Coordinate and Input Transformation for Feedback Linearization for a Class of Nonlinear Systems

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***Abstract*—This paper discusses the problem of finding the state transformation and feedback law for linearizing nonlinear systems. The construction of the state transformation and feedback law is proposed for two types of nonlinear systems. The steps are easy for practitioners to understand and follow. The result in this paper generalizes the well-known feedback linearization result for triangular systems by Meyer, Su, and Hunt. Two numerical examples are given to illustrate the state transformation and feedback law's construction process.**

***Index Terms*—feedback linearization; coordinate transformation; nonlinear systems; triangular systems**

I. Introduction

LINEAR system theories are relatively more mature, with many design and analysis tools available for use; in contrast, Controlling nonlinear systems poses greater challenges. Due to the inherent complexity of nonlinear systems, researchers began exploring advanced techniques, particularly in the field of differential geometry, to analyze and control such systems. This shift in focus was partly driven by the increasing prevalence of nonlinear dynamics in communication, networked systems, and other cyber-physical domains.

In the early 70's, researchers started to use differential geometry to analyze nonlinear systems partly driven by the needs in the aerospace industry. In the last three decades, many significant discoveries have been made in nonlinear geometric control theory [1]. An extensive review of the history of nonlinear geometric control up to the early 90's can be found in [2], [3]. Exact linearization of nonlinear systems is one of the major research topics in nonlinear control theory. In the late 70's, exact linearization of nonlinear systems via state transformation was first proposed by Krener [4]. A necessary and sufficient condition was derived for the existence of diffeomorphic nonlinear coordinate transformation to convert an affine nonlinear control system to a controllable linear system [4]. Brockett further introduced the concept of feedback invariance for nonlinear systems by adding feedback to the state coordinate transformation [5].

Consider an affine nonlinear system described by the equation

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in R^n$, $f(0) = 0$, and $f(x)$ and $g(x)$ are n-dimensional C^∞ vector fields. This system is said to be locally feedback linearizable if there exists a nonlinear coordinate transformation, a diffeomorphism defined in an open neighborhood of the origin $x = 0$,

$$z = T(x) \tag{2}$$

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and a feedback law

$$u = \alpha(x) + \beta(x)v \tag{3}$$

such that

$$\dot{z} = Az + bv \tag{4}$$

where (A, b) is a controllable pair in Brunovsky form [6], $z, b \in R^n, A \in R^{n\times n}$, and $u, v, \alpha(), \beta() \in R^1$. This definition can be easily extended to a multi-input case. If the linearization is valid in a given region, the system (1) is called globally feedback linearizable.

Since

$$\dot{z} = \frac{\partial T(x)}{\partial x}\{f(x) + g(x)[\alpha(x) + \beta(x)v]\}|_{x=T^{-1}(z)},$$

feedback linearization is equivalent to finding coordinate transformation $T(\,)$ in Equation (2), and $\alpha(\,)$ $\beta(\,)$ in the feedback law (3) such that

$$\frac{\partial T(x)}{\partial x}\{f(x) + g(x)\alpha(x)\}|_{x=T^{-1}(z)} = Az$$

$$\frac{\partial T(x)}{\partial x}\{g(x)\beta(x)\}|_{x=T^{-1}(z)} = b$$

The existence of such functions is highly non-trivial. Jakubczyk and Respondek [7], Su [8], and Su, Meyer, and Hunt [9] presented necessary and sufficient conditions for the existence of linearizing coordinate transformation and feedback law.

Theorem 1: [2] The affine nonlinear system (1) is locally feedback linearizable if and only if the following conditions hold in an open neighborhood of the origin:
(1) The following distribution is involutive

$$D = span\{g(x), ad_f g(x), ad_f^2 g(x), ..., ad_f^{n-2} g(x)\}$$

(2) $\quad rank[g(x), ad_f g(x), ad_f^2 g(x), ..., ad_f^{n-1} g(x)] = n$

The definitions for the Lie derivative $ad_f^i g(x)$, involutivity, and other terminologies in differential geometry can be found in [2], [3], [10].

Feedback linearization has been mostly viewed as a completely solved control problem, a mature field, and a great success in nonlinear control theory.

Meanwhile, global feedback linearization problem is significantly more complex than the local feedback linearization problem. Many results can be found in [11], [12], [13], [14].

The majority of the theoretical studies in this field have focused on the existence of the linearizing coordinate transformation and feedback law. However, finding the coordinate transformation and feedback law can be challenging. The construction of the coordinate transformation requires solving the following set of partial differential equations (see [2]):

$$< \frac{\partial \lambda(x)}{\partial x}, g(x) > = < \frac{\partial \lambda(x)}{\partial x}, ad_f g(x) > = \cdots$$

$$= < \frac{\partial \lambda(x)}{\partial x}, ad_f^{n-2} g(x) > = 0$$

with the constraint

$$< \frac{\partial \lambda(x)}{\partial x}, ad_f^{n-1} g(x) > \neq 0$$

where the operation $<\ ,\ >$ is the inner product of two vector fields.

Feedback linearization has been used in many nonlinear control problems, spanning motor control [15], robotics [16], [17], power system control [18], flight control [19], and unmanned aerial vehicle control [20]. A common challenge in these applications involves solving partial differential equations, which has been addressed through various approaches. In some instances, an output $y = h(x)$ naturally emerges as the solution to the partial differential equation. Alternatively, a judicious selection of coordinates, such as position and velocity derived from Newton's Second Law, may also serve as the solution. MATLAB, for instance, facilitates feedback linearization by necessitating the user to furnish the function $y = h(x)$, utilizing it as a solution to the partial differential equations.

Cheng *et al.* developed a sophisticated methodology for finding the coordinate transformation using the semi-tencor product [21], constructing the coordinate transformation by iteratively adding higher-order terms. This approach works well for local feedback linearization; however, it has limitations for global applications, where omitted higher-order terms may become significant in regions distant from the origin.

Solving partial differential equations with constraints can be more challenging than solving the original nonlinear differential equations. Consequently, the feedback linearization problem remains partially unsolved due to the lack of practical methodologies for coordinate transformation and feedback law construction.

For a specific class of nonlinear systems with a structured form known as triangular systems *triangular systems* [19], [22], explicit construction of the coordinate transformation and feedback law is possible. This paper focuses on a simplified version, *strict triangular systems*, for clarity and accessibility.

A strict triangular system has the following form

$$\begin{cases} \dot{x}_1 &= f_1(x_1) + x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + x_4 \\ & \vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \cdots, x_{n-1}) + x_n \\ \dot{x}_n &= f_n(x_1, x_2, \cdots, x_n) + g_n(x)u \end{cases} \tag{5}$$

with $f_i(0) = 0$ and $g_n(x) \neq 0$. Without loss of generality, $g_n(x)$ can be assumed to be equal to 1 since the function $g_n(x)$ can be taken care of by the input transformation $u = \frac{1}{g_n(x)} v$. The assumption $f_i(0) = 0$ is needed for 0 to be the equilibrium point with no input. $g_n(x) \neq 0$ is necessary because otherwise the system would not be controllable.

Theorem 2: [22] The strict triangular system (5) is globally feedback linearizable. The coordinate transformation is given by

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= \dot{z}_1 = f_1(x_1) + x_2 \\ z_3 &= \dot{z}_2 \\ &\vdots \\ z_n &= \dot{z}_{n-1} \end{aligned}$$

The coordinate transformation and feedback law for feedback linearization are also used in other areas such as nonlinear adaptive control. In the back-stepping design of an adaptive controller for nonlinear systems proposed by Kanellakopoulos *et al.* [23], it was assume that the nonlinear system was transformed into the strict triangular form before adaptive laws were derived. Without the actual coordinate transformation and feedback law, it is impractical to apply back-stepping design to nonlinear systems that are not in the form of triangular systems.

Triangular systems are also seen in other control system design problems [24]. However, many systems are not triangular systems.

Zhan and Wang found necessary and sufficient conditions for an affine nonlinear system (1) to be transformed to a triangular system by linear coordinate transformation [25]. However, their methodology only works when $g(x)$ is a constant vector. Apparently, this condition is not satisfied for all systems.

In this paper, a nonlinear system with a structure similar to that of strict triangular systems will be considered. This class of systems is not triangular systems, and $g(x)$ may not be constant; thus, the results in [19], [22], [25] are not applicable.

II. Extended Triangular Systems

Consider the following n-dimensional nonlinear system:

$$\begin{cases} \dot{x}_1 &= f_1(x_1) + x_2 + g(x_2)x_3 \\ \dot{x}_2 &= f_2(x_1, x_2) + x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + x_4 \\ & \vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \cdots, x_{n-1}) + x_n \\ \dot{x}_n &= f_n(x_1, x_2, \cdots, x_n) + g_n(x)u \end{cases} \tag{6}$$

where $g(0) = 0$ and $g_n(0) \neq 0$. The system (6) will be referred to as a *type A extended triangular system.* Strict triangular systems are a special case of the type A extended triangular system with $g(x_2) = 0$.

Theorem 3: The type A extended triangular system (6) is feedback linearizable with the following coordinate transformation:

$$\begin{aligned} z_1 &= x_1 - \int_0^{x_2} g(t) \mathrm{d}t \\ z_2 &= \dot{z}_1 = f_1(x_1) + x_2 - g(x_2) f_2(x_1, x_2) \\ &\vdots \\ z_n &= \dot{z}_{n-1} \end{aligned}$$

Proof: Define z_1 as

$$z_1 = x_1 - \int_0^{x_2} g(t) \mathrm{d}t$$

It follows that

$$\begin{aligned}\dot{z}_1 &= f_1(x_1) + x_2 + g(x_2)x_3 - g(x_2)[f_2(x_1, x_2) + x_3] \\ &= f_1(x_1) + x_2 - g(x_2)f_2(x_1, x_2)\end{aligned}$$

Define z_2 as

$$z_2 = f_1(x_1) + x_2 - g(x_2)f_2(x_1, x_2)$$

It follows that

$$\dot{z}_1 = z_2$$

Note that the transformation $(x_1, x_2, x_3, ..., x_n) \rightarrow (z_1, z_2, x_3, ..., x_n)$ is a valid coordinate transformation locally around the origin since the Jacobian matrix evaluated at 0 is nonsingular under the assumption $g(0) = 0$.

Taking the derivative of z_2 to get

$$\begin{gathered}\dot{z}_2 = \frac{df_1(x_1)}{dx_1}[f_1(x_1) + x_2 + g(x_2)x_3] + f_2(x_1, x_2) + \\ x_3 - g(x_2)\frac{\partial f_2(x_1,x_2)}{\partial x_1}[f_1(x_1) + x_2 + g(x_2)x_3] - \\ \frac{\partial(g(x_2)f_2(x_1,x_2))}{\partial x_2}[f_2(x_1, x_2) + x_3]\end{gathered} \tag{7}$$

it can be seen that $\dot{z}_2$ is an affine function of x_3, and $\dot{z}_2$ can be written in the following form

$$\dot{z}_2 = f_1^{'}(z_1, z_2) + f_2^{'}(z_1, z_2)x_3 + x_3,$$

where $f_1^{'}$ and $f_2^{'}$ are defined based on the right hand side of Equation (7), and $f_2^{'}(0, 0) = 0$. Let

$$z_3 = f_1^{'}(z_1, z_2) + f_2^{'}(z_1, z_2)x_3 + x_3$$

Since $f_2^{'}(0, 0) = 0$, the Jacobian matrix for the transformation $(x_1, x_2, x_3) \rightarrow (z_1, z_2, z_3)$ evaluated at 0 is nonsingular. Therefore, $(x_1, x_2, x_3, ..., x_n) \rightarrow (z_1, z_2, z_3, x_4, ..., x_n)$ is a valid coordinate transformation. This process of defining z_i as the derivative of z_{i-1} can be continued until $i = n$. The last equation can be written as

$$\dot{z}_n = f_{n-1}^{'}(z) + f_n^{'}(z)u + g_n(x)u,$$

where $f_n^{'}(0) = 0$. Therefore, the input transformation of

$$f_{n-1}^{'}(z) + f_n^{'}(z)u + g_n(x)u = v$$

completes the final step of feedback linearization.

Q.E.D.

Next, consider the following n-dimensional nonlinear system:

$$\begin{cases}\dot{x}_1 &= f_1(x_1) + x_2 + x_1 g(x_2)x_3 \\ \dot{x}_2 &= f_2(x_1, x_2) + x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + x_4 \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \cdots, x_{n-1}) + x_n \\ \dot{x}_n &= f_n(x_1, x_2, \cdots, x_n) + g_n(x)u\end{cases} \tag{8}$$

where $g_n(0) \neq 0$. The system (8) will be referred to as a *type B extended triangular system*. Strict triangular systems are a special case of the type B extended triangular system with $g(x_2) = 0$.

Theorem 4: The type B extended triangular system (8) is feedback linearizable with the following coordinate transformation:

$$\begin{aligned}z_1 &= x_1 e^{-\int_0^{x_2} g(t)\mathrm{d}t} \\ z_2 &= \dot{z}_1 = [f_1(x_1) + x_2 - x_1 g(x_2)f_2(x_1, x_2)]e^{-\int_0^{x_2} g(t)\mathrm{d}t} \\ &\vdots \\ z_n &= \dot{z}_{n-1}\end{aligned}$$

Proof: Define z_1 as

$$z_1 = x_1 e^{-\int_0^{x_2} g(t)\mathrm{d}t}$$

It follows that

$$\begin{aligned}\dot{z}_1 &= f_1(x_1) + x_2 + x_1 g(x_2)x_3 e^{-\int_0^{x_2} g(t)\mathrm{d}t} \\ &\quad - x_1 g(x_2)[f_2(x_1, x_2) + x_3]e^{-\int_0^{x_2} g(t)\mathrm{d}t} \\ &= [f_1(x_1) + x_2 - x_1 g(x_2)f_2(x_1, x_2)]e^{-\int_0^{x_2} g(t)\mathrm{d}t}\end{aligned}$$

Define z_2 as

$$z_2 = [x_2 + f_1(x_1) + x_2 - x_1 g(x_2)f_2(x_1, x_2)]e^{-\int_0^{x_2} g(t)\mathrm{d}t}$$

It follows that

$$\dot{z}_1 = z_2$$

Note that the transformation $(x_1, x_2, x_3..., x_n) \rightarrow (z_1, z_2, x_3..., x_n)$ is a valid coordinate transformation locally around the origin since the Jacobian matrix evaluated at 0 is nonsingular.

Taking the derivative of z_2, one can derive an equation similar to Equation (7) such that $\dot{z}_2$ is an affine function of x_3, and it can be written in the following form

$$\dot{z}_2 = f_1^{'}(z_1, z_2) + f_2^{'}(z_1, z_2)x_3 + x_3$$

with $f_2(0, 0) = 0$.

The rest of the proof is exactly the same as that of Theorem 3.

Q.E.D.

Remarks:
1. While strict triangular systems are always globally feedback linearizable in the entire state space, the types A and B extended triangular systems are in general only locally feedback linearizable. In the proofs, the Jacobian matrix is not zero in a small neighborhood of the origin, which only guarantees the existence of inverse transformation around the origin.
2. It can be shown that not only strict triangular systems, but also triangular systems more general can be extended to include the extra term $g(x_2)x_3$ or $x_1 g(x_2)x_3$. In this case, both triangular systems and types A and B extended triangular systems are locally feedback linearizable.

III. Two dimensional systems

In case of two dimensional systems, the variable x_3 in $g(x_2)x_3$ and $x_1 g(x_2)x_3$ will be replaced by u correspondingly for types A and B extended triangular systems.

A two dimensional type A extended triangular system has the following form:

$$\begin{cases}\dot{x}_1 &= f_1(x_1) + x_2 + g_2(x)g(x_2)u \\ \dot{x}_2 &= f_2(x) + g_2(x)u\end{cases} \tag{9}$$

where $g_2(x) \neq 0$, $x = (x_1, x_2)$, and $g(0) = 0$.

***Corollary** 1:* A two dimensional type A extended triangular system (9) is feedback linearizable with the following coordinate transformation:

$$\begin{aligned} z_1 = & \quad x_1 - \textstyle\int_0^{x_2} g(t)\mathrm{d}t \\ z_2 = & \quad \dot{z}_1 \end{aligned}$$

Proof: First, apply a control variable substitution $g_2(x)u = u'$ to have the following type A extended triangular system

$$\begin{cases} \dot{x}_1 = & f_1(x_1) + x_2 + g(x_2)u' \\ \dot{x}_2 = & f_2(x) + u' \end{cases} \tag{10}$$

Next, define a new coordinate z_1 as

$$z_1 = x_1 - \int_0^{x_2} g(t)\mathrm{d}t$$

It follows that

$$\begin{aligned} \dot{z}_1 &= f_1(x_1) + x_2 + g(x_2)u' - g(x_2)[f_2(x) + u'] \\ &= f_1(x_1) + x_2 - g(x_2)f_2(x) \end{aligned}$$

To simplify the notation, define $f_3(z_1, x_2)$ as

$$f_3(z_1, x_2) = f_1(x) + x_2 - g(x_2)f_2(x)|_{x_1 = z_1 + \int_0^{x_2} g(t)\mathrm{d}t}$$

and define the second coordinate z_2 as

$$z_2 = f_3(z_1, x_2)$$

then

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{\partial f_3(z_1, x_2)}{\partial z_1} z_2 + \frac{\partial f_3(z_1, x_2)}{\partial x_2} [f_2(x) + u'] \end{aligned}$$

It can be verified that this system is locally feedback linearizable since

$$\frac{\partial f_3(z_1, x_2)}{\partial x_2} \neq 0$$

Introducing a new control variable v

$$v = \frac{\partial f_3(z_1, x_2)}{\partial z_1} z_2 + \frac{\partial f_3(z_1, x_2)}{\partial x_2} [f_2(x) + u'],$$

one gets a controllable linear system

$$\begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= v \end{cases}$$

Combining all the steps, one gets the coordinate transformation

$$\begin{aligned} z_1 &= x_1 - \int_0^{x_2} g(t)\mathrm{d}t \\ z_2 &= f_3(z_1, x_2) \end{aligned}$$

and the feedback law

$$\begin{aligned} u = & \frac{-1}{g_2(x)} \left\{ f_2(x) + \left[\frac{\partial f_3(z_1, x_2)}{\partial x_2} \right]^{-1} \frac{\partial f_3(z_1, x_2)}{\partial z_1} z_2 \right\} \\ & - \frac{1}{g_2(x)} \left[\frac{\partial f_3(z_1, x_2)}{\partial x_2} \right]^{-1} v \end{aligned}$$

With z_1 and z_2 substituted as functions of x_1 and x_2 into the above formula, the feedback law takes the required form of $u = \alpha(x) + \beta(x)v$.

Q.E.D.

A similar result can be derived for two dimensional type B extended triangular systems. A two dimensional type B extended triangular system has the following form:

$$\begin{cases} \dot{x}_1 &= f_1(x_1) + x_2 + g_2(x)x_1 g(x_2)u \\ \dot{x}_2 &= f_2(x) + g_2(x)u \end{cases} \tag{11}$$

where $g_2(x) \neq 0$, $x = (x_1, x_2)$, and $g(0) = 0$.

***Corollary** 2:* A two dimensional type B extended triangular system (11) is feedback linearizable with the following coordinate transformation:

$$\begin{aligned} z_1 &= x_1 e^{-\int_0^{x_2} g(t)\mathrm{d}t} \\ z_2 &= \dot{z}_1 \end{aligned}$$

Proof: First, apply a control variable substitution $g_2(x)u = u'$ to have the following type A extended triangular system

$$\begin{cases} \dot{x}_1 = & f_1(x_1) + x_2 + x_1 g(x_2)u' \\ \dot{x}_2 = & f_2(x) + u' \end{cases} \tag{12}$$

Next, define a new coordinate z_1 as

$$z_1 = x_1 e^{-\int_0^{x_2} g(t)\mathrm{d}t}$$

It follows that

$$\dot{z}_1 = [f_1(x_1) + x_2 - x_1 g(x_2)f_2(x)]e^{-\int_0^{x_2} g(t)\mathrm{d}t} \tag{13}$$

To simplify the notation, define $f_3(z_1, x_2)$ as the righ hand side of Equation(13) with x_1 substituted by $z_1 e^{\int_0^{x_2} g(t)\mathrm{d}t}$.

Now define the second coordinate z_2 as

$$z_2 = f_3(z_1, x_2)$$

then

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{\partial f_3(z_1, x_2)}{\partial z_1} z_2 + \frac{\partial f_3(z_1, x_2)}{\partial x_2} [f_2(x) + u'] \end{aligned}$$

It can be verified that this system is locally feedback linearizable since

$$\frac{\partial f_3(z_1, x_2)}{\partial x_2} \neq 0$$

Introducing a new control variable v

$$v = \frac{\partial f_3(z_1, x_2)}{\partial z_1} z_2 + \frac{\partial f_3(z_1, x_2)}{\partial x_2} [f_2(x) + u'],$$

one gets a controllable linear system

$$\begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= v \end{cases}$$

Combining all the steps, one gets the coordinate transformation

$$\begin{aligned} z_1 &= x_1 - \int_0^{x_2} g(t)\mathrm{d}t \\ z_2 &= f_3(z_1, x_2) \end{aligned}$$

and the feedback law

$$\begin{aligned} u = & \frac{-1}{g_2(x)} \left\{ f_2(x) + \left[\frac{\partial f_3(z_1, x_2)}{\partial x_2} \right]^{-1} \frac{\partial f_3(z_1, x_2)}{\partial z_1} z_2 \right\} \\ & - \frac{1}{g_2(x)} \left[\frac{\partial f_3(z_1, x_2)}{\partial x_2} \right]^{-1} v \end{aligned}$$

With z_1 and z_2 substituted as functions of x_1 and x_2 into the above formula, the feedback law takes the required form of $u = \alpha(x) + \beta(x)v$.

Q.E.D.

Remarks 1. The feedback law looks complicated and the derivation steps are tedious, nonetheless, the process is fairly straightforward. The relatively simple construction process allows practitioners to use symbolic calculation such as in MATLAB [26] or other similar software packages [27].

2. The feedback linearization conditions can be expressed as a certain function not equal to zero instead of the involutivity of a distribution and non-singularity of another distribution. This is extremely helpful when trying to extend the result to the global feedback linearization problem.

3. Systems (10) and (12) have a non-constant coefficient of the control variable u; thus the results in [19], [22], [25] are not applicable.

IV. Numerical Examples

Two numerical examples are presented in this section to illustrate the step-by-step process of finding coordinate transformation for feedback linearization in the context of types A and B extended triangular systems. These examples serve to provide practical insights and enhance the understanding of the discussed methodology.

Example 1. Given the following nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(x_1 + x_2^2) - 2x_2(1 + x_1^2)u \\ \dot{x}_2 &= (1 + x_1^2)u\end{aligned}$$

Letting $u' = (1 + x_1^2)u$, the system is rewritten as

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(x_1 + x_2^2) - 2x_2 u' \\ \dot{x}_2 &= u'\end{aligned}$$

Define the new coordinate z_1:

$$z_1 = x_1 - \int_0^{x_2} -2t\mathrm{d}t = x_1 + x_2^2$$

It follows that

$$\dot{z}_1 = x_2 + \sin(x_1 + x_2^2)$$

Now define z_2 as

$$z_2 = x_2 + \sin(x_1 + x_2^2)$$

It follows that

$$\dot{z}_2 = u' + \cos(x_1 + x_2^2)[x_2 + \sin(x_1 + x_2^2)]$$

Letting

$$v = u' + \cos(x_1 + x_2^2)[x_2 + \sin(x_1 + x_2^2)]$$

one can solve for u:

$$u = \frac{-\cos(x_1 + x_2^2)[x_2 + \sin(x_1 + x_2^2)]}{1 + x_1^2} + \frac{1}{1 + x_1^2}v$$

With this feedback law and the coordinate transformation $(x_1, x_2) \to (z_1, z_2)$, one gets a linear controllable system

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= v\end{aligned}$$

The inverse transformation is given by:

$$\begin{aligned}x_1 &= z_1 - (z_2 - \sin(z_1))^2 \\ x_2 &= z_2 - \sin(z_1)\end{aligned}$$

For every point in the state space, the inverse of the coordinate transformation exists. The coordinate transformation and its inverse are both differentiable. The feedback law is also invertible. Therefore, the nonlinear system is globally feedback linearizable in the entire state space.

Example 2. Now change one term in Example 1 (inside the $\sin(\,)$ function)

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(x_1) - 2x_2(1 + x_1^2)u \\ \dot{x}_2 &= (1 + x_1^2)u\end{aligned}$$

Let $u' = (1 + x_1^2)u$, the system can be rewritten as

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(x_1) - 2x_2 u' \\ \dot{x}_2 &= u'\end{aligned}$$

Define the new coordinate z_1:

$$z_1 = x_1 - \int_0^{x_2} -2t\mathrm{d}t = x_1 + x_2^2$$

It follows that

$$\dot{z}_1 = x_2 + \sin(x_1)$$

Now define z_2 as

$$z_2 = x_2 + \sin(x_1)$$

It follows that

$$\dot{z}_2 = u' + \cos(x_1)[x_2 + \sin(x_1) - 2x_2 u']$$

Let

$$v = u' + \cos(x_1)[x_2 + \sin(x_1) - 2x_2 u']$$

one can solve for u:

$$u = \frac{-\cos(x_1)[x_2 + \sin(x_1)]}{(1 + x_1^2)[1 - 2x_2\cos(x_1)]} + \frac{1}{(1 + x_1^2)[1 - 2x_2\cos(x_1)]}v$$

With the feedback law defined in the above equation and the coordinate transformation defined as

$$\begin{aligned}z_1 &= x_1 + x_2^2 && (14) \\ z_2 &= x_2 + \sin(x_1) && (15)\end{aligned}$$

the nonlinear system is transformed into a linear controllable one. The control value cannot be infinity, therefore

$$2x_2\cos(x_1) \neq 1 \qquad (16)$$

The the inverse of the coordinate transformation must exist, i.e., one must be able to solve x_1, x_2 from Equations (14, 15) as functions of z_1, z_2. The Jacobian matrix at the origin can be calculated as

$$\begin{bmatrix}1 & 0 \\ 1 & 1\end{bmatrix}$$

Therefore, the coordinate transformation is a local diffeomorphism. The control variable u is also finite at the origin. One can conclude that the system is locally feedback linearizable.

Using numerical methods, one can search point-wise for the region over which feedback linearization is valid.

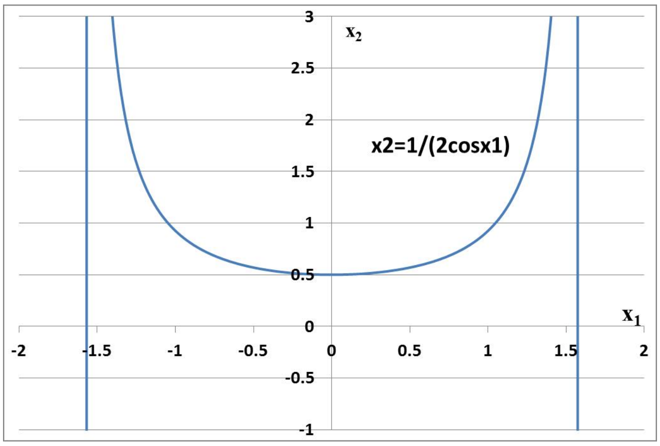

Fig. 1. Region of bounded control in x coordinates

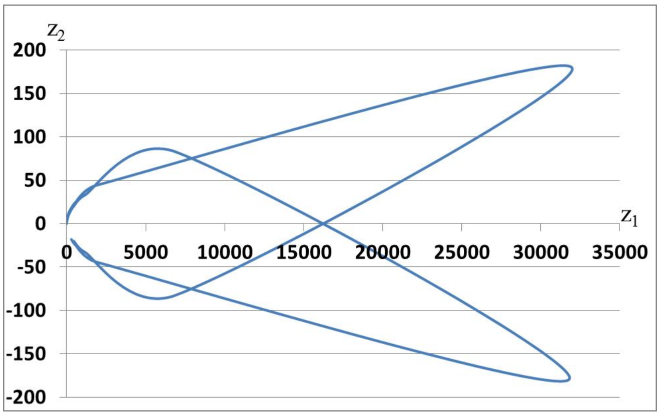

Fig. 2. Region of bounded control in z coordinates

It is straightforward to plot the constraint specified in (16). Fig. 1 shows the graph of the function

$$x_2 = \frac{1}{2\cos(x_1)}$$

The feedback law is valid as long as the coordinates (x_1, x_2) does not fall on the curve in Fig. 1. Using Equations (14, 15), one can easily map the region in (x_1, x_2) coordinates to a region in (z_1, z_2) coordinates, as illustrated in Fig. 2.

The invertability of coordinate transformation is more complicated. First, note that the existence of inverse of the mapping $(z_1, z_2) \to (x_1, x_2)$ is equivalent to the existence of a unique solution x_2 to the following equation, for any given value (z_1, z_2)

$$x_2 = z_2 - \sin(z_1 - x_2^2)$$

The existence of such a unique solution for x_2 for a given z coordinate (z_1, z_2) can be determined by finding the intersection of two curves:

$$x_2 = t$$

$$x_2 = z_2 - \sin(z_1 - t^2)$$

where t is a variable that takes values in the range of x_2 for the region that is being considered for feedback linearization. For example, if one is interested in feedback linearization for the system in the region $x_1 \in [-2, 2], x_2 \in [-3, 2]$, t can be varied from -3 to 2, as illustrated in Fig. 3. An algorithm was developed in MATLAB to search for the unique solution for

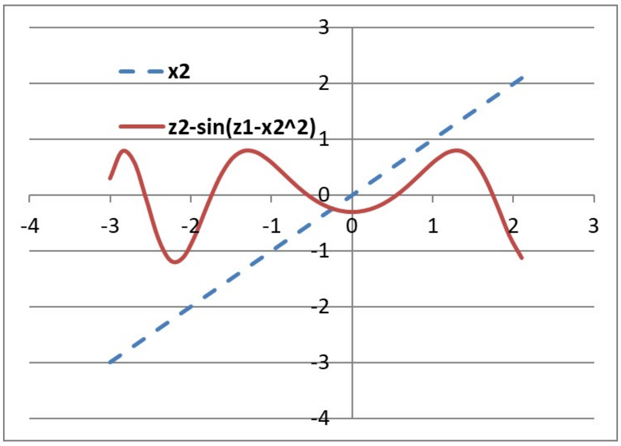

Fig. 3. Region where inverse transformation exists

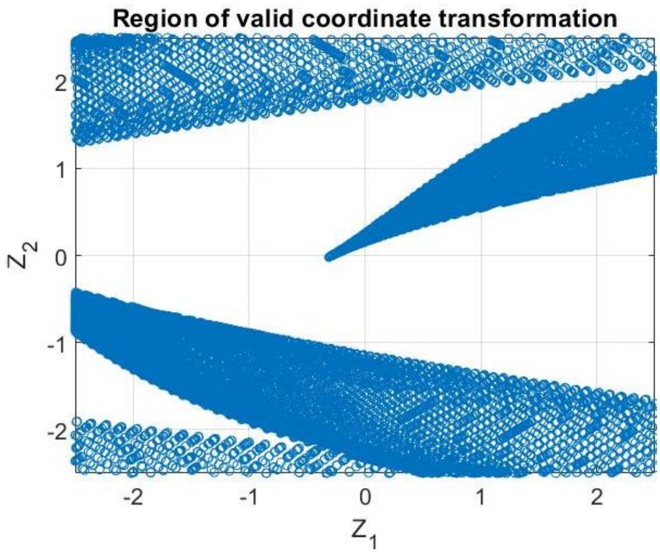

Fig. 4. Region of valid coordinate transformation

any given z coordinates (z_1, z_2) that can be mapped from (x_1, x_2). The result of the numerical search process for all (z_1, z_2) of interest is plotted in Fig. 4. The shaded area in Fig. 4 are values of (z_1, z_2) for which a unique solution does not exist. In the white area, the system is feedback linearizable.

Remarks:
1. Fig. 4 only shows that the particular coordinate transformation derived in Example 2 does not have an inverse transformation in the dark area. It is possible that there exists other coordinate transformations that would work in some area inside the dark area. In other words, the white area in Fig. 4 is a sufficient but not necessary condition for feedback linearization.
2. Once the search algorithm is developed, a different region can be easily checked for feedback linearization by changing the search scope.

V. Conclusions

In this paper, the problem of finding the coordinate transformation and feedback law for feedback linearization is studied. For systems with some special structures defined as types A and B extended triangular systems, the coordinate transformation and feedback law can be readily derived without the need to solve partial differential equations. The triangular system is a special case for the extended triangular systems. Therefore, the result in this paper generalizes the well-known feedback linearization result for triangular systems by Meyer, Su, and Hunt [19], [22].

For two dimensional extended triangular systems, coefficient of the control variable u can be non-constant; thus the results in [19], [22], [25] are not applicable. When the coefficient of u contains higher order terms, the systems cease to be bilinear. As a consequence, the results presented in [28] are not applicable.

While the derivation of the coordinate transformation and feedback law may appear tedious, the use of symbolic computation software streamlines the process.

While this paper only presents the single input case, a similar result can be easily derived for multi-input case.

Future work includes development of a dedicated software package designed for feedback linearization in extended triangular systems. Such software holds the potential to make feedback linearization more accessible to practicing engineers in the field of nonlinear system control.

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