A Modulus-Based Shamanskii-Like Levenberg–Marquardt Method for Solving Nonlinear Complementary Problems

Defeng Ding, Minglei Fang, Min Wang, and Yuting Sheng

Abstract—**In this paper, a modulus-based Shamanskii-Like Levenberg-Marquardt method is proposed for solving nonlinear complementarity problems (NCPs). First, the NCP is reformulated in the form of an equivalent non-smooth system of equations. Then, a non-smooth Shamanskii-Like Levenberg-Marquardt method using a non-monotone r-order Armijo line search is developed by generalizing a smooth Levenberg-Marquardt method to solve the resulting system. Global convergence of the proposed method is achieved under some suitable assumptions. Numerical experiments verify the feasibility and efficiency of the proposed method.**

Index Terms—**Armijo line search, Levenberg-Marquardt method, modulus-based manipulation, nonlinear complementarity problem.**

I. Introduction

FOR a given smooth mapping $F: \mathbb{R}^n \to \mathbb{R}^n$, the nonlinear complementarity problem (NCP) is finding a vector $z \in \mathbb{R}^n$ that satisfies the following conditions:

$$z \geq 0,\ F(z) \geq 0,\ z^{\mathrm{T}}F(z) = 0, \tag{1}$$

where $F: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.
If $F(z) = Mz + q$, the NCP degenerates to a linear complementarity problem (LCP). Assume that a solution set to (1) denoted by Z^* is nonempty, and $\|\cdot\|$ represents a two-norm in all cases.

The NCP has been widely used in engineering, economics, and mechanics. Many problems in scientific computing and engineering applications, such as elastic contact, economic equilibrium, and free boundary problems in fluid dynamics, can be categorized as nonlinear complementary problems [1], [2], [3]. Many numerical algorithms have been developed to solve the NCPs, including Fixed point iterative methods, Newton methods, Conjugate gradient methods, and Levenberg-Marquardt methods [4], [5], [6], [7], [8], [9], [10]. Semi-smooth equations methods that use nonlinear complementarity functions have been popular methods in recent years. Common forms of a complementary function ψ are as follows:

$$\psi_{min}(s,\ t) = min\{s,\ t\},$$
$$\psi_{FB}(s,\ t) = \sqrt{s^2+t^2} - s - t,$$

which are called the minimum function [11] and the Fischer Burmeister function [12], respectively. Recently, Bai Zhongzhi proposed a modulus-based iteration method for solving the LCP and analyzed the global convergence of the proposed method [13]. By applying modulus-based manipulations to solving the other complementary problems, various modulus-based methods have been developed [14], [15], [16]. This study considers the Levenberg-Marquardt (LM) method, which computes the search direction by:

$$d_k = -\left(J_k^{\mathrm{T}} J_k + t_k I\right)^{-1} J_k^{\mathrm{T}} F_k, \tag{2}$$

where $F_k = F(x_k)$, $J_k = F'(x_k)$ is the Jacobian matrix of F at x_k, and t_k is a non-negative regularized parameter used to prevent the iteration point from moving in wrong directions when approaching the saddle point.
Hu Yaning, Peng Zheng, et al. [16] proposed a modulus-based adaptive multi-step LM method for NCPs. In this method, at each iteration, d_k is obtained by performing approximate LM steps in addition to the classical LM step (2) as follows:

$$\begin{aligned} &d_{k,\ i} = -\left(J_k^{\mathrm{T}} J_k + t_k I\right)^{-1} J_k^{\mathrm{T}} F\left(x_{k,\ i}\right), \\ &x_{k,\ i} = x_{k,\ i-1} + d_{k,\ i-1}, \end{aligned} \tag{3}$$

where i is a positive integer, and $i = 1,\ 2,\ \cdots,\ r$; $x_{k,\ 0} = x_k$; $d_{k,\ 0} = d_k$.
Further, global convergence of the modulus-based non-smooth LM method has been proven using the trust region techniques. However, instead of using the trust region techniques, Chen Liang and Ma Yanfang [17] presented a smooth Shamanskii-Like Levenberg-Marquardt (SLM) method using a non-monotone r-order Armijo line search for solving nonlinear equations, which is expressed by:

$$\begin{aligned} \left\|F\left(x_k + \alpha_k s_k\right)\right\|^2 \leq (1+\omega_k)\left\|F_k\right\|^2 - \xi_0 \alpha_k^2 \left\|s_k\right\|^2 \\ - \xi_1 \alpha_k^2 \left\|F_k\right\|^2, \end{aligned}$$

where ξ_0 and ξ_1 are positive constants, and $\{\omega_k\}$ is a sequence that satisfies the conditions of:

$$\sum_{k=0}^{\infty} \omega_k < \infty,\ \omega_k > 0,$$

and,

$$s_k = \begin{cases} \sum_{i=0}^{r-1} d_{k,\ i}, & F_k^{\mathrm{T}} J_k \sum_{i=0}^{r-1} d_{k,\ i} \leq -\lambda, \\ d_{k,\ 0}, & \text{else}, \end{cases}$$

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where λ is a small positive constant, and $d_{k,\ i},\ i = 1,\ 2,\ \cdots,\ (r-1)$ satisfies (3). The LM parameter of the method t_k is computed by:

$$t_k = \mu \left\| F_k \right\|,$$

where $\mu > 0$ is a constant.
The method's convergence rate is proven to be $(r+1)$. By generalizing the method presented in [17] to the non-smooth case, this study proposes a modulus-based non-smooth Shamanskii-Like Levenberg-Marquardt (NSLM) method with r-order Armijo line search for NCPs. The LM parameter of the proposed method t_k is computed by:

$$t_k = \mu \left\| F_k \right\|^{\delta},$$

where $\delta \in [1,\ 2]$ and $\mu > 0$ are constants.
Under suitable conditions, global convergence of the proposed method is proven.

The rest of this paper is organized as follows. In Section II, a modulus-based manipulation is used to translate the NCP into a non-smooth system of nonlinear equations, and several common lemmas and definitions in the non-smooth case are introduced. In Section III, the proposed NSLM method is described. In Section IV, the global convergence of the proposed method is proven under suitable conditions. In Section IV, the preliminary numerical results are presented and discussed. Finally, the main conclusions and remarks are given in Section V.

II. Preliminaries

In this paper, absolute value $|\cdot|$ is component-wise, Clarke's subdifferential of F at x is denoted by $\partial F(x)$, and the identity matrix is denoted by I.

Consider a vector $x = (x_1,\ x_2,\ \cdots,\ x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and denote $|x| = (|x_1|,\ |x_2|,\ \cdots,\ |x_n|)^{\mathrm{T}} \in \mathbb{R}^n_+$. Next, let

$$z = |x| + x, F(z) = |x| - x, \tag{4}$$

then, it can be obtained that:

$$z \geq 0,\ F(z) \geq 0,\ z^{\mathrm{T}} F(z) = 0,$$

which is called the modulus-base manipulation.

Further, consider the following non-smooth nonlinear system of equations

$$H(x) = 0, \tag{5}$$

where $H(x) = F(x + |x|) + x - |x|$,

Theorem 1: If $x \in \mathbb{R}^n$ is a solution to (5), then z dedined by $z = |x| + x$ is the solution to (1). In other words, solving (1) is equivalent to solving the non-smooth nonlinear system of equations (5).

Proof. Suppose that $x \in \mathbb{R}^n$ is a solution of (5); then, $H(x) = F(x + |x|) + x - |x| = 0$. Based on (4), z is the solution to (1). □

Definition 1: [18] Suppose $H : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. Assume Ω_H represents a set of points at which H is differentiable. Then Clarke's generalized Jacobian matrix of H in the neighborhood of x is expressed by:

$$\partial H(x) = co\left\{\lim H'(x_k) :\ x_k \to x,\ x_i \in \Omega_H\right\},$$

where $H'(x_k)$ is Jaccobian matrix of $H(x)$ at x_k, and co denotes the convex hull of a collection.

Clearly, $|x|$ is locally Lipschitz continuous. According to Definition 1, Clarke's generalized Jacobian matrix of $|x|$ is given by:

$$\partial |x| = diag\left(\partial |x_1|,\ \partial |x_2|,\ \cdots,\ \partial |x_n|\right),$$

where,

$$\partial |x_i| = \begin{cases} -1, & x_i < 0, \\ \alpha, & x_i = 0, \\ 1, & x_i > 0, \end{cases}$$

where $\alpha \in [-1,\ 1]$, and $i = 1,\ 2,\ \cdots,\ n$.
Then, $V_k \in \partial H_k$, and the Jacobian matrix of H at x_k is given by:

$$V_k = F'(z_k)\left(\partial |x_k| + I\right) + \left(I - \partial |x_k|\right). \tag{6}$$

Further, let $f(x) = \frac{1}{2}\left\|H(x)\right\|^2$; then,

$$\partial f(x) = \left\{V^{\mathrm{T}} H(x) :\ V \in \partial H(x)\right\}$$

For convenience, denote $\partial H_k = \partial H(x_k)$, $H_k = H(x_k)$, $f_k = f(x_k)$, and set $V_k \in \partial H_k$.

Definition 2: [19] For any non-zero vector $x \in \mathbb{R}^n$, if there is a component $x_j \neq 0$ such that $x_j (Mx)_j > 0$, then $M \in \mathbb{R}^{n\times n}$ is called a P-matrix.

Definition 3: [19] Suppose $D \subset \mathbb{R}^n$ is nonempty, $g = (g_1,\ g_2,\ \cdots, g_n)^{\mathrm{T}} :\ D \to \mathbb{R}^n$. If there is a constant $\alpha > 0$, for any $x,\ y \in D,\ x \neq y$, there is a subscript $k = k(x,\ y),\ 1 \leq k \leq n$ satisfying the condition of:

$$(x_k - y_k)\left(g_k(x) - g_k(y)\right) \geq \alpha \left\|x - y\right\|^2,$$

then, g is called a uniformly P-mapping on D.

Lemma 1: [20] If $H : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous at $x \in \mathbb{R}^n$, then the following statements hold:

1) $\partial H(x) \subset \mathbb{R}^n$ is a convex compact subset;
2) A set-valued mapping $x \to \partial H(x)$ is upper continuous, that is, for any $\omega > 0$, there is $\kappa > 0$ such that
$$\partial H(y) \subset \partial H(x) + \omega B_{n\times n}, \quad \forall y \in x + \kappa B_{n\times n}(0,\ 1),$$
where $B_{n\times n}(0,\ 1)$ is the unit ball in space $\mathbb{R}^{n\times n}$;
3) Let l_H denote the Lipschitz constant of $H(x)$ in the neighborhood of x; then, $\partial H(x) \subset l_H B_{n\times n}(0,\ 1)$.

Lemma 2: [8] Suppose $H :\ \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous; then, the following statements hold:

1) H is semismooth at x;
2) For any $V \in \partial H(x + h),\ h \to 0$,
$$\left\|Vh - H'(x; h)\right\| = o\left(\|h\|\right);$$
3) For any $V \in \partial H(x + h),\ h \to 0$,
$$\left\|H(x + h) - H(x) - Vh\right\| = o\left(\|h\|\right).$$

III. NSLM Method

In this section, an NSLM method is proposed. For convenience, the proposed method is denoted as the NSLM algorithm.

Assumption 1: The following statement holds:

1) The nonlinear mapping F is a uniformly continuous P-mapping;
2) Suppose x^* is a stationary point of $f(x) = \frac{1}{2}\left\|H(x)\right\|^2$. If $z^* = |x^*| + x^*$, then the Jaccobian matrix of F at z^*, $F'(z^*)$, is a P-matrix.

Algorithm NSLM

Input The starting point $x_0 \in \mathbb{R}^n$, $\mu > 0,\ \lambda > 0,\ \xi_0,\ \xi_1 > 0,\ \rho,\ \upsilon \in (0,\ 1),\ \delta \in [1,\ 2]$;

Step 1 Compute $H_k = H(x_k)$ and select $V_k \in \partial H_k$; set $k = 0$;

Step 2 If $\|V_k^{\mathrm{T}} F_k\| = 0$, stop; otherwise, set $x_{k,\ 0} = x_k,\ d_{k,\ 0} = d_k$ and compute

$$d_{k,\ i} = -\left(V_k^{\mathrm{T}} V_k + t_k I\right)^{-1} V_k^{\mathrm{T}} H_{k,\ i}, \tag{7}$$

with $x_{k,\ i} = x_{k,\ i-1} + d_{k,\ i-1}$, $t_k = \mu \|F_k\|^{\delta}$ to obtain $d_{k,\ i}$, where $i = 1,\ 2,\ \cdots,\ r-1$, $H_{k,\ i} = H(x_{k,\ i})$.

Set,

$$s_k = \sum_{i=0}^{r-1} d_{k,\ i}\ ;$$

Step 3 If it holds that:

$$\|H(x_{k,\ i})\| \le \rho \|H(x_k)\|, \tag{8}$$

then, set $\alpha_k = 1$ and go to Step 5;

Step 4 Set

$$s_k = \begin{cases} \sum_{i=0}^{r-1} d_{k,\ i} & H_k^{\mathrm{T}} V_k \sum_{i=0}^{r-1} d_{k,\ i} \le -\lambda, \\ d_{k,\ 0}, & \text{else.} \end{cases} \tag{9}$$

Solve $\alpha_k = \max\{1,\ \upsilon,\ \upsilon^2,\ \cdots\}$ with $\alpha_k = \upsilon^j,\ j \in N$ such that

$$\begin{aligned} \|H(x_k + \alpha_k s_k)\|^2 \le (1 + \omega_k)\|H_k\|^2 - \xi_0 \alpha_k^2 \|s_k\|^2 \\ - \xi_1 \alpha_k^2 \|H_k\|^2, \end{aligned} \tag{10}$$

where $\omega_k > 0$ satisfies

$$\sum_{k=0}^{\infty} \omega_k < \infty,\ \omega_k > 0; \tag{11}$$

Step 5 Let $x_{k+1} = x_k + \alpha_k s_k$, $k = k+1$, go to step 2.

IV. NSLM Method and Global Convergence

This section explains the global convergence of the proposed NSLM algorithm. First, the definitions and lemmas are introduced.

Lemma 3: [21] If $F(F_1,\ F_2,\ \cdots,\ F_n)$ is a uniformly continuous P-mapping, then the level set

$$L(x_0) = \{x \in \mathbb{R}^n :\ f(x) \le f(x_0)\} \tag{12}$$

is bounded.

Assumption 2: Suppose the level set $L(x_0)$ is bounded, and $H(x)$ is semismooth. For a given $x \in \partial H(x+d)$, $h \in \mathbb{R}^n$, $W \in \partial H(x+d)$, define

$$\Phi(x,\ h,\ W) = H(x) + Wh - H(x+d).$$

Then, let:

$$\begin{aligned} &\beta_1 = \max \|H(x)\|,\ x \in L(x_0), \\ &\beta_2 = \max \|V\|,\ V \in \partial H(x),\ x \in L(x_0), \\ &\beta_3 = \max \|V^{\mathrm{T}} V\|,\ V \in \partial H(x),\ x \in L(x_0), \end{aligned}$$

Lemma 4: Suppose Assumption 1 is satisfied. For any $\omega > 0$, if there exists a constant $\delta_0 > 0$ such that for all $x \in L(x_0)$ and $\|h\| \le \delta_0$ satisfying $x + h \in L(x_0)$, it holds that:

$$\max_{W \in \partial H(x+h)} \|\Phi(x,\ h,\ W)\| \le \min\left[\frac{\sqrt{\omega\|h\|}}{4\sqrt{2}},\ \frac{\omega}{32\eta(\omega)}\|h\|\right], \tag{13}$$

where $\eta(\omega) = \beta_1 + \beta_3^{-1}\beta_2\omega$.

Proof. See Lemma 3.2 in [8]. □

Lemma 5: Suppose Assumption 1 is satisfied. For any $\omega > 0$, there exists a constant $\delta_1 > 0$ such that for all $x,\ x+h \in L(x_0)$, $\|h\| \le \delta_1$ satisfying

$$\|W - V\| \le \min\left[\frac{\sqrt{\omega}}{4\sqrt{2}},\ \frac{\omega}{32\eta(\omega)}\right], \tag{14}$$

where $W \in \partial H(x+h),\ V \in \partial H(x)$, and $\eta(\omega) = \beta_1 + \beta_3^{-1}\beta_2\omega$.

Proof. See Lemma 1 (2) in this paper. □

Lemma 6: [20] Let $\{\alpha_k\}$ and $\{\upsilon_k\}$ be two positive sequences, where $\alpha_{k+1} \le (1 + \upsilon_k)\alpha_k + \upsilon_k$ and $\sum_{k=0}^{\infty} \upsilon_k < \infty$; then, $\{\alpha_k\}$ is convergent.

Lemma 7: Suppose $\{x_k\}$ is updated by the NSLM algorithm.

1) For any $x_k \in L(x_0)$, $k \ge 0$, $\{\|H_k\|\}$ is convergent;
2) For any $x_k \in L(x_0)$ |, $\{\|H_k\|\}$ is bounded, that is, there is a positive constant $M > 0$ satisfying
$$\|H_k\| \le M,\ \forall k \ge 0; \tag{15}$$
3) If $\|H(x_k + s_k)\| \le \rho \|H_k\|$ is satisfied for all $k > 0$; then, $\{\|H_k\|\}$ will converge to zero.

Proof. See Lemma 3.2 in [22] for the proof of (1) and Lemma 3.3 in [23] for the proof of (2). □

Theorem 2: Suppose Assumption 2 is satisfied and $\{x_k\}$ is obtained by the NSLM algorithm; then, the NSLM algorithm terminates after a finite number of iteration steps, or there is $\lim_{k\to\infty} \left\|V_k^{\mathrm{T}} H_k\right\| = 0,\ \forall V_k \in \partial H_k$.

Proof. Proceed with the proof by contradiction. Suppose that there exists $\tau > 0$ and a larger integer k' such that

$$\left\|V_k^{\mathrm{T}} H_k\right\| \ge \tau, \quad \forall k > k'. \tag{16}$$

Denote $K = \{k \in N_+ :\ \|H(x_k + s_k)\| \le \rho\|H_k\|\}$. If K is an infinite set of integers, then $\|H_k\| \to 0$, which contradicts (16). Therefore, K is a finite set of integers. Based on (7), (12), and (15), it can be obtained that $\sum_{k=0}^{\infty} \alpha_k^2 \|H_k\|^2 < \infty$; then, it holds that:

$$\lim_{k\to\infty} \alpha_k = 0. \tag{17}$$

Further, let $\bar{\alpha}_k = \alpha_k/\upsilon$. If $H_k^{\mathrm{T}} V_k \sum_{i=0}^{r-1} d_k^i \le -\lambda$, according to (7), it holds that:

$$\begin{aligned} &\|H(x_k + \bar{\alpha}_k s_k)\|^2 - \|H_k\|^2 \\ &> -\bar{\alpha}_k^2\left(\xi_0\|s_k\|^2 + \xi_1\|H_k\|^2\right) + \omega_k\|H_k\|^2 \\ &\ge -\bar{\alpha}_k^2\left(\xi_0\|s_k\|^2 + \xi_1\|H_k\|^2\right), \end{aligned}$$

which means,

$$\begin{aligned} &\bar{\alpha}_k^2\left(\xi_0\|s_k\|^2 + \xi_1\|H_k\|^2\right) \\ &\ge -\left(\|H(x_k + \bar{\alpha}_k s_k)\|^2 - \|H_k\|^2\right) \\ &\ge -2H_k^{\mathrm{T}}\left(H(x_k + \bar{\alpha}_k s_k) - H_k\right) \\ &\quad - \|H(x_k + \bar{\alpha}_k s_k) - H_k\|^2. \end{aligned}$$

Consider the right side of the above equation; one element is

$$\begin{aligned} & H_k^{\mathrm{T}}\left(H\left(x_k+\bar{\alpha}_k s_k\right)-H_k\right) \\ & =H_k^{\mathrm{T}}\left(W_k \bar{\alpha}_k s_k-\Phi\left(x,\ \bar{\alpha}_k s_k,\ W\right)\right) \\ & =H_k^{\mathrm{T}} V_k \bar{\alpha}_k s_k+H_k^{\mathrm{T}}\left(W_k-V_k\right) \bar{\alpha}_k s_k \\ & \quad -H_k^{\mathrm{T}} \Phi\left(x,\ \bar{\alpha}_k s_k,\ W\right) \\ & \leq-\bar{\alpha}_k \lambda+\frac{\omega}{16 \eta(\omega)} \beta_1 \bar{\alpha}_k\left\|s_k\right\|, \end{aligned}$$

and another element is

$$\begin{aligned} & \left\|H\left(x_k+\bar{\alpha}_k s_k\right)-H_k\right\|^2 \\ & =\left\|W \bar{\alpha}_k s_k-\Phi\left(x,\ \bar{\alpha}_k s_k,\ W\right)\right\|^2 \\ & \leq\left\|V \bar{\alpha}_k s_k\right\|^2+\left\|(W-V) \bar{\alpha}_k s_k\right\|^2 \\ & +\left\|\Phi\left(x,\ \bar{\alpha}_k s_k,\ W\right)\right\|^2 \\ & \quad +2\left\|V \bar{\alpha}_k s_k\right\|\left\|(W-V) \bar{\alpha}_k s_k\right\| \\ & \quad +2\left\|V \bar{\alpha}_k s_k\right\|\left\|\Phi\left(x,\ \bar{\alpha}_k s_k,\ W\right)\right\| \\ & \quad +2\left\|(W-V) \bar{\alpha}_k s_k\right\|\left\|\Phi\left(x,\ \bar{\alpha}_k s_k,\ W\right)\right\| \\ & \leq\left(\beta_2^2+\left(\frac{\omega}{16 \eta(\omega)}\right)^2+\frac{\beta_2 \omega}{8 \eta(\omega)}\right)\left\|\bar{\alpha}_k s_k\right\|^2 \\ & =C \bar{\alpha}_k^2\left\|s_k\right\|^2, \end{aligned}$$

where $C=\beta_2^2+\left(\frac{\omega}{16 \eta(\omega)}\right)^2+\frac{\beta_2 \omega}{8 \eta(\omega)}$.
Combining the above two elements yields:

$$\begin{aligned} & \bar{\alpha}_k^2\left(\xi_0\left\|s_k\right\|^2+\xi_1\left\|H_k\right\|^2\right) \\ & \geq-\left(\left\|H\left(x_k+\bar{\alpha}_k s_k\right)\right\|^2-\left\|H_k\right\|^2\right) \\ & \geq-2 H_k^{\mathrm{T}}\left(H\left(x_k+\bar{\alpha}_k s_k\right)-H_k\right) \\ & \quad-\left\|H\left(x_k+\bar{\alpha}_k s_k\right)-H_k\right\|^2 \\ & \geq 2 \bar{\alpha}_k \lambda-\frac{\omega}{8 \eta(\omega)} \beta_1 \bar{\alpha}_k\left\|s_k\right\|-C \bar{\alpha}_k^2\left\|s_k\right\|^2, \end{aligned}$$

which means that:

$$\begin{aligned} & \left(2 \lambda-\frac{\omega}{8 \eta(\omega)} \beta_1\left\|s_k\right\|\right) \bar{\alpha}_k \\ & \leq\left(C\left\|s_k\right\|^2+\xi_0\left\|s_k\right\|^2+\xi_1\left\|H_k\right\|^2\right) \bar{\alpha}_k^2; \end{aligned}$$

then, it holds that:

$$\bar{\alpha}_k \geq \frac{\left(C\left\|s_k\right\|^2+\xi_0\left\|s_k\right\|^2+\xi_1\left\|H_k\right\|^2\right) \bar{\alpha}_k^2}{\left(2 \lambda-\frac{\omega}{8 \eta(\omega)} \beta_1\left\|s_k\right\|\right)}. \tag{18}$$

Similarly, if $H_k^{\mathrm{T}} V_k \sum_{i=0}^{r-1} d_{k,\ i}>-\lambda$, there exists

$$\begin{aligned} & \bar{\alpha}_k^2\left(\xi_0\left\|d_{k, 0}\right\|^2+\xi_1\left\|H_k\right\|^2\right) \\ & \geq-\left(\left\|H\left(x_k+\bar{\alpha}_k d_{k, 0}\right)\right\|^2-\left\|H_k\right\|^2\right) \\ & \geq-2 H_k^{\mathrm{T}}\left(H\left(x_k+\bar{\alpha}_k d_{k, 0}\right)-H_k\right) \\ & \quad-\left\|H\left(x_k+\bar{\alpha}_k d_{k, 0}\right)-H_k\right\|^2 \\ & \geq-2 H_k^{\mathrm{T}} V_k \bar{\alpha}_k d_{k, 0}-2 H_k^{\mathrm{T}}\left(W_k-V_k\right) \bar{\alpha}_k d_{k, 0} \\ & \quad+2 H_k^{\mathrm{T}} \Phi\left(x,\ \bar{\alpha}_k d_{k, 0},\ W\right)-C \bar{\alpha}_k^2\left\|d_{k, 0}\right\|^2 \\ & \geq-2 \bar{\alpha}_k H_k^{\mathrm{T}} V_k d_{k,\ 0}-\frac{\omega}{8 \eta(\omega)} \beta_1 \bar{\alpha}_k\left\|d_{k, 0}\right\|-C \bar{\alpha}_k^2\left\|d_{k, 0}\right\|^2 \\ & =2 \bar{\alpha}_k d_{k,\ 0}\left(V_k^{\mathrm{T}} V_k+t_k I\right) d_{k,\ 0}-\frac{\omega}{8 \eta(\omega)} \beta_1 \bar{\alpha}_k\left\|d_{k, 0}\right\| \\ & \quad-C \bar{\alpha}_k^2\left\|d_{k, 0}\right\| \end{aligned}$$

and then, it holds that:

$$\begin{aligned} & \bar{\alpha}_k\left(\xi_0\left\|d_{k, 0}\right\|^2+\xi_1\left\|H_k\right\|^2+C\left\|d_{k, 0}\right\|^2\right) \\ & \geq 2 d_{k,\ 0}\left(V_k^{\mathrm{T}} V_k+t_k I\right) d_{k,\ 0}-\frac{\omega}{8 \eta(\omega)} \beta_1\left\|d_{k, 0}\right\|, \end{aligned}$$

that is,

$$\bar{\alpha}_k \geq \frac{2 t_k d_{k,\ 0}^2-\frac{\omega}{16 \eta(\omega)} \beta_1\left\|d_k\right\|}{\left(C+\xi_0\right)\left\|d_k\right\|^2+\xi_1\left\|H_k\right\|^2}. \tag{19}$$

Further, suppose the singular value decomposition (SVD) of V_k is $V_k=P_k \Sigma_k Q_k$, where P_k and Q_k are orthogonal matrices, and $\Sigma_k=\operatorname{diag}\left(\sigma_1, \sigma_2, \cdots, \sigma_n\right)$, $\sigma_i \geq 0$, $i=1,\ 2,\ \cdots,\ n$ is the singular value of v_k. Then, it can be written that:

$$\begin{aligned} \left\|\left(V_k^{\mathrm{T}} V_k+t_k I\right)^{-1}\right\| & =\left\|Q_k\left(\Sigma_k^2+t_k I\right)^{-1} Q_k^{\mathrm{T}}\right\| \\ & =\left\|\left(\Sigma_k^2+t_k I\right)^{-1}\right\| \\ & =\max _{i=\{1,\ 2,\ \cdots,\ n\}}\left(\sigma_i^2+t_k\right)^{-1} \leq t_k^{-1}, \end{aligned}$$

which means that:

$$\begin{aligned} \left\|d_{k,\ 0}\right\| & =\left\|-\left(V_k^{\mathrm{T}} V_k+t_k I\right)^{-1} V_k H_k\right\| \\ & \leq\left\|\left(V_k^{\mathrm{T}} V_k+t_k I\right)^{-1}\right\|\left\|V_k\right\|\left\|H_k\right\| \leq \frac{\beta_2}{\mu t_k^{\frac{\delta-1}{\delta}}} \end{aligned}$$

and,

$$\begin{aligned} & \left\|d_{k,\ i}\right\|=\left\|-\left(V_k^{\mathrm{T}} V_k+t_k I\right)^{-1} V_k H_{k,\ i}\right\| \\ & \leq \sum_{j=1}^i\left\|\left(V_k^{\mathrm{T}} V_k+t_k I\right)^{-1} V_k\left(H_{k,\ j}-H_{k,\ j-1}\right)\right\| \\ & \quad+\left\|\left(V_k^{\mathrm{T}} V_k+t_k I\right)^{-1} V_k H_{k,\ 0}\right\| \\ & \leq \beta_2 t_k^{-1} \sum_{j=1}^i\left\|d_{k,\ j-1}\right\|+\left\|d_{k,\ 0}\right\|, \end{aligned}$$

where $i=1,\ 2,\ \cdots,\ r-1$.
Thus, for any k large enough, it holds that:

$$\left\|d_{k,\ i}\right\| \leq\left\|d_{k,\ 0}\right\| \sum_{j=0}^i\left(\beta_2 t_k^{-1}\right)^j \leq C_1\left\|d_{k,\ 0}\right\|,$$

where C_1 is a positive constant.
If $\liminf_{k \rightarrow \infty}\left\|d_{k,\ 0}\right\|=0$, then $\liminf_{k \rightarrow \infty}\left\|{V_k}^{\mathrm{T}} H_k\right\|=\liminf_{k \rightarrow \infty}\left\|\left(V_k^{\mathrm{T}} V_k+t_k I\right) d_{k,\ 0}\right\|=0$. This contradicts (16), so there exists a constant $\tau_1>0$ such that $\lim_{k \rightarrow \infty}\left\|d_{k,\ 0}\right\| \geq \tau_1$. Due to the arbitrariness of ω, (18) and (19), $\alpha_k>0$ has a non-zero lower bound if an appropriate $\mu,\ \lambda$ value is selected, which contradicts (17). Thus, the assumption is false. □

V. Numerical Results

To assess the effectiveness of the improved algorithm, the proposed NSLM algorithm was evaluated on five numerical problems constructed based on [24] and compared with Algorithm 1 of [16].
Let $g(z)=0$ be a differentiable system of nonlinear

equations [24] and denote $z^* = (1,\ 0,\ 1,\ 0,\ \cdots)^T \in \mathbb{R}^n$. For all $i = 1,\ \cdots,\ n$, set

$$G_i(z) = \begin{cases} g_i(z) - g_i(z^*), & \text{if } i \text{ odd or } i > \frac{1}{2}n, \\ g_i(z) - g_i(z^*) + 1, & \text{else }. \end{cases}$$

Obviously, z^* is a solution to the corresponding nonlinear complementarity problems.

The proposed algorithm was developed in MatlabR2021a and ran on a PC with the 11th generation Intel(R)Core(TM)i5-11300H@3. 10GHz3. 11GHz RAM16G). The stopping criterion was $\|H(x_k)\| \le 10^{-5}$ or the number of iteration exceeded 100. When the iteration number exceeded 100, the test was regarded as "failed" and denoted by "F". It was set that: $\alpha = 0,\ r = 4$, and $\omega_k = 0.01^k/10$, and V_k was computed by (2.3). The parameters of the NSLM algorithm were set as follows: $\mu_0 = 1e-5,\ \delta = 1.8,\ \lambda = 1e-5,\ \xi_0 = \xi_1 = 0.05,\ \rho = 0.9,\ \upsilon = 0.5$; x_0 is selected according to the initial point suggested in [24]. The parameters of Algorithm 1 were consistent with those used in [16]. The algorithms' numerical performances were tested in solving the problems with 600, 1,500, 5,000, and 10,000 dimensions, in turn. Further details of the symbol descriptions are provided in Table I. The numerical results are presented in Table II.

TABLE I: Description of Symbols

Symbol	Symbol description
Dim	The dimension of a function
NI	The number of iterations
NF	The function calculations
NJ	The function's Jacobian calculations
Tcpu	The running time of the problem, expressed in seconds
Problem1	Extended Rosenbrock function
Problem2	Extended Powell singular function
Problem3	Extended Cragg and Levy function
Problem4	Broyden banded problem
Problem5	Broyden tridiagonal problem

Based on the numerical results presented in Table II, the NI, NF, and $\|H_k\|$ values of the proposed NSLM were overall lower than those of Algorithm 1. The NJ and Tcpu values of the proposed NSLM were significantly lower than those of Algorithm 1. This difference was even more obvious for n=5,000 and n=10,000. Generally, the algorithm mainly spent the most time solving the Jacobian matrices. Therefore, although the proposed NSLM might require performing more function calculations and iterations to converge than Algorithm 1, its total computational time is shorter, especially for high-dimensional test problems.

VI. Conclusions

Translating the considered problems into a system of nonlinear equations has been a common strategy for solving the NCPs. In this paper, modulus-based manipulation is used to complete the aforementioned conversion. A Levenberg–Marquardt method with the standard r-order Amijio line search technique is developed to solve the resulting nonsmooth equations. Numerical results demonstrate that, in comparison to Algorithm 1, the proposed NSLM method has fewer calculations of Jacobian matrices and a shorter running time. The proposed NSLM method can be considered competitive in solving large-scale nonlinear complementarity problems with the existing methods. However, in practical applications, the efficiency of the proposed NSLM method depends on the parameters' values, which could be challenging to select appropriately.

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TABLE II: Comparison Results of the Proposed NSLM Algorithm and Algorithm 1

		NSLM algorithm					Algorithm 1				
Function	Dim	NI	NF	NJ	$\|H_k\|$	Tcpu	NI	NF	NJ	$\|H_k\|$	Tcpu
Problem 1	600	8	10	2	6.06E-06	0.10	20	27	7	3.13E-08	0.28
	1,500	9	12	3	4.30E-15	0.82	21	28	7	2.75E-09	2.12
	5,000	9	12	3	9.08E-11	24.55	24	32	8	1.83E-04	67.73
	10,000	9	12	3	1.09E-07	174.23	26	34	8	3.51E-04	508.55
Problem 2	600	F	F	F	1.35E+04	5.68	52	76	24	2.66E-05	0.89
	1,500	F	F	F	7.06E+01	42.87	25	35	10	6.74E-06	2.88
	5,000	33	43	8	1.19E-06	78.54	55	81	26	5.28E-04	5,999.26
	10,000	10	13	3	1.25E-06	220.17	66	96	30	1.06E-04	2,237.64
Problem 3	600	21	28	6	5.49E-07	0.24	69	95	26	1.08E-04	1.01
	1,500	17	23	5	6.20E-12	2.07	35	51	11	3.42E-04	10.56
	5,000	16	20	4	9.71E-07	30.82	72	99	27	1.20E-07	231.67
	10,000	15	19	4	3.94E-06	309.67	51	59	15	7.74E-04	2,944.54
Problem 4	600	40	55	10	1.96E-06	0.41	51	73	21	2.35E-04	0.82
	1,500	53	71	13	6.29E-06	5.39	33	49	15	6.28E-04	3.78
	5,000	34	46	9	2.55E-07	69.16	48	69	20	9.08E-04	152.53
	10,000	38	50	10	4.82E-06	546.49	31	45	13	1.06E-04	690.83
Problem 5	600	26	34	7	4.53E-10	0.27	24	34	9	5.35E-06	0.43
	1,500	23	30	6	2.90E-06	2.53	24	34	9	8.82E-04	2.76
	5,000	20	25	5	2.87E-06	39.77	23	32	9	3.59E-06	66.19
	10,000	9	12	3	4.48E-07	168.18	25	35	10	7.85E-07	517.51

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