Local Lie Triple Derivations on Triangular Algebras

Jinhong Zhuang, Yanping Chen

Abstract—**In this paper, by using the structure properties of algebras and the technique of matrix operation, we investigate the structure of local Lie triple derivations on triangular algebras. Under some mild conditions, we prove that each local Lie triple derivation of triangular algebras can be expressed as the sum of a derivation and a linear map from the triangular algebra to its center that vanishes at Lie triple products. Finally, we apply the main result to the problem of characterizing local Lie triple derivations on an upper triangular matrix algebra.**

Index Terms—**Triangular algebra, Derivation, Local Lie triple derivation.**

I. Introduction

Let $\mathcal{U}$ be an unital algebra over a commutative ring R, $Z(\mathcal{U}) = \{u \in \mathcal{U} | uv = vu, \forall v \in \mathcal{U}\}$ be the center of $\mathcal{U}$. Given $u, v, w \in \mathcal{U}$, $[u, v] = uv - vu$ indicates the Lie product of u and v; the notation $[[u, v], w]$ represents the Lie triple product of u, v and w. Let $d : \mathcal{U} \to \mathcal{U}$ is an R-linear map. d is called a derivation of $\mathcal{U}$ if

$$d(uv) = d(u)v + ud(v), \forall u, v \in \mathcal{U}.$$

d is called a Lie derivation of $\mathcal{U}$ if

$$d([u, v]) = [d(u), v] + [u, d(v)], \forall u, v \in \mathcal{U}.$$

d is called a Lie triple derivation of $\mathcal{U}$ if $\forall u, v \in \mathcal{U}$,

$$d([[u, v], w]) = [[d(u), v], w] + [[u, d(v)], w] + [[u, v], d(w)].$$

The derivation can be understood as an algebraic extension of differentiation, and it holds significance in the analysis of rings or algebraic structures. The renowned Posner's theorem [1] effectively illustrates the strong correlation between derivations on prime rings and the commutativity of those rings. Subsequently, many scholars studied the structures of derivations on different algebras or rings [2–5,17]. The investigation of local derivations originated from the groundbreaking research conducted by R. Kadison, D. Larson and A. Sourour [6,7]. In [6], Kadison initially introduced the concept of local derivation. Recall that an R-linear mapping d of $\mathcal{U}$ is called a local derivation if, for any $u \in \mathcal{U}$, there exists a derivation d_u of $\mathcal{U}$ that depends on u satisfying $d(u) = d_u(u)$. He proved that each local continuous derivation from a von Neumann algebra to its dual bimodule can be categorized as a derivation. In a subsequent study, Larson and Sourour [7] conducted an examination of local derivations on standard operator algebras $B(X)$. Their findings demonstrated that each local derivation of $B(X)$ is a derivation. Johnson [8] considered this problem in C^*-algebra and obtained the same result. Recently, Brešar [9] extended this conclusion to the algebra generated by all their idempotents.

Inspired by the study of local derivation, many researchers have studied local Lie derivation. Recall that an R-linear mapping d of $\mathcal{U}$ is called a local Lie derivation if, for any $u \in \mathcal{U}$, there exists a Lie derivation d_u of $\mathcal{U}$ that depends on u satisfying $d(u) = d_u(u)$. In [10], Chen et al. proved that each local Lie derivation of operator algebras on a Banach space may be classified as a Lie derivation. After that, the similar conclusion has been obtained for nest algebras on Hilbert spaces[11], for factor von Neumann algebras [12] and for one particular kind of triangular algebras [13]. Liu studied local Lie derivations on generalized matrix algebras in [14] and shown that every local Lie derivation of generalized matrix algebras is standard when certain conditions are met. This means that each local Lie derivation takes the form $d + \tau$, in which d is a derivation, τ is the central-valued mapping.

More generally, we say that d is a local Lie triple derivation of $\mathcal{U}$ if, for any $u \in \mathcal{U}$, there exists a Lie triple derivation d_u of $\mathcal{U}$ that depends on u satisfying $d(u) = d_u(u)$. Xiao and Wei [15] demonstrated that each Lie triple derivation on triangular algebras is standard. It is natural to question whether or not each local Lie triple derivation on triangular algebras is standard. Based on these results, we investigate local Lie triple derivations in triangular algebra and demonstrate that they can be expressed in a standard form under certain conditions.

II. Preliminaries

This section will introduce the fundamental concepts and lemmas used in the following sections.

Assume that $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring R, $\mathcal{M}$ be a nonzero $(\mathcal{A}, \mathcal{B})$-bimodule. Under matrix-like addition and multiplication, the set

$$\mathcal{T} = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

forms an algebra on R . This is referred to triangular algebra. This algebra was first introduced by Chueung [16]. Furthermore, we assume that $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ is faithful, that is, if $a\mathcal{M} = \{0\}$ implies $a = 0$ for every $a \in \mathcal{A}$ and if $\mathcal{M}b = \{0\}$ implies $b = 0$ for every $b \in \mathcal{B}$.

Define the projections $\pi_{\mathcal{A}} : \mathcal{T} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{T} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a, \ \pi_{\mathcal{B}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b.$$

In the last section of the paper, the next lemma will be used.

Manuscript received August 28, 2023; revised January 19, 2024. This work was supported by the Natural Science Foundation of Fujian Province (2021J011252).

Jinhong Zhuang is a lecturer at the College of Information Engineering, Fujian Business University, Fuzhou, Fujian 350102, China. (e-mail: zhuangjh8@163.com).

Yanping Chen is a professor at the College of Information Engineering, Fujian Business University, Fuzhou, Fujian 350102, China. (e-mail: cypfjfz@163.com).

Lemma 2.1.[15] Let $\mathcal{T}$ be a triangular algebra. Then

$$Z(\mathcal{T})=\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\middle|\, am=mb,\forall m\in\mathcal{M}\right\}.$$

Lemma 2.2.[14] Let $\mathcal{T}$ be a 2-torsion-free triangular algebra (i.e. $\forall x\in\mathcal{T}$, $2x=0$ implies $x=0$). Suppose that

$$\pi_{\mathcal{A}}(Z(\mathcal{T}))=Z(\mathcal{A}),\ \pi_{\mathcal{B}}(Z(\mathcal{T}))=Z(\mathcal{B}) \tag{C1}$$

and

$$\begin{aligned}&Z(\mathcal{A})=\{a|[[a,x],y]=0,\forall x,y\in\mathcal{A}\},\ or\\ &Z(\mathcal{B})=\{b|[[b,x],y]=0,\forall x,y\in\mathcal{B}\}\end{aligned} \tag{C2}$$

hold. If Δ is a Lie triple derivation of $\mathcal{T}$, then Δ is of standard form. More precisely, there exists a derivation d of $\mathcal{T}$ and a linear map $\tau:\mathcal{T}\to Z(\mathcal{T})$ that vanishes at all Lie triple products of $\mathcal{T}$ such that $\Delta=d+\tau$.

Throughout this article, $J(\mathcal{A})$ be the subalgebra of $\mathcal{A}$, which is generated by all idempotents of $\mathcal{A}$, $1_{\mathcal{A}}$ be the identity of $\mathcal{A}$ and $1_{\mathcal{B}}$ be the identity of $\mathcal{B}$. Set $p_1=\begin{pmatrix}1_{\mathcal{A}} & 0\\ 0 & 0\end{pmatrix}$, $p_2=\begin{pmatrix}0 & 0\\ 0 & 1_{\mathcal{B}}\end{pmatrix}$, then $I=p_1+p_2$ is the identity of $\mathcal{T}$.

Set $\mathcal{T}_{11}=p_1\mathcal{T}p_1$, $\mathcal{T}_{12}=p_1\mathcal{T}p_2$ and $\mathcal{T}_{22}=p_2\mathcal{T}p_2$. Then we can obtain $\mathcal{T}=\mathcal{T}_{11}\dotplus\mathcal{T}_{12}\dotplus\mathcal{T}_{22}$, where $\mathcal{T}_{11}$ is a subalgebra of $\mathcal{T}$ isomorphic to $\mathcal{A}$, $\mathcal{T}_{22}$ is a subalgebra of $\mathcal{T}$ isomorphic to $\mathcal{B}$ and $\mathcal{T}_{12}$ is a $(\mathcal{T}_{11},\mathcal{T}_{22})$-bimodule isomorphic to the $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$. We can also see that $\pi_{\mathcal{A}}(Z(\mathcal{T}))$ is isomorphic to $Z(\mathcal{T}_{11})$ and $\pi_{\mathcal{B}}(Z(\mathcal{T}))$ is isomorphic to $Z(\mathcal{T}_{22})$.

III. Main results

Theorem 3.1. Let $\mathcal{T}$ be a 2-torsion-free triangular algebra, Δ be a local Lie triple derivation of $\mathcal{T}$. Assume (C1) and (C2) hold, further suppose

$$\mathcal{A}=J(\mathcal{A})\ and\ \mathcal{B}=J(\mathcal{B}) \tag{C3}$$

holds, then $\Delta=d+h$, in which $d:\mathcal{T}\to\mathcal{T}$ is a derivation, $h:\mathcal{T}\to Z(\mathcal{T})$ is a linear map that vanishes at Lie triple products of $\mathcal{T}$.

We will use a series of lemmas to prove Theorem 3.1. Throughout the discussion, set Δ be a local Lie triple derivation of $\mathcal{T}$. For every $x\in\mathcal{T}$, the notation Δ_x represents a Lie triple derivation of $\mathcal{T}$ satisfying $\Delta(x)=\Delta_x(x)$. To enhance convenience, we'll use the notation x_{ij} to represent the element in the set $\mathcal{T}_{ij}$ which corresponds to an element in $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{M}$.

Lemma 3.2. $p_1\Delta(p_1)p_1+p_2\Delta(p_1)p_2\in Z(\mathcal{T})$.

Proof. For every $x_{12}\in\mathcal{T}$, there exists a Lie triple derivation Δ_{p_1} of $\mathcal{T}$ satisfying

$$\begin{aligned}\Delta_{p_1}(x_{12})&=\Delta_{p_1}([[x_{12},p_1],p_1])\\ &=[[\Delta_{p_1}(x_{12}),p_1],p_1]+[[x_{12},\Delta(p_1)],p_1]\\ &+[[x_{12},p_1],\Delta(p_1)]\\ &=p_1\Delta_{p_1}(x_{12})p_2\\ &-2(x_{12}p_2\Delta(p_1)p_2-p_1\Delta(p_1)p_1x_{12}).\end{aligned}$$

Multiply x on the left side of the equality and y on the right side, we have

$$p_1\Delta(p_1)p_1x_{12}=x_{12}p_2\Delta(p_1)p_2.$$

By Lemma 2.1, $p_1\Delta(p_1)p_1+p_2\Delta(p_1)p_2\in Z(\mathcal{T})$. ☐

Define the linear map $\delta:\mathcal{T}\to\mathcal{T}$ as $\delta(x)=\Delta(x)-[x,p_1\Delta(p_1)p_2]$. It's not difficult to confirm that δ is a local Lie triple derivation and $\delta(p_1)=p_1\Delta(p_1)p_1+p_2\Delta(p_1)p_2$. According to Lemma 3.2, we obtain $\delta(p_1)\in Z(\mathcal{T})$.

Lemma 3.3. $\delta(x_{12})\in\mathcal{T}_{12}$ for any $x_{12}\in\mathcal{T}_{12}$.

Proof. For every $x_{12}\in\mathcal{T}_{12}$, we have

$$\begin{aligned}\delta(x_{12})&=\delta_{x_{12}}(x_{12})=\delta_{x_{12}}([[x_{12},p_1],p_1])\\ &=[[\delta(x_{12}),p_1],p_1]+[[x_{12},\delta_{x_{12}}(p_1)],p_1]\\ &+[[x_{12},p_1],\delta_{x_{12}}(p_1)]\\ &=p_1\delta(x_{12})p_2\\ &-2(x_{12}p_2\delta_{x_{12}}(p_1)p_2-p_1\delta_{x_{12}}(p_1)p_1x_{12}).\end{aligned}$$

Multiply x on the left side of the equality and y on the right side, we get

$$2(x_{12}p_2\delta_{x_{12}}(p_1)p_2-p_1\delta_{x_{12}}(p_1)p_1x_{12})=0.$$

Thus $\delta(x_{12})=p_1\delta(x_{12})p_2\in\mathcal{T}_{12}$. ☐

Lemma 3.4. Assume that e and f be idempotents of $\mathcal{T}$. Let $e^{\perp}$ denotes $I-e$, $f^{\perp}$ denotes $I-f$. Then for any $x\in\mathcal{T}$, there exist linear mappings $\tau_1,\tau_2,\tau_3,\tau_4:\mathcal{T}\to Z(\mathcal{T})$ that vanishe at Lie triple products such that

$$\begin{aligned}\delta(exf)&=\delta(ex)f-e\delta(x)f+e\delta(xf)+e^{\perp}\tau_1(exf)f^{\perp}\\ &-e\tau_2(e^{\perp}xf)f^{\perp}+e\tau_3(e^{\perp}xf^{\perp})f-e^{\perp}\tau_4(exf^{\perp})f.\end{aligned}$$

Proof. For every idempotents $e,f\in\mathcal{T}$ and $x\in\mathcal{T}$, by Lemma 2.2, there exist derivations $d_i:\mathcal{T}\to\mathcal{T}(i=1,2,3,4)$ and linear mappings $\tau_i:\mathcal{T}\to Z(\mathcal{T})(i=1,2,3,4)$ that vanishe at Lie triple products such that

$$\delta(exf)=\delta_{exf}(exf)=d_1(exf)+\tau_1(exf). \tag{1}$$

$$\delta(e^{\perp}xf)=\delta_{e^{\perp}xf}(e^{\perp}xf)=d_2(e^{\perp}xf)+\tau_2(e^{\perp}xf). \tag{2}$$

$$\begin{aligned}\delta(e^{\perp}xf^{\perp})&=\delta_{e^{\perp}xf^{\perp}}(e^{\perp}xf^{\perp})\\ &=d_3(e^{\perp}xf^{\perp})+\tau_3(e^{\perp}xf^{\perp}).\end{aligned} \tag{3}$$

$$\delta(exf^{\perp})=\delta_{exf^{\perp}}(exf^{\perp})=d_4(exf^{\perp})+\tau_4(exf^{\perp}). \tag{4}$$

Since $d_1(exf)=d_1(e)xf+ed_1(x)f+exd_1(f)$ and the fact $e^{\perp}e=0, ff^{\perp}=0$, then $e^{\perp}d_1(exf)f^{\perp}=0$. By the equality (1),we have

$$e^{\perp}\delta(exf)f^{\perp}=e^{\perp}\tau_1(exf)f^{\perp}.$$

Similarly, from the equality (2)-(4), we get

$$e\delta(e^{\perp}xf)f^{\perp}=e\tau_2(e^{\perp}xf)f^{\perp},$$

$$e\delta(e^{\perp}xf^{\perp})f=e\tau_3(e^{\perp}xf^{\perp})f,$$

$$e^{\perp}\delta(exf^{\perp})f=e^{\perp}\tau_4(exf^{\perp})f,$$

then

$$\begin{aligned}e^{\perp}\delta(exf)&=e^{\perp}\delta(exf)f+e^{\perp}\delta(exf)f^{\perp}\\ &=e^{\perp}\delta(ex)f-e^{\perp}\delta(exf^{\perp})f+e^{\perp}\delta(exf)f^{\perp}\\ &=e^{\perp}\delta(ex)f-e^{\perp}\tau_4(exf^{\perp})f+e^{\perp}\tau_1(exf)f^{\perp},\end{aligned}$$

and

$$\begin{aligned}e\delta(e^{\perp}xf)&=e\delta(e^{\perp}xf)f+e\delta(e^{\perp}xf)f^{\perp}\\ &=e\delta(e^{\perp}x)f-e\delta(e^{\perp}xf^{\perp})f+e\delta(e^{\perp}xf)f^{\perp}\\ &=e\delta(e^{\perp}x)f-e\tau_3(e^{\perp}xf^{\perp})f+e\tau_2(e^{\perp}xf)f^{\perp}\\ &=e\delta(x)f-e\delta(ex)f\\ &-e\tau_3(e^{\perp}xf^{\perp})f+e\tau_2(e^{\perp}xf)f^{\perp}.\end{aligned}$$

From $e^{\perp}\delta(ex)f + e\delta(ex)f = \delta(ex)f$, we have

$$\begin{aligned}\delta(exf) &= e^{\perp}\delta(exf) + e\delta(exf)\\ &= e^{\perp}\delta(exf) + e\delta(xf) - e\delta(e^{\perp}xf)\\ &= \delta(ex)f - e\delta(x)f + e\delta(xf) + e^{\perp}\tau_1(exf)f^{\perp}\\ &\quad - e\tau_2(e^{\perp}xf)f^{\perp} + e\tau_3(e^{\perp}xf^{\perp})f - e^{\perp}\tau_4(exf^{\perp})f.\end{aligned}$$

□

Lemma 3.5. For every $x_{11} \in \mathcal{T}_{11}$, $x_{22} \in \mathcal{T}_{22}$, we get
(1)$\delta(x_{11}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$ and $p_2\delta(x_{11})p_2 \in Z(\mathcal{T}_{22})$;
(2)$\delta(x_{22}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$ and $p_1\delta(x_{22})p_1 \in Z(\mathcal{T}_{11})$.

Proof. Let $x_{11} \in \mathcal{T}_{11}$ and e_1 be any idempotent of $\mathcal{T}_{11}$. Putting $e = e_1, x = x_{11}$ and $f = p_1$ in Lemma 3.4, from the facts $exf^{\perp} = e_1x_{11}p_2 = 0$ and $e^{\perp}xf^{\perp} = e_1^{\perp}x_{11}p_2 = 0$, we have $\tau_3(exf^{\perp}) = 0$, $\tau_4(e^{\perp}xf^{\perp}) = 0$. Then

$$\begin{aligned}\delta(e_1x_{11}) &= \delta(e_1x_{11})p_1 - e_1\delta(x_{11})p_1 + e_1\delta(x_{11})\\ &\quad + e_1^{\perp}\tau_1(e_1x_{11})p_2 - e_1\tau_2(e_1^{\perp}x_{11})p_2\\ &= \delta(e_1x_{11})p_1 + e_1\delta(x_{11})p_2 + \tau_1(e_1x_{11})p_2,\end{aligned}$$

where we have used the facts $e_1\tau_1(e_1x_{11})p_2 = 0$ and $e_1\tau_2(e_1^{\perp}x_{11})p_2 = 0$ in the second equality.
This implies that

$$p_1\delta(e_1x_{11})p_2 = e_1\delta(x_{11})p_2 \tag{5}$$

and

$$p_2\delta(e_1x_{11})p_2 = p_2\tau_1(e_1x_{11})p_2 \in p_2Z(\mathcal{T})p_2 = Z(\mathcal{T}_{22}).$$

In particular,

$$p_2\delta(x_{11})p_2 = p_2\tau_1(x_{11})p_2 \in Z(\mathcal{T}_{22}). \tag{6}$$

By the equality (5) and the fact $p_1\delta(p_1)p_2 = 0$, we have

$$\begin{aligned}p_1\delta(e_1e_2\cdots e_n)p_2 &= p_1\delta(e_1e_2\cdots e_np_1)p_2\\ &= e_1\delta(e_2\cdots e_np_1)p_2\\ &= e_1e_2\cdots e_n\cdot p_1\delta(p_1)p_2\\ &= 0\end{aligned}$$

for any idempotents $e_1, e_2, \cdots, e_n \in \mathcal{T}_{11}$.

From the fact $\mathcal{A} = J(\mathcal{A})$, we can represent x_{11} as a linear combination of idempotents. Thus, $p_1\delta(x_{11})p_2 = 0$ for all $x_{11} \in \mathcal{T}_{11}$. So

$$\delta(x_{11}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22} \text{ and } p_2\delta(x_{11})p_2 \in Z(\mathcal{T}_{22}).$$

With the similar argument, we can prove that $\delta(x_{22}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$ and

$$p_1\delta(x_{22})p_1 = p_1\tau_2(x_{22})p_1 \in Z(\mathcal{T}_{11}) \tag{7}$$

for all $x_{22} \in \mathcal{T}_{22}$. □

For any $x_{11} \in \mathcal{T}_{11}$ and $x_{22} \in \mathcal{T}_{22}$, let us define $h_1(x_{11}) = p_2\delta(x_{11})p_2$ and $h_2(x_{22}) = p_2\delta(x_{22})p_2$. It follows from (6) that $h_1 : \mathcal{T}_{11} \to Z(\mathcal{T}_{22})$ is a linear mapping such that $h_1([[a_{11}, b_{11}], c_{11}]) = 0$ for any $a_{11}, b_{11}, c_{11} \in \mathcal{T}_{11}$. It follows from (7) that $h_2 : \mathcal{T}_{22} \to Z(\mathcal{T}_{11})$ is a linear mapping such that $h_2([[a_{22}, b_{22}], c_{22}]) = 0$ for any $a_{22}, b_{22}, c_{22} \in \mathcal{T}_{22}$. For any $x = x_{11} + x_{12} + x_{22} \in \mathcal{T}$, by the hypothesis (2) of Theorem 3.1 and Lemma 2.1, we can define $h : \mathcal{T} \to \mathcal{T}$ by $h(x) = \eta^{-1}(h_1(x_{11})) + h_1(x_{11}) + h_2(x_{22}) + \eta(h_2(x_{22}))$. It can be easily proved that $h(x) \in Z(\mathcal{T})$ and $h([[x, y], z]) = 0$ for any $x, y, z \in \mathcal{T}$.

Define a linear map $\beta : \mathcal{T} \to \mathcal{T}$ by $\beta(x) = \delta(x) - h(x)$ for any $x \in \mathcal{T}$. As a result of Lemmas 3.3 and 3.5, we have

Lemma 3.6. For any $x_{11} \in \mathcal{T}_{11}$, $x_{12} \in \mathcal{T}_{12}$, $x_{22} \in \mathcal{T}_{22}$, we have
(1) $\beta(p_1) = 0$;
(2) $\beta(x_{11}) = \delta(x_{11}) - h(x_{11}) \in \mathcal{T}_{11}$;
(3) $\beta(x_{12}) = \delta(x_{12}) \in \mathcal{T}_{12}$;
(4) $\beta(x_{22}) = \delta(x_{22}) - h(x_{22}) \in \mathcal{T}_{22}$.

Lemma 3.7. For every $x_{11} \in \mathcal{T}_{11}$, $x_{12} \in \mathcal{T}_{12}$, $x_{22} \in \mathcal{T}_{22}$, we get
(1) $\beta(x_{11}x_{12}) = \beta(x_{11})x_{12} + x_{11}\beta(x_{12})$;
(2) $\beta(x_{12}x_{22}) = \beta(x_{12})x_{22} + x_{12}\beta(x_{22})$.

Proof. Let $x_{11} \in \mathcal{T}_{11}$, $x_{12} \in \mathcal{T}_{12}$ and $e_1 \in \mathcal{T}_{11}$ be any idempotent of $\mathcal{T}_{11}$. Take $e = e_1, x = x_{11}$ and $f = p_2 + x_{12}$ in Lemma 3.4.

For any $y_{12} \in \mathcal{T}_{12}$, since y_{12} can be written as Lie triple product $y_{12} = [[y_{12}, p_1], p_1]$, thus we have

$$\tau_i(y_{12}) = 0, i = 1, 2, 3, 4. \tag{8}$$

From the facts $exf \in \mathcal{T}_{12}$ and $e^{\perp}xf \in \mathcal{T}_{12}$, we have $\tau_1(exf) = 0$, $\tau_2(e^{\perp}xf) = 0$, $\tau_3(e^{\perp}xf^{\perp}) = \tau_3(e^{\perp}x)$, $\tau_4(exf^{\perp}) = \tau_4(ex)$.

Using Lemma 3.3 and 3.5 into Lemma 3.4, one can deduce

$$\begin{aligned}&\delta(e_1x_{11}x_{12})\\ &= \delta(e_1x_{11}(p_2 + x_{12}))\\ &= \delta(e_1x_{11})(p_2 + x_{12}) - e_1\delta(x_{11})x_{12} + e_1\delta(x_{11}x_{12})\\ &+ e_1\tau_3(x_{11} - e_1x_{11})(p_2 + x_{12}) - e_1^{\perp}\tau_4(e_1x_{11})(p_2 + x_{12})\\ &= \delta(e_1x_{11})p_2 + \delta(e_1x_{11})x_{12} - e_1\delta(x_{11})x_{12}\\ &+ e_1\delta(x_{11}x_{12}) + e_1\tau_3(x_{11} - e_1x_{11})x_{12}\\ &- \tau_4(e_1x_{11})p_2 - \tau_4(e_1x_{11})x_{12} + e_1\tau_4(e_1x_{11})x_{12}.\end{aligned}$$

This implies that

$$p_2\delta(e_1x_{11})p_2 = p_2\tau_4(e_1x_{11})p_2 \tag{9}$$

and

$$\begin{aligned}\delta(e_1x_{11}x_{12}) &= \delta(e_1x_{11})x_{12} - e_1\delta(x_{11})x_{12}\\ &\quad + e_1\delta(x_{11}x_{12}) + e_1\tau_3(x_{11} - e_1x_{11})x_{12}\\ &\quad - \tau_4(e_1x_{11})x_{12} + e_1\tau_4(e_1x_{11})x_{12}.\end{aligned} \tag{10}$$

From (3), $\delta(e^{\perp}xf^{\perp}) = d_3(e^{\perp}xf^{\perp}) + \tau_3(e^{\perp}xf^{\perp})$. By (8), we can obtain

$$\begin{aligned}&\delta(x_{11} - e_1x_{11} - x_{11}x_{12} + e_1x_{11}x_{12})\\ &= d_3(x_{11} - e_1x_{11} - x_{11}x_{12} + e_1x_{11}x_{12})\\ &+ \tau_3(x_{11} - e_1x_{11}).\end{aligned}$$

According to [11, Lemma 5], we get

$$d_3(\mathcal{T}_{11}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{12}, d_3(\mathcal{T}_{12}) \subseteq \mathcal{T}_{12}.$$

Since $\delta(\mathcal{T}_{12}) \subseteq \mathcal{T}_{12}$, we have

$$p_2\delta(x_{11} - e_1x_{11})p_2 = p_2\tau_3(x_{11} - e_1x_{11})p_2.$$

Then by Lemma 3.6, we have

$$\begin{aligned}&\tau_3(x_{11} - e_1x_{11})x_{12}\\ &= x_{12}\tau_3(x_{11} - e_1x_{11}) = x_{12}\delta(x_{11} - e_1x_{11})\\ &= x_{12}(\beta(x_{11} - e_1x_{11}) + h(x_{11} - e_1x_{11}))\\ &= x_{12}h(x_{11} - e_1x_{11})\\ &= x_{12}h(x_{11}) - x_{12}h(e_1x_{11})\\ &= h(x_{11})x_{12} - h(e_1x_{11})x_{12}.\end{aligned} \tag{11}$$

From (9) and Lemma 3.6, we have

$$\begin{aligned}\tau_4(e_1x_{11})x_{12} &= x_{12}\tau_4(e_1x_{11}) \\ &= x_{12}\delta(e_1x_{11}) \\ &= x_{12}(\beta(e_1x_{11})+h(e_1x_{11})) \\ &= x_{12}h(e_1x_{11}) \\ &= h(e_1x_{11})x_{12}. \end{aligned} \tag{12}$$

Using (11) and (12) into (10), we can obtain

$$\begin{aligned}&\beta(e_1x_{11}x_{12}) = \delta(e_1x_{11}x_{12}) \\ &= \delta(e_1x_{11})x_{12} - e_1\delta(x_{11})x_{12} + e_1\delta(x_{11}x_{12}) \\ &+ e_1h(x_{11})x_{12} - h(e_1x_{11})x_{12} \\ &= \beta(e_1x_{11})x_{12} - e_1\beta(x_{11})x_{12} + e_1\beta(x_{11}x_{12}). \end{aligned} \tag{13}$$

In particular, putting $x_{11} = p_1$, we have

$$\beta(e_1x_{12}) = \beta(e_1)x_{12} + e_1\beta(x_{12}).$$

By induction based on (13), we can prove that

$$\beta(e_1e_2\cdots e_nx_{12}) = \beta(e_1e_2\cdots e_n)x_{12} + e_1e_2\cdots e_n\beta(x_{12})$$

for any idempotents $e_1, e_2, \cdots, e_n \in \mathcal{T}_{11}$ and $x_{12} \in \mathcal{T}_{12}$. From $\mathcal{A} = J(\mathcal{A})$, we obtain

$$\beta(x_{11}x_{12}) = \beta(x_{11})x_{12} + x_{11}\beta(x_{12})$$

for all $x_{11} \in \mathcal{T}_{11}$, $x_{12} \in \mathcal{T}_{12}$.

With the similar argument, we can show that

$$\beta(x_{12}x_{22}) = \beta(x_{12})x_{22} + x_{12}\beta(x_{22})$$

for all $x_{12} \in \mathcal{T}_{12}$, $x_{22} \in \mathcal{T}_{22}$. □

Lemma 3.8. $\beta(x_{11}y_{11}) = \beta(x_{11})y_{11} + x_{11}\beta(y_{11})$ for any $x_{11}, y_{11} \in \mathcal{T}_{11}$; $\beta(x_{22}y_{22}) = \beta(x_{22})y_{22} + x_{22}\beta(y_{22})$ for any $x_{22}, y_{22} \in \mathcal{T}_{22}$.

Proof. For any $x_{11}, y_{11} \in \mathcal{T}_{11}$ and $x_{12} \in \mathcal{T}_{12}$, by Lemma 3.7, we have

$$\begin{aligned}\beta(x_{11}y_{11}x_{12}) &= \beta((x_{11}y_{11})x_{12}) \\ &= \beta(x_{11}y_{11})x_{12} + x_{11}y_{11}\beta(x_{12}). \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}&\beta(x_{11}y_{11}x_{12}) \\ &= \beta(x_{11}(y_{11}x_{12})) \\ &= \beta(x_{11})y_{11}x_{12} + x_{11}\beta(y_{11}x_{12}) \\ &= \beta(x_{11})y_{11}x_{12} + x_{11}(\beta(y_{11})x_{12} + y_{11}\beta(x_{12})). \end{aligned} \tag{15}$$

Combining (14) with (15), we take

$$\beta(x_{11}y_{11})x_{12} = (\beta(x_{11})y_{11} + x_{11}\beta(y_{11}))x_{12}$$

for any $x_{12} \in \mathcal{T}_{12}$. Since $\mathcal{T}_{12}$ is a $(\mathcal{T}_{11}, \mathcal{T}_{22})$-bimodule, we get $\beta(x_{11}y_{11}) = \beta(x_{11})y_{11} + x_{11}\beta(y_{11})$ for all $x_{11}, y_{11} \in \mathcal{T}_{11}$.

With the similar argument, by considering $\beta(x_{12}x_{22}y_{22})$, we can prove that $\beta(x_{22}y_{22}) = \beta(x_{22})y_{22} + x_{22}\beta(y_{22})$ for all $x_{22}, y_{22} \in \mathcal{T}_{22}$. □

Lemma 3.9. β is a derivation of $\mathcal{T}$.

Proof. For any $x = x_{11} + x_{12} + x_{22} \in \mathcal{T}, y = y_{11} + y_{12} + y_{22} \in \mathcal{T}$. By Lemmas 3.6-3.8, we have

$$\begin{aligned}\beta(xy) &= \beta(x_{11}y_{11} + x_{11}y_{12} + x_{12}y_{22} + x_{22}y_{22}) \\ &= \beta(x_{11})y_{11} + x_{11}\beta(y_{11}) + \beta(x_{11})y_{12} + x_{11}\beta(y_{12}) \\ &+ \beta(x_{12})y_{22} + x_{12}\beta(y_{22}) + \beta(x_{22})y_{22} + x_{22}\beta(y_{22}) \\ &= \beta(x)y + x\beta(y), \end{aligned}$$

then β is a derivation of $\mathcal{T}$. □

Proof of Theorem 3.1. Set $d(x) = [x, p_1\Delta(p_1)p_2] + \beta(x)$ for any $x \in \mathcal{T}$. Then based on the definitions of Δ and δ, we obtain

$$\Delta(x) = [x, p_1\Delta(p_1)p_2] + \beta(x) + h(x) = d(x) + h(x)$$

holds for all $x \in \mathcal{T}$. From Lemma 3.9, one can show that d is a derivation of $\mathcal{T}$. This ends the proof. □

Theorem 3.1 is then used to investigate upper triangular matrix algebra.

Corollary 3.10. Assume that R be a 2-torsion free commutative ring with identity, $T_n(R)(n \geq 4)$ be the upper triangular matrix algebra over R. If Δ is a local Lie triple derivation of $T_n(R)$, then $\Delta = d + h$, in which d is a derivation of $T_n(R)$, $h : \mathcal{T} \to Z(R) \cdot I_n$ is a linear map that vanishes at Lie triple products, where I_n is the unit of $T_n(R)$.

Proof. Given $n \geq 4$, the expression $T_n(R)$ may also be written as a triangular algebra $\begin{pmatrix} T_l(R) & M_{l\times(n-l)}(R) \\ 0 & T_{n-l}(R) \end{pmatrix}$ for $2 \leq l < n-1$. $T_l(R)$ and $T_{n-l}(R)$ are generated by their idempotents as shown in example (iii) of [7]. As a result, condition (C1) is met. Since $Z(T_k(R)) = Z(R) \cdot I_k$ for any positive integer k, the condition (C2) has been met. $T_l(R)$ satisfies the condition (C3), according to the finding of [15]. By using Theorem 3.1, we get the conclusion of Corollary 3.10. □

IV. Conclusion

This study aims to examine the characteristics of local Lie triple derivations within a category of triangular algebras. Under certain conditions, it is demonstrated that each local Lie triple derivation of a triangular algebra may be written as the combination of a derivation and a linear central-valued map that vanishes for Lie triple products. As an application of the main results, we describe the structure of upper triangular matrix algebra.

References

[1] E. Posner, "Derivations in Prime Rings," *Proceedings of the American Mathematical Society*, vol. 8, pp. 1093-1100, 1957.

[2] J. Zhang, W. Yu, "Jordan Derivations of Triangular Algebras," *Linear Algebra and its Applications*, vol. 419, no. 1, pp. 251-255, 2006.

[3] K. Zhu, J. Wang and Y. Yang, "On Generalized Derivations in Residuated Lattices," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp. 330-335, 2020.

[4] K. Zhu, J. Wang and Y. Yang, "On Derivations of State Residuated Lattices," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 4, pp. 751-759, 2020.

[5] M. Wang, T. Qian and J. Wang, "Some Results of Derivations on FI-lattices," *IAENG International Journal of Computer Science*, vol. 48, no. 3, pp. 559-563, 2021.

[6] R. V. Kadison, "Local Derivations," *Journal of Algebra*, vol. 130, no. 2, pp. 494-509, 1990.

[7] D. R. Larson, A. R. Sourour, "Local Derivations and Local Automorphisms of $B(X)$," *Proceedings of Symposia in Pure Mathematics*, vol. 51, pp. 187-194, 1990.

[8] B. E. Johnson, "Local Derivations on C$^*-$algebras are Derivations," *Transactions of the American Mathematical Society*, vol. 353, no. 1, pp. 313-325, 2001.

[9] M. Brešar, "Characterizing Homomorphisms, Derivations and Multipliers in Rings with Idempotents," *Proceedings of the Royal Society of Edinburgh Section A-Mathematics*, vol. 137, no. 1, pp. 9-21, 2007.

[10] L. Chen, F. Lu and T. Wang, "Local and 2-local Lie Derivations of Operator Algebras on Banach Spaces," *Integral Equations and Operator Theory*, vol. 77, no. 1, pp. 109-121, 2013.

[11] L. Chen, F. Lu, "Local Lie Derivations of Nest Algebras," *Linear Algebra and its Applications*, vol. 475, pp. 62-72, 2015.

[12] D. Liu, J. Zhang, "Local Lie Derivations of Factor von Neumann Algebras," *Linear Algebra and its Applications*, vol. 519, pp. 208-218, 2017.

[13] D. Liu, J. Zhang, "Local Lie Derivations on Certain Operator Algebras," *Annals of Functional Analysis*, vol. 8, no. 2, pp. 270-280, 2017.

[14] D. Liu, J. Zhang and M. Song, "Local Lie derivations of Generalized Matrix Algebras," *AIMS Mathematics*, vol. 8, no. 3, pp. 6900-6912, 2023.

[15] Z. Xiao, F. Wei, "Lie Triple Derivations of Triangular Algebras," *Linear Algebra and its Applications*, vol. 437, no. 5, pp. 1234-1249, 2012.

[16] W. S. Cheung, "Lie Derivations of Triangular Algebras," *Linear and Multilinear Algebra*, vol. 51, no. 3, pp. 299-310, 2003.

[17] L. Lu, Y. Yang, "Generalized Additive Derivations on MV-algebras," *Engineering Letters*, vol. 29, no. 2, pp. 789-794, 2021.