol. 68, no. 2, pp269-282, 2016

[24] V.L. Kamynin, "On Convergence of the Solutions of Inverse Problems for Parabolic Equations with Weakly Converging Coefficients," Florida, United States: Chapman and Hall/CRC, 1995

[25] T.E. Oussaeif, B. Antara, A. Ouannas, I.M. Batiha, K.M. Saad, H. Jahanshahi, A.M. Aljuaid and A.A. Aly, "Existence and uniqueness of the solution for an inverse problem of a fractional diffusion equation with integral condition," Journal of Function Spaces, vol. 2022, Article ID 7667370, 2022

[26] A. Benguesmia, I.M. Batiha, T.E. Oussaeif, A. Ouannas and W.G. Alshanti, "Inverse problem of a semilinear parabolic equation with an integral overdetermination condition," Nonlinear Dynamics and Systems Theory, vol. 23, no. 3, pp249-260, 2023

[27] A. Bouziani, "Solution forte d'un problème mixte avec conditions non locales pour une classe d'équations hyperboliques," Bulletins de l'Académie Royale de Belgique, vol. 8, no. 1, pp53-70, 1997

[28] J.H. He, "Some applications of nonlinear fractional differential equations and their approximations," Bulletin of Science, Technology & Society, vol. 15, no. 2, pp86-90, 1999

[29] I.M. Batiha, A. Ouannas, R. Albadarneh, A.A. Al-Nana and S. Momani, "Existence and uniqueness of solutions for generalized Sturm–Liouville and Langevin equations via Caputo–Hadamard fractional-order operator," Engineering Computations, vol. 39, no. 7, pp2581-2603, 2022

[30] I.M. Batiha, Z. Chebana, T.E. Oussaeif, A. Ouannas and I.H. Jebril, "On a weak solution of a fractional-order temporal equation," Mathematics and Statistics, vol. 10, no. 5, pp1116-1120, 2022

[31] I.M. Batiha, "Solvability of the solution of superlinear hyperbolic Dirichlet problem," International Journal of Analysis and Applications, vol. 20, Article ID 62, 2022

[32] I.M. Batiha, N. Barrouk, A. Ouannas and W.G. Alshanti, "On global existence of the fractional reaction-diffusion system's solution," International Journal of Analysis and Applications, vol. 21, Article ID 11, 2023

[33] N. Anakira, Z. Chebana, T.E. Oussaeif, I.M. Batiha and A. Ouannas, "A study of a weak solution of a diffusion problem for a temporal fractional differential equation," Nonlinear Functional Analysis and Applications, vol. 27, no. 3, pp679-689, 2022

[34] I.M. Batiha, N. Barrouk, A. Ouannas and A. Farah, "A study on invariant regions, existence and uniqueness of the global solution for tridiagonal reaction-diffusion systems," Journal of Applied Mathematics and Informatics, vol. 41, no. 4, pp893-906, 2023

[35] Z. Chebana, T.E. Oussaeif, S. Dehilis, A. Ouannas and I.M. Batiha, "On nonlinear Neumann integral condition for a semilinear heat problem with blowup simulation," Palestine Journal of Mathematics, vol. 12, no. 3, pp378-394, 2023

[36] M.N. Lamya, I.M. Batiha, I. Rezzoug, A. Ouannas and T.E. Ouassaeif, "On sentinel method of one-phase Stefan problem," Journal of the Nigerian Society of Physical Sciences, vol. 5, no. 4, Article ID 1772, 2023

[37] K. Oldham and J. Spanier, "The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order," Amsterdam, Netherlands: Elsevier Science, 1974

[38] J.H. He, "Approximate analytical solution for seepage flow with fractional derivatives in porous media," Computer Methods in Applied Mechanics and Engineering, vol. 167, no. 1-2, pp57-68, 1998

[39] A.I. Prilepko, D.G. Orlovsky and I.A. Vasin, "Methods for Solving Inverse Problems in Mathematical Physics," Florida, United States: CRC Press, 2000

[40] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," Physics Reports, vol. 339, no. 1, pp1-77, 2000

[41] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, "Theory and Applications of Fractional Differential Equations," Amsterdam, Netherlands: Elsevier Science, 2006

[42] X. Li and C. Xu, "Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation," Communications in Computational Physics, vol. 8, no. 5, pp1016-1051, 2010