Almost Periodic Solutions in Shifts Delta(+/-) of Nonlinear Dynamic Equations with Impulses on Time Scales

Lili Wang, Pingli Xie

Abstract—**Based on the estimation of the Cauchy matrix of linear impulsive differential equation, by using Banach fixed point theorem and exponential dichotomy, sufficient conditions for the existence of almost periodic solutions in shifts $\delta_\pm$ of some nonlinear dynamic equations with impulses on time scales are established. Finally, two impulsive ecosystems defined on some specific time scales are studied to illustrate the effectiveness of the main results.**

Index Terms—**Almost periodic solution; Nonlinear dynamic equation; Shift operator; Impulse; Time scale.**

I. Introduction

IN this paper, we study the following impulsive dynamic equations on time scales:

$$\begin{cases} y^{\Delta}(x) = D(x, y(x))y(x) + \xi(x, y(\delta_-(\tau, x))), \\ x \neq x_k, k \in \mathbb{Z}, \\ y(x^+) = y(x^-) - B_k y(x) + I(y(x)) + \gamma_k, \\ x = x_k, k \in \mathbb{Z}, \end{cases} \tag{1}$$

and

$$\begin{cases} y^{\Delta}(x) = D(x)y(x) + \xi(x, y(\delta_-(\tau, x))), \\ x \neq x_k, k \in \mathbb{Z}, \\ y(x^+) = y(x^-) - B_k y(x) + I(y(x)) + \gamma_k, \\ x = x_k, k \in \mathbb{Z}, \end{cases} \tag{2}$$

and

$$\begin{cases} y^{\Delta}(x) = D(y(x))y(x) + \xi(x, y(\delta_-(\tau, x))), \\ x \neq x_k, k \in \mathbb{Z}, \\ y(x^+) = y(x^-) - B_k y(x) + I(y(x)) + \gamma_k, \\ x = x_k, k \in \mathbb{Z}, \end{cases} \tag{3}$$

where $x \in \mathbb{T}$, $\mathbb{T}$ is a time scale; $D_{n\times n}$ and $\xi_{n\times 1}$ are continuous functions.

Choose appropriate functions of $D_{n\times n}$ and $\xi_{n\times 1}$, the equations (1), (2) and (3) can be used to describe many phenomena in physics and biology and so on under some special time scales.

It is well known that periodic dynamic systems in nature world may exhibit almost periodicity due to the influence of external or human factors; see, for example [1-11].

A general time scale is usually not closed to the addition operation, the theory of almost periodic differential (difference) equation on $\mathbb{R}$ ($\mathbb{Z}$) is no longer applicable to the study of almost periodicity of dynamic equations on general time scales. In recent years, by means of the shift operators $\delta_\pm$, the concepts and properties of periodic and almost periodic functions in shift $\delta_\pm$ on time scales have been defined and studied in [12-15] and [16-18], respectively. Since then the theory of periodic and almost periodic dynamic equations in shift $\delta_\pm$ on time scales have been rapidly developed and applied.

On the basis of the above works, to further construct the theory of almost periodicity in shift $\delta_\pm$ of dynamic equations with impulses on time scales, the main work of this paper is to explore the existence theorems of almost periodic solutions in shift $\delta_\pm$ of (1), (2) and (3). Furthermore, based on the obtained results, we bring two dynamic systems under investigation on some specific time scales to obtain more general results.

Let $y(x) = y(x; x_0, y_0)$. The initial value of the equations (1), (2) and (3) is defined by

$$y(x_0 + 0; x_0, y_0) = y_0. \tag{4}$$

II. Preliminaries

The theory of time scales and its applications on dynamic equations, see [19].

Let $\mathbb{B} = \{\{x_k\} \subset \mathbb{T} : x_k < x_{k+1},\ k \in \mathbb{Z},\ \lim\limits_{k\to\pm\infty} x_k = \pm\infty\}$ is the set of all sequences that are unbounded and strictly increasing, $\theta = \inf\limits_{k\in\mathbb{Z}}\{x_k - x_{k-1} : x_k \in \mathbb{B}\}$.

Let $PC(\mathbb{T}, \mathbb{R}^n)$ is the set of all piecewise continuous functions from $\mathbb{T}$ to $\mathbb{R}^n$ with the first kind discontinuous points $x_k (k \in \mathbb{Z})$, at which it is continuous from the left.

Definition 1. *The set of sequences $\{x_k^j\}, x_k^j = x_{k+j} - x_k, k, j \in \mathbb{Z}, \{x_k\} \in \mathbb{B}$ is said to be uniformly almost periodic in shift $\delta_\pm$, if for any $\varepsilon > 0$, there exists a common relatively dense set of ε-almost periodic in shift $\delta_\pm$ for any sequences.*

Definition 2. *The function $\psi(x) \in PC(\mathbb{T}, \mathbb{R}^n)$ is said to be almost periodic in shift $\delta_\pm$, if the following hold:*

(a) *The set of sequences $\{x_k^j\}, x_k^j = x_{k+j} - x_k, k, j \in \mathbb{Z}, \{x_k\} \in \mathbb{B}$ is uniformly almost periodic in shift $\delta_\pm$.*

(b) *For any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points x' and x'' belong to one and the same interval of continuity of $\psi(x)$ and satisfy the inequality $|x' - x''| < \delta$, then $|\psi(x') - \psi(x'')| < \varepsilon$.*

(c) *For any $\varepsilon > 0$ there exists a relatively dense set $\mathbb{E}$ such that if $p \in \mathbb{E}$, then $|\psi(\delta_\pm^p(x)) - \psi(x)| < \varepsilon$ for all $x \in \mathbb{T}$ satisfying the condition $|x - x_k| > \varepsilon$, $k \in \mathbb{Z}$.*

Consider the following system

$$\begin{cases} y^{\Delta}(x) = D(x)y(x), x \neq x_k, \\ y(x^+) = y(x^-) - B_k y(x), x = x_k, k \in \mathbb{Z}, \end{cases} \tag{5}$$

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where $D_{n\times n}$ is an almost periodic function in shift $\delta_{\pm}$, $\{B_k\}$ is an almost periodic sequences in shift $\delta_{\pm}$, that is, for any $\varepsilon > 0$, there exists a $q > 0$ such that $|B_{\delta^q_{\pm}(k)} - B_k| < \varepsilon$, and $0 < |I - B_k| < 1$.

If $U_k(x, \sigma(z))$ is the Cauchy matrix of the linear system

$$y^{\Delta}(x) = D(x)y(x), x_{k-1} < x < x_k, \{x_k\} \in \mathbb{B},$$

then system (5) has the Cauchy matrix

$$W(x,\sigma(z)) = \begin{cases} U_k(x,\sigma(z)), x_{k-1} < \sigma(z) < x < x_k, \\ U_{k+1}(x, x_k+0)(I-B_k)U_k(x,\sigma(z)), \\ x_{k-1} < \sigma(z) \le x_k < x \le x_{k+1}, \\ U_{k+1}(x, x_k+0)(I-B_k)U_k(x_k, x_k+0) \\ \cdots (I-B_i)U_i(x_i,\sigma(z)), \\ x_{i-1} < \sigma(z) \le x_i < x_k < x \le x_{k+1}, \end{cases}$$

and the solution of system (5) can be written as

$$y(x; x_0, y_0) = W(x, x_0)y_0.$$

Remark 1. *Consider the corresponding nonimpulsive dynamic equation of (5), that is,*

$$u^{\Delta}(x) = D(x)u(x), x \in \mathbb{T}, \tag{6}$$

and $\prod_{0\le x_k<x}(I - B_k)u(x) = y(x)$. *If (6) satisfies exponential dichotomy on* $\mathbb{T}$, *then (5) satisfies exponential dichotomy on* $\mathbb{T}$.

Lemma 1. ([18]) *If* $b \in \mathcal{R}^+$, *then*

$$e_b(x_1, x_2) \le \exp\left(\int_{x_2}^{x_1} b(\tau)\Delta\tau\right),$$

for all $x_1 \ge x_2$.

Lemma 2. ([19]) *Suppose that* $\psi : \mathbb{T} \to \mathbb{R}$ *is strictly increasing,* $\tilde{\mathbb{T}} := \psi(\mathbb{T})$ *is a time scale. If* $g : \tilde{\mathbb{T}} \to \mathbb{R}$, $\psi^{\Delta}(x)$ *and* $g^{\tilde{\Delta}}(\psi(x))$ *exist for* $x \in \mathbb{T}^k$, *then*

$$(g \circ \psi)^{\Delta} = (g^{\tilde{\Delta}} \circ \psi)\psi^{\Delta}.$$

Similar to the proof of Lemma 2.3 in [20], we can get the following lemma.

Lemma 3. *If there exist positive constants* β *and* α *such that*

$$W(x, \sigma(z)) \le \beta e_{\ominus\alpha}(x, \sigma(z)), z, x \in \mathbb{T}, x \ge \sigma(z),$$

then for any $\varepsilon > 0$, $x \ge \sigma(z)$, $x, z \in \mathbb{T}$, $|x - x_k| > \varepsilon$, $|z - x_k| > \varepsilon$, $k \in \mathbb{Z}$, *there exist a relatively dense set* $\mathbb{E}$ *of the function* $D(t)$ *and a positive constant* Γ, *such that*

$$|W(\delta^p_{\pm}(x), \delta^p_{\pm}(\sigma(z))) - W(x, \sigma(z))| \le \varepsilon\Gamma e_{\ominus\frac{\alpha}{2}}(x, \sigma(z))$$

for all $p \in \mathbb{E}$.

III. Main results

Throughout this paper, we assume that

(H_1) $D(x, y(x)), D(x), D(y(x))$ are almost periodic functions in shift $\delta_{\pm}$ with respect to x.

(H_2) The set of sequences $\{x^j_k\}$, $x^j_k = x_{k+j} - x_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{x_k\} \in \mathbb{B}$ is uniformly almost periodic in shift $\delta_{\pm}$ and there exists $\theta > 0$ such that $\inf_{k\in\mathbb{Z}} x^1_k = \theta > 0$.

(H_3) The function $\xi(x, y(x))$ is Δ-almost periodic in shift $\delta_{\pm}$ with respect to x, and there exists a positive constant L_{ξ} such that for $u, v \in \mathbb{R}^n$, $|\xi(x, u) - \xi(x, v)| \le L_{\xi}|u - v|$, and $\xi(x, 0) = 0$.

(H_4) The function $I_k(y(x))$ is almost periodic in shift $\delta_{\pm}$ with respect to x, and there exists a positive constant L_I such that for $u, v \in \mathbb{R}$, $|I_k(u) - I_k(v)| \le L_I|u - v|$, and $I_k(0) = 0$.

(H_5) $\{B_k\}$ and $\{\gamma_k\}$ are almost periodic sequences in shift $\delta_{\pm}$, $0 < |I - B_k| < 1$ and $\max_k\{\gamma_k\} = \hat{\gamma}$.

Theorem 1. *If*

(I) *The conditions* (H_1)-(H_5) *hold;*

(II) *Suppose that the linear system*

$$y^{\Delta}(x) = D(x, \varphi(x))y(x)$$

satisfies exponential dichotomy on $\mathbb{T}$ *with projection* P *and positive constants* β *and* α, *and the dichotomy constants* β *and* α *does not depend on* φ, *where* $\varphi(x)$ *is a bounded continuous function;*

(III) $\lambda < 1$, *where* $\lambda = \beta(\frac{1}{\alpha} + \frac{1}{\inf|\ominus\alpha|})L_{\xi} + \beta(\frac{1}{\inf|1-e^{-\alpha\theta}|} + \frac{1}{\inf|1-e^{\ominus\alpha\theta}|})L_I$;

then (1) exists a unique almost periodic solution in shifts $\delta_{\pm}$.

Proof: Define the set $\mathbb{X} = \{\varphi(x) : \varphi(x) \in PC(\mathbb{T}, \mathbb{R}^n)$ is an almost periodic function $\}$ and the norm

$$\|\varphi\| = \sup_{x\in\mathbb{T}} |\varphi(x)|,$$

then $(\mathbb{X}, \|\cdot\|)$ is a Banach space.

For $\varphi \in \mathbb{X}$, consider the following system

$$\begin{cases} y^{\Delta}(x) = D(x, \varphi(x))y(x) + \xi(x, \varphi(\delta_-(\tau, x))), \\ x \ne x_k, k \in \mathbb{Z} \\ y(x^+) = y(x^-) - B_k\varphi(x) + I_k(\varphi(x)) + \gamma_k, \\ x = x_k, k \in \mathbb{Z}. \end{cases} \tag{7}$$

Let $W(x, \sigma(z))$ is the Cauchy matrix of

$$\begin{cases} y^{\Delta}(x) = D(x, \varphi(x))y(x), \\ x \ne x_k, k \in \mathbb{Z} \\ y(x^+) = y(x^-) - B_k y(x), \\ x = x_k, k \in \mathbb{Z}. \end{cases} \tag{8}$$

By Remark 1, (II) and (H_5), (8) satisfies exponential dichotomy. Suppose that $\Psi(x)$ is the fundamental solution matrix of (8), and

$$W_1(x, \sigma(z)) = \Psi(x)P\Psi^{-1}(\sigma(z)),$$
$$W_2(x, \sigma(z)) = \Psi(x)(I - P)\Psi^{-1}(\sigma(z)),$$

then (7) has a solution $y_{\varphi}(x)$, and

$$\begin{aligned} y_{\varphi}(x) &= \int_{-\infty}^{x} W_1(x, \sigma(z))\xi(z, \varphi(\delta_-(\tau, z)))\Delta z \\ &\quad - \int_{x}^{+\infty} W_2(x, \sigma(z))\xi(z, \varphi(\delta_-(\tau, z)))\Delta z \\ &\quad + \sum_{-\infty}^{x} W_1(x, x^+_k)(I_k(\varphi(x)) + \gamma_k) \\ &\quad - \sum_{x}^{+\infty} W_2(x, x^+_k)(I_k(\varphi(x)) + \gamma_k) \\ &= \int_{-\infty}^{x} W_1(x, \sigma(z))\xi(z, \varphi(\delta_-(\tau, z)))\Delta z \\ &\quad - \int_{x}^{+\infty} W_2(x, \sigma(z))\xi(z, \varphi(\delta_-(\tau, z)))\Delta z \end{aligned}$$

$$+\sum_{-\infty<x_k<x} W_1(x,x_k^+)(I_k(\varphi(x))+\gamma_k)$$
$$-\sum_{x<x_k<+\infty} W_2(x,x_k^+)(I_k(\varphi(x))+\gamma_k).$$

Define the mapping $\Phi:\mathbb{X}\to\mathbb{X}$ by

$$(\Phi\varphi)(x) \;=\; y_\varphi(x).$$

Let $\mathbb{X}^*$ is a subset of $\mathbb{X}$, and

$$\mathbb{X}^*=\{\varphi\in\mathbb{X}:\|\varphi-\varphi_0\|\le\frac{\lambda A}{1-\lambda}\},$$

where

$$\varphi_0=\sum_{-\infty<x_k<x} W_1(x,x_k^+)\gamma_k-\sum_{x<x_k<+\infty} W_2(x,x_k^+)\gamma_k,$$

by Lemma 1 and (II),

$$\begin{aligned}
&\|\varphi_0\|\\
=\;& \sup_{t\in\mathbb{R}}\Big|\sum_{-\infty<x_k<x} W_1(x,x_k^+)\gamma_k\\
&-\sum_{x<x_k<+\infty} W_2(x,x_k^+)\gamma_k\Big|\\
\le\;& \beta\Big(\frac{1}{\inf|1-e^{-\alpha\theta}|}+\frac{1}{\inf|1-e^{\ominus\alpha\theta}|}\Big)\hat{\gamma}\\
:=\;& A. \qquad (9)
\end{aligned}$$

For any $\varphi\in\mathbb{X}^*$, from (7) and (9), we have

$$\|\varphi\|\le\|\varphi-\varphi_0\|+\|\varphi_0\|\le\frac{\lambda A}{1-\lambda}+A=\frac{A}{1-\lambda}.$$

Now we prove that Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Firstly, we show that $\Phi\varphi\in\mathbb{X}^*$, $\forall\varphi\in\mathbb{X}^*$. In fact

$$\begin{aligned}
&\|\Phi\varphi-\varphi_0\|\\
=\;& \sup_{x\in\mathbb{T}}\Big|\int_{-\infty}^{x} W_1(x,\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&-\int_{x}^{+\infty} W_2(x,\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&+\sum_{-\infty<x_k<x} W_1(x,x_k^+)I_k(\varphi(x))\\
&-\sum_{x<x_k<+\infty} W_2(x,x_k^+)I_k(\varphi(x))\Big|\\
\le\;& \Big(\beta\Big(\frac{1}{\alpha}+\frac{1}{\inf|\ominus\alpha|}\Big)L_\xi\\
&+\beta\Big(\frac{1}{\inf|1-e^{-\alpha\theta}|}+\frac{1}{\inf|1-e^{\ominus\alpha\theta}|}\Big)L_I\Big)\|\varphi\|\\
=\;& \lambda\|\varphi\|\le\frac{\lambda A}{1-\lambda}. \qquad (10)
\end{aligned}$$

Next, we show that $\Phi\varphi$ is almost periodic in shifts $\delta_\pm$. In fact, for $x\in\mathbb{T}$, $p>0$, by Lemmas 2 and 3,

$$\begin{aligned}
&\int_{-\infty}^{\delta_\pm^p(x)} W_1(\delta_\pm^p(x),\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
=\;& \int_{-\infty}^{x} W_1(\delta_\pm^p(x),\delta_\pm^p(\sigma(z)))\\
&\times\xi(\delta_\pm^p(z),\varphi(\delta_-(\tau,\delta_\pm^p(z))))\delta_\pm^{\Delta p}(z)\Delta z; \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&\int_{\delta_\pm^p(x)}^{+\infty} W_2(\delta_\pm^p(x),\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
=\;& \int_{x}^{+\infty} W_2(\delta_\pm^p(x),\delta_\pm^p(\sigma(z)))\\
&\times\xi(\delta_\pm^p(z),\varphi(\delta_-(\tau,\delta_\pm^p(z))))\delta_\pm^{\Delta p}(z)\Delta z. \qquad (12)
\end{aligned}$$

Consider the $\delta_+^p(x)$ case,

$$\begin{aligned}
&|(\Phi\varphi)(\delta_+^p(x))-(\Phi\varphi)(x)|\\
=\;& \Big|\int_{-\infty}^{\delta_+^p(x)} W_1(\delta_+^p(x),\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&-\int_{\delta_+^p(x)}^{+\infty} W_2(\delta_+^p(x),\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&+\sum_{-\infty<x_k<\delta_+^p(x)} W_1(\delta_+^p(x),x_k^+)(I_k(\varphi(\delta_+^p(x)))\\
&+\gamma_k)\\
&-\sum_{\delta_+^p(x)<x_k<+\infty} W_2(\delta_+^p(x),x_k^+)(I_k(\varphi(\delta_+^p(x)))\\
&+\gamma_k)\\
&-\int_{-\infty}^{x} W_1(x,\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&+\int_{x}^{+\infty} W_2(x,\sigma(z))\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&-\sum_{-\infty<x_k<x} W_1(x,x_k^+)(I_k(\varphi(x))+\gamma_k)\\
&+\sum_{x<x_k<+\infty} W_2(x,x_k^+)(I_k(\varphi(x))+\gamma_k)\Big|\\
=\;& \Big|\int_{-\infty}^{x} W_1(\delta_+^p(x),\delta_+^p(\sigma(z)))\\
&\times[\xi(\delta_+^p(z),\varphi(\delta_-(\tau,\delta_+^p(z))))\delta_+^{\Delta p}(z)\Delta z\\
&-\xi(z,\varphi(\delta_-(\tau,z)))]\Delta z\\
&+\int_{-\infty}^{x}[W_1(\delta_+^p(x),\delta_+^p(\sigma(z)))-W_1(x,\sigma(z))]\\
&\times\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&-\int_{x}^{+\infty} W_2(\delta_+^p(x),\delta_+^p(\sigma(z)))\\
&\times[\xi(\delta_+^p(z),\varphi(\delta_-(\tau,\delta_+^p(z))))\delta_+^{\Delta p}(z)\Delta z\\
&-\xi(z,\varphi(\delta_-(\tau,z)))]\Delta z\\
&-\int_{x}^{+\infty}[W_2(\delta_+^p(x),\delta_+^p(\sigma(z)))-W_2(x,\sigma(z))]\\
&\times\xi(z,\varphi(\delta_-(\tau,z)))\Delta z\\
&+\sum_{-\infty<x_k<x} W_1(\delta_+^p(x),x_{k+q}^+)(I_{k+q}(\varphi(x))\\
&+\gamma_{k+q})\\
&-\sum_{x<x_k<+\infty} W_2(\delta_+^p(x),x_{k+q}^+)(I_{k+q}(\varphi(x))\\
&+\gamma_{k+q})\\
&-\sum_{-\infty<x_k<x} W_1(x,x_k^+)(I_k(\varphi(x))+\gamma_k)\\
&+\sum_{x<x_k<+\infty} W_2(x,x_k^+)(I_k(\varphi(x))+\gamma_k)\Big|
\end{aligned}$$

$$\leq \Bigg(\frac{\beta+2\Gamma L_\xi\|\varphi\|}{\alpha}+\frac{\beta+2\Gamma L_\xi\|\varphi\|}{\inf|\ominus\alpha|} +\frac{2\Gamma}{1-e^{\frac{\alpha}{2}\theta}}(L_I\|\varphi\|+\hat{\gamma}) +2\beta\Big(\frac{1}{\inf|1-e^{-\alpha\theta}|}+\frac{1}{\inf|1-e^{\ominus\alpha\theta}|}\Big)\Bigg)\varepsilon. \quad (13)$$

It follows from (10) and (13) that $\Phi\varphi\in\mathbb{X}^*$.
For any $\varphi\in\mathbb{X}^*$, $\psi\in\mathbb{X}^*$, we can get

$$\begin{aligned}
&\|\Phi\varphi-\Phi\psi\|\\
=\ &\sup_{x\in\mathbb{T}}\Big|\int_{-\infty}^{x}W_1(x,\sigma(z))[\xi(z,\varphi(\delta_-(\tau,z)))\\
&-\xi(z,\psi(\delta_-(\tau,z)))]\Delta z\\
&-\int_{x}^{+\infty}W_2(x,\sigma(z))[\xi(z,\varphi(\delta_-(\tau,z)))\\
&-\xi(z,\psi(\delta_-(\tau,z)))]\Delta z\\
&+\sum_{-\infty<x_k<x}W_1(x,x_k^+)(I_k(\varphi(x))-I_k(\psi(x)))\\
&-\sum_{x<x_k<+\infty}W_2(x,x_k^+)(I_k(\varphi(x))-I_k(\psi(x)))\Big|\\
=\ &\lambda\|\varphi-\psi\|<\|\varphi-\psi\|. \qquad (14)
\end{aligned}$$

By (III) and (14), Φ is a contraction mapping in $\mathbb{X}^*$, then Φ has a unique nonzero fixed point $\varphi^*\in\mathbb{X}^*$ such that $\Phi\varphi^*=\varphi^*$, that is, system (1) has exactly one nonzero almost periodic solution in shifts $\delta_\pm$. This completes the proof. ■

Theorem 2. *If*

(I) *The conditions* (H_1)-(H_5) *hold;*

(II') *Suppose that the linear system*

$$y^{\Delta}(x)=D(x)y(x)$$

satisfies exponential dichotomy on $\mathbb{T}$ *with projection* P *and positive constants* β *and* α*;*

(III) $\lambda<1$, *where* $\lambda=\beta(\frac{1}{\alpha}+\frac{1}{\inf|\ominus\alpha|})L_\xi+\beta(\frac{1}{\inf|1-e^{-\alpha\theta}|}+\frac{1}{\inf|1-e^{\ominus\alpha\theta}|})L_I$*;*

then (2) exists a unique almost periodic solution in shifts $\delta_\pm$.

Theorem 3. *If*

(I) *The conditions* (H_1)-(H_5) *hold;*

(II'') *Suppose that the linear system*

$$y^{\Delta}(x)=D(\varphi(x))y(x)$$

satisfies exponential dichotomy on $\mathbb{T}$ *with projection* P *and positive constants* β *and* α*, and the dichotomy constants* β *and* α *does not depend on* φ*, where* $\varphi(x)$ *is a bounded continuous function;*

(III) $\lambda<1$, *where* $\lambda=\beta(\frac{1}{\alpha}+\frac{1}{\inf|\ominus\alpha|})L_\xi+\beta(\frac{1}{\inf|1-e^{-\alpha\theta}|}+\frac{1}{\inf|1-e^{\ominus\alpha\theta}|})L_I$*;*

then (3) exists a unique almost periodic solution in shifts $\delta_\pm$.

IV. Applications

Example 1. Consider the following impulsive dynamic equation

$$\begin{cases}
y^{\Delta}(x)=A(x,y(x))y(x)\\
\qquad+\int_{-\infty}^{t}C(x,z)y(z)\Delta z\\
\qquad+g(y(x-\tau(x))),\\
\qquad x_0\in\mathbb{T},x\neq x_k,k\in\mathbb{Z},\\
y(x^+)=y(x^-)-B_ky(x)+I_k(y(x))+\gamma_k,\\
x=x_k,k\in\mathbb{Z}.
\end{cases} \qquad (15)$$

where $y(x)=(y_1(x),y_2(x))^T$, and

$$\begin{aligned}
&y(x-\tau(x))=\begin{pmatrix}y_1(x-\sin(x))\\ y_2(x-\cos(x))\end{pmatrix},\\
&A(x,y(x))\\
&=\begin{pmatrix}-12-\frac{1}{40}\sin(x)+\frac{1}{4}y_2^2(x) & 12-y_1^2(x)\\ 44-y_2^2(x) & -50-\frac{1}{4}\sin(x)+2y_1^2(x)\end{pmatrix},\\
&C(x,z)\\
&=\begin{pmatrix}\frac{1}{800}\frac{\cos(-(x-z))}{\sqrt{1-\frac{1}{4}\sin^2(-(x-z))}} & \frac{1}{1000}\frac{\cos(-(x-z))}{\sqrt{1-\frac{1}{9}\sin^2(-(x-z))}}\\ \frac{1}{1500}\frac{\cos(-(x-z))}{\sqrt{1-\frac{1}{16}\sin^2(-(x-z))}} & \frac{1}{2000}\frac{\cos(-(x-z))}{\sqrt{1-\frac{1}{25}\sin^2(-(x-z))}}\end{pmatrix},\\
&g(y(x-\tau(x)))\\
&=\begin{pmatrix}\frac{1}{45}\sin(y_1(x-\sin(x)))\\ \frac{1}{40}\sin(y_2(x-\cos(x)))\end{pmatrix},\\
&B_k=\begin{pmatrix}0 & -\frac{1}{4}\cos(\sqrt{2}k)\\ 1+\cos(\sqrt{2}k) & 1\end{pmatrix},\\
&I_k(y(x))=\begin{pmatrix}\frac{1}{8}(1-\cos(\frac{y_1(x)}{2^k}))\\ \frac{1}{8}\sin(\frac{y_2(x)}{2^k})\end{pmatrix},\\
&\gamma_k=\begin{pmatrix}2\sin(k)\\ \cos(2k)\end{pmatrix}.
\end{aligned}$$

Let $\mathbb{T}=\mathbb{R}$. It is easy to check that the conditions $(H_1)-(H_5)$ and (I)-(II) hold, and

$$\lambda=0.0172<1.$$

According to Theorem 1, (15) exists a positive almost periodic solution in shift $\delta_\pm$.

The dynamic simulations of (15), see Figure 1.

Example 2. Consider the following Schoener's competition system with impulses

$$\begin{cases}
y_1^{\Delta}(x)\\
=\frac{1.8-0.1\cos(\sqrt{3}x)}{\exp\{y_1(\delta_-(\tau_1,x))\}+(5+0.2\cos(x))}\\
-(0.2+0.01\cos(x))\exp\{y_1(x)\}\\
-(0.003-0.001\sin(x))\exp\{y_2(x)\}\\
-0.001,\\
y_2^{\Delta}(x)\\
=\frac{1+0.1\sin(\sqrt{5}x)}{\exp\{y_2(\delta_-(\tau_2,x))\}+(6+0.1\sin(x))}\\
-(0.004-0.001\sin(x))\exp\{y_1(x)\}\\
-(0.1+0.01\sin(x))\exp\{y_2(x)\}\\
-0.005,x\neq x_k,k\in\mathbb{Z},\\
y_1(x^+)=y_1(x^-)+0.3y_1(x),\\
y_2(x^+)=y_2(x^-)+0.3y_2(x),\\
x=x_k,k\in\mathbb{Z}.
\end{cases} \qquad (16)$$

with the initial conditions

$$y(x_0)=(y_{10},y_{20})^T,y_{10}>0,y_{20}>0,x_0\in\mathbb{T}.$$

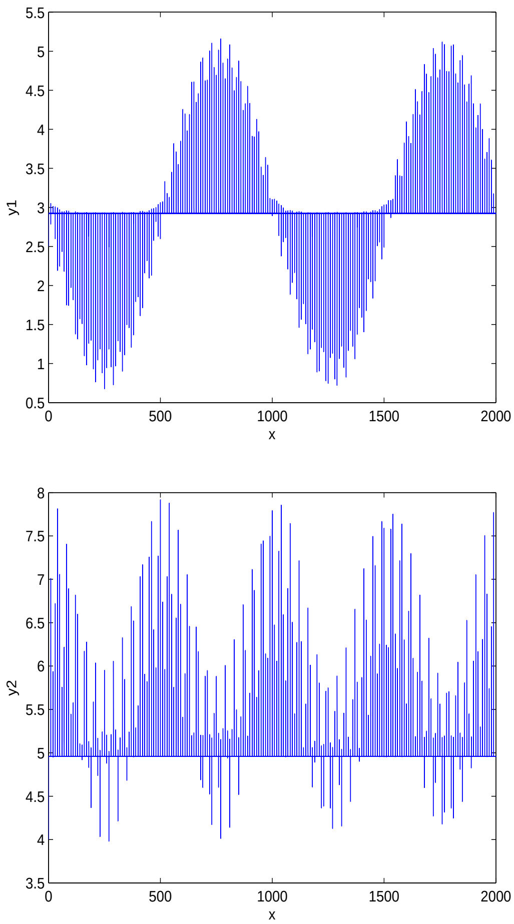

Fig. 1. Numerical solution of (15) (Example 1) with the initial value $y(0) = (2.5, 4)^T$.

System (16) can be written as

$$\begin{cases} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}^{\Delta} \\ = \begin{pmatrix} -(0.2+0.01\cos(x))\frac{\exp\{y_1(x)\}}{y_1(x)} \\ -(0.004-0.001\sin(x))\frac{\exp\{y_1(x)\}}{y_1(x)} \end{pmatrix. \\ \begin{matrix} -(0.003-0.001\sin(x))\frac{\exp\{y_2(x)\}}{y_2(x)} \\ -(0.1+0.01\sin(x))\frac{\exp\{y_2(x)\}}{y_2(x)} \end{matrix}\Bigg) \\ \times \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \\ + \begin{pmatrix} \frac{1.8-0.1\cos(sqrt(3)x)}{\exp\{y_1(\delta_-(\tau_1,x))\}+(5+0.2\cos(x))} - 0.001 \\ \frac{1+0.1\sin(sqrt(5)x)}{\exp\{y_2(\delta_-(\tau_2,x))\}+(6+0.1\sin(x))} - 0.005 \end{pmatrix}, \\ x \neq x_k, k \in \mathbb{Z}, \\ y_1(x^+) = y_1(x^-) + 0.3y_1(x), \\ y_2(x^+) = y_2(x^-) + 0.3y_2(x), \\ x = x_k, k \in \mathbb{Z}. \end{cases}$$

Let

$$\mathbb{T} = \overline{\bigcup_{n\in\mathbb{Z}} [2n, 2n+1]}.$$

It is easy to check that the conditions $(H_1)-(H_5)$ and (I)-(II) hold, and

$$\lambda = 0.1086 < 1.$$

According to Theorem 1, (16) exists a positive almost periodic solution in shift $\delta_\pm$.

Let $\delta_-(\tau_i, x)) = x - 1, i = 1, 2$, the dynamic simulations of (16), see Figure 2.

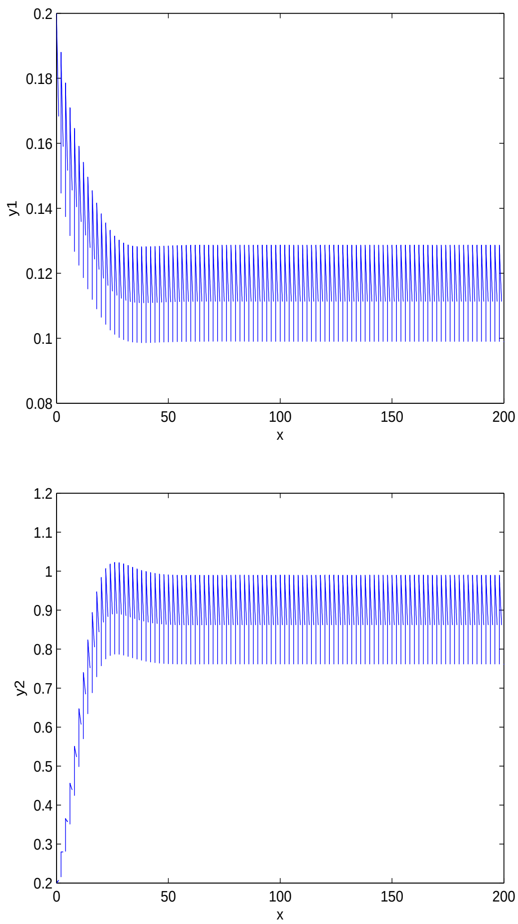

Fig. 2. Numerical solution of (16) (Example 2) with the initial value $y(0) = (0.2, 0.2)^T$.

V. Conclusion

By means of the shift operators, this paper deals with the existence of almost periodic solutions in shifts $\delta_\pm$ for three kinds of impulsive dynamic equations on time scales. The new approach will enable researchers to investigate almost periodicity notion on a large class of time scales whose members may not to be closed under the operation $x \pm z$ for a fixed $z \in \mathbb{T}$ or to be unbounded. Besides, the results of this paper lay a foundation for further exploring the influence of impulsive effects on the dynamic behaviors of one system.

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