Methods of Solving Linear Fractional Programming Problem - an interval approach

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***Abstract*—This article demonstrates techniques to solve the linear fractional programming (LFP) problem using an interval approach. This approach addresses uncertainties as intervals and employs interval arithmetic for robustness. In this paper, a reasonable attempt is made to construct a mathematical model of interval linear fractional programming, and various approaches were employed to solve it. The proposed process emphasizes solving the ILFP problem in different optimization techniques and uses interval arithmetic to obtain a better range of intervals. The study illustrates the practical aspects of this approach and its effectiveness in solving real-world situations when uncertainties are significant. The methods, process, solutions, and time consumption are analyzed later to show our proposed method's real-life application and efficiency.**

***Index Terms*—Interval linear fractional programming(ILFP), Uncertainty, Intervals, Time consumption, Efficiency.**

I. Introduction

LINEAR fractional programming problem (LFP) is a situation where the variables are related linearly, the objective function to be optimized is a ratio of two linear functions, and the given constraints are linearly expressed. LFP problems occur when it is necessary to maximize the effectiveness of a specific activity, such as the company's profit per unit of labor expense, production cost per unit of production, the nutritional ratio per unit of charge, and so on. Many researchers worked diligently on the linear fractional programming problem since it provides a more practical method than the linear programming problem. The key reason for their interest in fractional programming is the wide range of programming models that could apply to real-life scenarios if optimization problems are evaluated as a ratio of physical quantities.

In real-world situations, data measurement and observational ambiguity are frequent occurrences. Particularly in the case of optimization problems, the parameters can be ambiguous, in which case they can be expressed as intervals. One such approach is interval linear fractional programming (ILFP), a specialized optimization methodology combining interval analysis and fractional programming to deal with problem-solving under uncertainty. Providing decision-makers with more reliable and robust solutions offers a practical approach to dealing with inaccurate objective function and constraints data. ILFP is employed in various domains, including engineering, finance, economics, and operations research. It is helpful in situations involving uncertainty, such as production planning, portfolio optimization, resource allocation, and supply chain management. This work aims to thoroughly understand ILFP, its formulation, and its practical applications.

The literature discusses different methods for solving various models of the linear fractional programming problem. In 1960, Hungarian mathematician B. Matros developed the area of LFP. R.E. Moore et al. [1] analyzed the interval and its computational algorithms with various applications. Erik B. Bajalinov [2] studied the different types, methods, and applications of linear fractional programming problems. Charnes et al., [3] transformed the linear fractional programming problem into a linear one. To tackle issues involving linear fractional function programming without converting them into LP problems, Swarup [4] improved the well-known simplex method. Bitran and Magnanti [5] studied sensitivity analysis, duality, and algorithms for optimization problems. Singh.C. [6] investigated the optimality requirement in fractional programming. S. Effati and Morteza Pakdaman [7] dealt with an interval-valued objective function with linear fractional bounds. Rasha Jalal Mitlif [8] used the development Lagrange method to solve problems with constrained and unconstrained linear fractional programming with interval coefficients in the objective function. Majeed Amir S [9] applied interval values to solve problems in linear fractional bounded variable programming. Sapan Kumar Das et al. [10] proposed a method to solve fuzzy linear fractional programming problems under non-negative fuzzy variables. Veeramani Chinnadurai and Sumathi Muthukumar [11] proposed a procedure to solve a fully fuzzy linear fractional programming problem using the upper and lower bounds. To obtain the pareto optimum solution, B Stanojevic and M Stanojevic [12] developed a technique for dealing with linear fractional programming problems with uncertain coefficients in the objective function. M. Borza and A.S. Rambely [13] discussed a method for solving linear fractional programming with fuzzy coefficients based on α-cuts and max-min. Suvasis Nayak and S. Maharana [14] studied multi-objective fractional programming problems under a fuzzy environment and analyzed the solutions in uncertain conditions.

Section (II) in this article describes some basic concepts for interval parameters, arithmetic, and ranking functions. Likewise, section (III) & sections (IV) discusses the formulation of ILFPP, methods, algorithms, and theorems for solving ILFPP. Furthermore, the article's numerical example and a graphical illustration, strategic decision-making with ILFPP and result analysis are shown in sections (V), (VI) & (VII). Finally, conclusion and future work is addressed in section (VIII) to analyze the efficiency of the proposed method.

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II. PRELIMINARIES

A. Interval number

Let $\tilde{u} = [u_1, u_2] = \{x \in R : u_1 \leq x \leq u_2$ and $u_1, u_2 \in R\}$ be an interval on the real line R. If $u_1 = u_2 = u$, then $\tilde{u} = [u, u]$ is a real number(or a degenerate interval). The intervals are identified with an ordered pair $\langle m, w\rangle$ defined as follows: Let $\tilde{u} = [u_1, u_2] \subseteq R$. Define $m(\tilde{u}) = \left(\frac{u_1+u_2}{2}\right)$ and $w(\tilde{u}) = \left(\frac{u_2-u_1}{2}\right)$ and hence $\tilde{u} \to \langle m(\tilde{u}), w(\tilde{u})\rangle$ is unique. Conversely, when $\langle m(\tilde{u}), w(\tilde{u})\rangle$ is given we know that $m(\tilde{u}) - w(\tilde{u})$ gives the left end point of $\tilde{u}$ and $m(\tilde{u}) + w(\tilde{u})$ gives the right end point of $\tilde{u}$ and hence given $\langle m(\tilde{u}), w(\tilde{u})\rangle$, the interval is unique.

B. Interval vector

An interval vector $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, ...\tilde{v}_n)^t$ is a vector whose components are closed intervals. We use IR^n to denote the set of all $n-$ component interval vectors. By $m(\tilde{v})$ we denote a vector whose entries are the corresponding midpoints of the entries of $\tilde{v}$. (i.e.) $m(\tilde{v}) = (m(\tilde{v_1}), m(\tilde{v_2}), ..., m(\tilde{v_n}))^t$ and the width of interval vector is defined by $w(\tilde{v}) = (w(\tilde{v}_1), w(\tilde{v}_2), ..., w(\tilde{v_n}))^t$.

C. Interval arithmetic

Ming Ma et al.[19] suggested a new fuzzy arithmetic focused on the index of locations and the index function of fuzziness. For the ordinary arithmetic the position index number is taken, while in the lattice L the fuzziness index functions are assumed to obey the lattice law which is the least upper bound and the greatest lower bound(i.e.) for $x, y \in L$ we define $x \vee y = max\{x, y\}$ and $x \wedge y = min\{x, y\}$.
For any two intervals, $\tilde{x}, \tilde{y} \in IR$ and for $* \in \{+, -, \cdot, \div\}$, the arithmetic operations on IR is defined as:

$$\tilde{x} * \tilde{y} = \langle m(\tilde{x}) * m(\tilde{y}),\ max\{w(\tilde{x}), w(\tilde{y})\}\rangle$$

In particular,

- $\tilde{x} + \tilde{y} = \langle m(\tilde{x}) + m(\tilde{y}),\ max\{w(\tilde{x}), w(\tilde{y})\}\rangle$
- $\tilde{x} - \tilde{y} = \langle m(\tilde{x}) - m(\tilde{y}),\ max\{w(\tilde{x}), w(\tilde{y})\}\rangle$
- $\tilde{x} \cdot \tilde{y} = \langle m(\tilde{x}) \cdot m(\tilde{y}),\ max\{w(\tilde{x}), w(\tilde{y})\}\rangle$
- $\tilde{x} \div \tilde{y} = \langle m(\tilde{x}) \div m(\tilde{y}),\ max\{w(\tilde{x}), w(\tilde{y})\}\rangle$, provided $m(\tilde{y}) \neq 0$

D. Ranking of Interval Numbers

Sengupta and Pal [20] suggested a easy and powerful index to compare any two intervals on IR through the satisfaction of decision-makers. Let $\preceq$ be an extended order relation between the interval numbers. For any two intervals $\tilde{u} = [u_1, u_2], \tilde{v} = [v_1, v_2] \in IR$ then for $m(\tilde{u}) < m(\tilde{v})$, we construct a premise($\tilde{u} \preceq \tilde{v}$) which implies that $\tilde{u}$ is inferior to $\tilde{v}$ (or $\tilde{v}$ is superior to $\tilde{u}$).

An acceptability function $A_{\preceq} : IR \times IR \to [0, \infty)$ is defined as: $A_{\preceq}(\tilde{u}, \tilde{v}) = A(\tilde{u} \preceq \tilde{v}) = \frac{m(\tilde{v})-m(\tilde{u})}{w(\tilde{v})+w(\tilde{u})}$,
where $w(\tilde{v}) + w(\tilde{u}) \neq 0$

$A_{\preceq}$ may be interpreted as the grade of acceptability of the first interval number to be inferior to the second interval number.

E. Bounded and Unbounded solution

Let $\tilde{x}$ be the feasible region defined by the system of linear inequalities and non-negativity constraints. An interval bounded solution $\tilde{x}^*$ is a point in the feasible region $\tilde{x}$ such that each of its decision variables $\tilde{x}_i^*$ lies within an interval $[L_i, R_i]$ for all i = 1, 2,..., n.

III. INTERVAL LINEAR FRACTIONAL PROGRAMMING PROBLEM

General form: Consider linear fractional programming problem involving interval numbers as follows:

$$\textbf{Max/Min } \tilde{\phi}(\mathbf{\tilde{x}}) \approx \frac{\tilde{P}(\tilde{x})}{\tilde{Q}(\tilde{x})} \approx \frac{\sum_{j=1}^{n} \tilde{p}_j^t \tilde{x}_j + \tilde{p}_0}{\sum_{j=1}^{n} \tilde{q}_j^t \tilde{x}_j + \tilde{q}_0} \quad (1)$$

Subject to,

$$\sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j \preceq \tilde{b}_i \qquad , i = 1, 2, ..., m \quad (2)$$

$$m(\tilde{x}_j) \geq 0 \qquad , j = 1, 2, ..., n \quad (3)$$

where $\tilde{A} \in IR^{m\times n}$, $\tilde{b} \in IR^m$, $\tilde{p}$, $\tilde{q} \in IR^n$ and $\tilde{p}_0, \tilde{q}_0 \in IR$ consisting of interval numbers.

IV. METHODS OF SOLVING ILFPP

A. Charnes and Cooper method

In 1962, A. Charnes and W.W. Cooper developed a technique to transform linear-fractional programming problem into a linear programming problem. This article focuses on interval numbers to achieve the required outcomes in an unambiguous environment. This method transforms by using a suitable variable transformation, $\tilde{y}_0 = \frac{1}{\tilde{q}^t \tilde{x} + \tilde{q}_0}$ and $\tilde{y} = \tilde{x}\tilde{y}_0$ in the problem (1)-(3). Then the problem transforms into ILPP,

$$\textbf{Max/Min } \tilde{\phi}(\mathbf{\tilde{x}}) \approx \frac{\sum_{j=1}^{n} \tilde{p}_j^t \frac{\tilde{y}_j}{\tilde{y}_0} + \tilde{p}_0}{\sum_{j=1}^{n} \tilde{q}_j^t \frac{\tilde{y}_j}{\tilde{y}_0} + \tilde{q}_0} = \frac{\sum_{j=1}^{n} \tilde{p}_j^t \tilde{y}_j + \tilde{p}_0 \tilde{y}_0}{\sum_{j=1}^{n} \tilde{q}_j^t \tilde{y}_j + \tilde{q}_0 \tilde{y}_0} \quad (4)$$

$$\textbf{Maximum } \tilde{\zeta}(\mathbf{\tilde{y}}) = \sum_{j=1}^{n} \tilde{p}_j^t \tilde{y}_j + \tilde{p}_0 \tilde{y}_0 \quad (5)$$

$$(Assuming \quad \sum_{j=1}^{n} \tilde{q}_j \tilde{y} + \tilde{q}_0 \tilde{y}_0 \approx \tilde{1})$$

Subject to,

$$\sum_{j=1}^{n} \tilde{q}_j \tilde{y}_j + \tilde{q}_0 \tilde{y}_0 \approx \tilde{1} \quad (6)$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} \tilde{y}_j - \tilde{b}_i \tilde{y}_0 \preceq \tilde{0} \qquad , i = 1, 2, ..., m \quad (7)$$

$$m(\tilde{y}_j), m(\tilde{y}_0) \geq 0 \quad (8)$$

Now, the problem can be solved using classical two-phase method with interval parameters.

Theorem 4.1 If $\tilde{x}$ is a interval feasible solution of interval linear fractional programming problem then $\tilde{y}$ is a interval feasible solution to ILPP and the objective functions are equivalent at these points (i.e.) $\tilde{\phi}(\tilde{x}) = \tilde{\zeta}(\tilde{y})$

Proof Let us consider the feasibility of ILFP to show the feasibility of the solution of ILPP.

- To begin with, proving all the constraints of ILFP compatible with the corresponding linear analogue, we consider $\tilde{x}$ be the feasible solution of ILFP problem with $\tilde{y}_0 = \frac{1}{\tilde{q}^t\tilde{x}+\tilde{q}_0}$ and $\tilde{y} = \tilde{x}\tilde{y}_0$

$$\tilde{A}\tilde{y} - \tilde{b}\tilde{y}_0 \approx \tilde{A}\tilde{y}_0\tilde{x} - \tilde{y}_0\tilde{x} \approx \tilde{y}_0(\tilde{A}\tilde{x} - \tilde{b}) \approx \tilde{y}_0 * 0.$$
$$\tilde{q}^t\tilde{y} + \tilde{q}_0\tilde{y}_0 \approx \tilde{y}_0(\tilde{q}^t\tilde{x} + \tilde{q}_0) \approx \tilde{y}_0(\tilde{q}^t\tilde{x} + \tilde{q}_0) \approx \frac{\tilde{q}^t\tilde{x}+\tilde{q}_0}{\tilde{q}^t\tilde{x}+\tilde{q}_0} \approx \tilde{1}$$

this proves $(\tilde{y}, \tilde{y}_0)$ is the feasible solution of ILPP. In addition to, for non-negative restrictions, $m(\tilde{y}) \geq 0$ and $\tilde{y}_0 = \frac{1}{\tilde{Q}(\tilde{x})} \succeq \tilde{0}$

- To prove objective functions are equal at these points, we consider,

$$\tilde{\zeta}(\tilde{y}, \tilde{y}_0) \approx \tilde{p}^t\tilde{y} + \tilde{p}_0\tilde{y}_0 \approx \tilde{y}_0(\tilde{p}^t\tilde{x} + \tilde{p}_0) \approx \frac{1}{\tilde{q}^t\tilde{x}+\tilde{q}_0}(\tilde{p}^t\tilde{x} + \tilde{p}_0) \approx \tilde{\phi}(\tilde{x})$$

Therefore, this shows if ILFP is feasible then ILPP is also feasible and the objective functions at these points are equivalent.

Theorem 4.2 If an interval vector $\tilde{y} = (\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_n)^t$is an interval feasible solution of the ILFP problem(1)-(3) then $\tilde{y}_0$, is an interval number that is positive.

Proof Let us suppose $\tilde{x}' = (\tilde{x}'_1, \tilde{x}'_2, \cdots, \tilde{x}'_n)^t$ and $\tilde{y}' = (\tilde{y}'_0, \tilde{y}'_1, \tilde{y}'_2, ..., \tilde{y}'_n)^t$ as the interval feasible solutions of ILFPP and ILPP respectively.

Assuming $(\tilde{y}', \tilde{y}'_0) = (\tilde{0}, \tilde{y}')$ $i.e.,$ $\tilde{y}'_0 \approx \tilde{0}$ with the interval feasible solutions $\tilde{x}' \in \tilde{F}$ and $\tilde{y}' \in \tilde{M}$ where $\tilde{F}, \tilde{M}$ are the interval feasible sets of ILFP and ILP problems respectively. Then,

$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}'_j \preceq \tilde{b}_i \qquad i = 1, 2, ..., m \tag{9}$$
$$m(\tilde{x}'_j) \geq 0 \qquad j = 1, 2, ..., n \tag{10}$$
$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{y}'_j \preceq \tilde{0} \qquad i = 1, 2, ..., m \tag{11}$$
$$m(\tilde{y}'_j) \geq 0 \qquad j = 1, 2, ..., n \tag{12}$$

On multiplying constraint (10) with an arbitrary interval $m(\tilde{\mu})$ and adding it to constraint (8) of the system. Similarly with the non-negative restrictions (11) and (9) which gives,

$$\sum_{j=1}^{n} \tilde{a}_{ij}(\tilde{x}'_j + \tilde{\mu}\tilde{y}'_j) \preceq \tilde{b}_i \qquad i = 1, 2, ..., m \tag{13}$$
$$m(\tilde{x}'_j) + m(\tilde{\mu}\tilde{y}'_j) \geq 0 \qquad j = 1, 2, ..., n \tag{14}$$

Then, $m(\tilde{x}'_j) + m(\tilde{\mu}\tilde{y}'_j)$ is in $\tilde{F}$ for $m(\tilde{\mu}) > 0$. But $m(\tilde{\mu})$ value may be required as large as possible then $\tilde{F}$ is unbounded which is a contradiction to our assumption on the feasible set $\tilde{F}$. Hence $\tilde{y}_0$ is a positive interval number.

B. Kanti Swarup method

In 1975, Kanti Swarup developed a simplex-like technique to solve linear fractional programming problem without converting them into linear programming problem. This algorthim is applied to the interval case to solve the system without converting them into classical form.

Algorithm

1) Determine if the provided interval linear fractional programming problem (ILFPP) should be maximized or minimized.
2) For all basically viable solutions, the denominator of ILFPP is non-negative, i.e., $\tilde{q}(\tilde{x}) + \tilde{q}_0 \succ \tilde{0}$.
3) Verify that all of the $(\tilde{b}i)$ are positive. If not, in the right-hand side of the restriction, multiply both sides by $(-\tilde{1})$ to make them positive.
4) To find $\tilde{\phi}$ since $\tilde{\phi} \approx \frac{\tilde{p}_j\tilde{x}+\tilde{p}_0}{\tilde{q}_j\tilde{x}+\tilde{q}_0}$, where $\tilde{\phi}_1 = \tilde{p}_B\tilde{x} + \tilde{p}_0$ and $\tilde{\phi}_2 = \tilde{q}_B\tilde{x} + \tilde{q}_0$
5) Compute the value of $\tilde{\Delta} \approx \tilde{\phi}_2(\tilde{\Delta}_1) - \tilde{\phi}_1(\tilde{\Delta}_2)$, where $\tilde{\Delta}_1 = \tilde{z}_{j1} - \tilde{p}_j$ and $\tilde{\Delta}_2 = \tilde{z}_{j2} - \tilde{q}_j$, $j = 1, 2, ..., n$ then examine,
   - If all $\tilde{\Delta}_j \succeq \tilde{0}$, the present basic feasible interval solution $\tilde{x}_B$ is optimal.
   - If at least one of $\tilde{\Delta}_j \prec \tilde{0}$, the present basic feasible interval solution is not optimal and continue the next step.
6) To maximize (minimize), choose the most positive (negative) $\tilde{\Delta}_j$ entering variable.
7) Choose the intersection cell with the minimum ratio as of the simplex method to find the leaving and entering variable.
8) Continue the iterations and (5) until an optimal interval solution is obtained.

C. Denominator objective restriction method

In this method, we construct single linear programming problem with its numerator $\tilde{P}$ (C) and denominator $\tilde{Q}$ (D) as the two single objective functions.

(C) **Max** $\tilde{P} = \sum_{j=1}^{n} \tilde{p}^t_j\tilde{x}_j + \tilde{p}_0$

subject to,

$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \preceq \tilde{b}_i \quad , i = 1, 2, ..., m$
$m(\tilde{x}_j) \geq 0 \quad , j = 1, 2, ..., n$

(D) **Min** $\tilde{Q} = \sum_{j=1}^{n} \tilde{q}^t_j\tilde{x}_j + \tilde{q}_0$

subject to,

$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \preceq \tilde{b}_i \quad , i = 1, 2, ..., m$
$m(\tilde{x}_j) \geq 0 \quad , j = 1, 2, ..., n$

Theorem 4.3 Let $\tilde{X}_0$ represent the optimal interval solution to the problem (C). If the problem (D) with $\tilde{X}_n$ is a sequence of feasible interval solutions with $\tilde{\phi}(\tilde{X}_k) \preceq \tilde{\phi}(\tilde{X}_{k+1})$, $\tilde{X}_n$ is the interval optimal solution of the problem $\tilde{\phi}$ with $\tilde{X}_0$ as the initial feasible interval solution for all $k = 0, 1, 2, \cdots n-1$ and $\tilde{\phi}(\tilde{X}_n) \succeq \tilde{\phi}(\tilde{X}_{n+1})$.

Proof Let the problem $(\tilde{\phi})$ have the feasible interval solution $\tilde{X}_n$. Considering $\tilde{R}$ be a feasible interval solution to the problem $(\tilde{\phi})$. This implies two possibilities $\tilde{Z}(\tilde{X}_n) \prec \tilde{Z}(\tilde{R})$ (or) $\tilde{Z}(\tilde{R}) \preceq \tilde{Z}(\tilde{X}_n)$.

case 1: By hypothesis, we have $\tilde{\phi}(\tilde{X}_k) \preceq \tilde{\phi}(\tilde{X}_{k+1})$ for all $k = 0, 1, 2, ..., n-1$ and $\tilde{\phi}(\tilde{X}_n) \succeq \tilde{\phi}(\tilde{X}_{n+1})$ and it tends to $\tilde{Z}(\tilde{R}) \preceq \tilde{Z}(\tilde{X}_n)$ since the problem (D) is of the minimization problem. Then $\tilde{X}_n$ is the optimal interval solution to the problem.

case 2: Implying the same as case 1, $\tilde{Z}(\tilde{X}_n) \prec \tilde{Z}(\tilde{R})$ do not occur as the problem (D) is the minimization problem. Hence, $\tilde{X}_n$ is an optimal interval solution to the problem $\tilde{\phi}$.

Theorem 4.4 Let $\tilde{X}_0$ represent the optimal interval solution to the problem (C). If the problem (D) with $\tilde{X}_n$ is a sequence of feasible solutions with interval numbers that have $\tilde{\phi}(\tilde{X}_k) \preceq \tilde{\phi}(\tilde{X}_{k+1})$ for every $k = 0, 1, 2, ..., n-1$. $\tilde{X}_{n+1}$ is an optimal solution with interval numbers to the problem $\tilde{\phi}$ where $\tilde{X}_0$ be the initial feasible interval solution.

Proof Let $\tilde{X}_{n+1}$ be a interval feasible solution to the problem $\tilde{\phi}$. Considering $\tilde{S}$ as the interval feasible solution to $\tilde{\phi}$. From the theorem, $\tilde{\phi}(\tilde{X}_k) \preceq \tilde{\phi}(\tilde{X}_{k+1})$ for all $k = 0, 1, 2, ..., n-1$ and $\tilde{X}_{n+1}$ is an optimal solution with interval numbers to the problem (D) (i.e.,) $\tilde{Z}(\tilde{X}_n + 1) \succeq \tilde{Z}(\tilde{S})$.

Algorithm

1) Create the problems (C) and (D) from the preceding problem($\tilde{\phi}$), into two single objective linear programming problems.
2) Calculate the optimal interval solution to the problem (C) using the classical simplex technique with interval numbers. Let $\tilde{X}_0$ represent the optimal interval solution to the problem (C), and suppose $\tilde{\phi}(X)_0 = \tilde{\phi}_0$.
3) Use the optimum table of problem (C) as an initial simplex table to problem (D) to create a sequence of improved basic feasible interval solutions. Then, for each of the improved basic feasible interval solutions, use the simplex method for intervals to determine the value of $\tilde{\phi}$.
   - If $\tilde{\phi}(\tilde{X}_k) \preceq \tilde{\phi}(\tilde{X}_{k+1})$, for all instances of $k = 0, 1, 2, ..., n-1$ and $\tilde{\phi}(\tilde{X}_n) \succeq \tilde{\phi}(\tilde{X}_{n+1})$ stop the computing process for some n and move on to Step(4) instead.
   - If $\tilde{\phi}(\tilde{X}_k) \preceq \tilde{\phi}(\tilde{X}_{k+1})$, for all instances of $k = 0, 1, 2, ..., n-1$ and $\tilde{\phi}(\tilde{X}_n) \preceq \tilde{\phi}(\tilde{X}_{n+1})$ stop the computing process for some n and move on to Step(5) instead.
4) According to the theorem (4.3), Max $\tilde{\phi}(\tilde{X})$= $\tilde{\phi}(\tilde{X}_n)$, $\tilde{X}_n$ is the optimal interval solution to the problem ($\tilde{\phi}$).
5) According to the theorem (4.4), Max $\tilde{\phi}(\tilde{X})$= $\tilde{\phi}(\tilde{X}_{n+1})$, $\tilde{X}_{n+1}$ is the optimal interval solution to the problem ($\tilde{\phi}$).

D. Graphical method

A linear fractional programming problem can be solved graphically. In interval case, to solve in graphical method the mid-points of the interval is taken into account. Considering as crisp case: the problem is solved.

Algorithm

Step 1 Let us consider the general form:

$$Optimize\ \tilde{\phi}(\tilde{\mathbf{x}}) \approx \frac{m(\tilde{p}_1)m(\tilde{x}_1)+m(\tilde{p}_2)m(\tilde{x}_2)+m(\tilde{p}_0)}{m(\tilde{q}_1)m(\tilde{x}_1)+m(\tilde{q}_2)m(\tilde{x}_2)+m(\tilde{q}_0)}$$

Subject to,

$$\sum_{j=1}^{n} m(\tilde{a}_{ij})m(\tilde{x}_j) \leq m(\tilde{b}_i), \quad i = 1, 2, ..., m$$

$$m(\tilde{x}_j) \geq 0,\ j = 1, 2, ..., n$$

Step 2 Convert the problem into its standard form

$$Optimize\ \tilde{\phi}(\tilde{\mathbf{x}}) \approx \frac{m(\tilde{p}_1)m(\tilde{x}_1)+m(\tilde{p}_2)m(\tilde{x}_2)+m(\tilde{p}_0)}{m(\tilde{q}_1)m(\tilde{x}_1)+m(\tilde{q}_2)m(\tilde{x}_2)+m(\tilde{q}_0)}$$

Subject to,

$$\sum_{j=1}^{n} m(\tilde{a}_{ij})m(\tilde{x}_j) \approx m(\tilde{b}_i), \quad i = 1, 2, ..., m$$

$$m(\tilde{x}_j) \geq 0,\ j = 1, 2, ..., n$$

Step 3 To solve ILFPP,
$\tilde{P}(\tilde{x}) = m(\tilde{p}_1)m(\tilde{x}_1) + m(\tilde{p}_2)m(\tilde{x}_2) = -m(\tilde{p}_0)$
$\tilde{Q}(\tilde{x}) = m(\tilde{q}_1)m(\tilde{x}_1) + m(\tilde{q}_2)m(\tilde{x}_2) = -m(\tilde{q}_0)$

Step 4 To determine the extreme points, individually solve $\tilde{P}(\tilde{x})$ and $\tilde{Q}(\tilde{x})$ and the constraints.
Step 5 Plot A, B, C, and D as the extreme points.
Step 6 The feasible region can be found via,

- If the constraint has $\geq$, the shaded region would then be below the line.
- If the constraint has $\leq$, the shaded region would then be above the line.

Step 7 By substituting the extreme points in the objective function $\tilde{\phi}(m(\tilde{x}))$, we get $\tilde{\phi}(A), \tilde{\phi}(B), \tilde{\phi}(C), \tilde{\phi}(D)$
Step 8 Determine the maximum/minimum of the objective function based on the values of $\tilde{\phi}(A), \tilde{\phi}(B), \tilde{\phi}(C), \tilde{\phi}(D)$
Step 9 Join the points and shade the feasible region for both maximum and minimum case.
Step 10 Depending on whether the shaded region is closed or open, we tend to identify whether the solution is bounded feasible or unbounded feasible.

V. Real-life application of ILFPP

Numerical example: Suppose a corporation manufactures two different types of products, $\tilde{x}_1$and $\tilde{x}_2$. The objective is to maximize total profit $\tilde{\phi}$, which depends on the product quantities, and uncertain production costs, represented as intervals. Let [3,5] and [1,4] represent the profits that each unit of the products $\tilde{x}_1$ and $\tilde{x}_2$ contributes, respectively. Let [0.5,2] and [1,2] be the total production cost per each unit of $\tilde{x}_1$ and $\tilde{x}_2$, including variable costs, respectively. Furthermore, [7,11] and [4,6] are the fixed profit and cost of the products $\tilde{x}_1$ and $\tilde{x}_2$.

$\tilde{x}_1 + 3\tilde{x}_2 \preceq 30$ (is the available machine hours for the production of x and y)
$-\tilde{x}_1 + 2\tilde{x}_2 \preceq 5$ (represent the minimum level of production required to maintain a steady supply)
$\tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}$ (ensure that product quantities cannot be negative)

Solution: The real-life application is formulated into an ILFP problem as,

$$\textbf{Max}\ \tilde{\phi} = \frac{[3,5]\tilde{x}_1+[1,4]\tilde{x}_2+[7,11]}{[0.5,2]\tilde{x}_1+[1,2]\tilde{x}_2+[4,6]}$$

Subject to,

$$\tilde{x}_1 + 3\tilde{x}_2 \preceq 30$$
$$-\tilde{x}_1 + 2\tilde{x}_2 \preceq 5$$
$$\tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}$$

With the interval arithmetic mentioned in section 2C, the problem is written in its mid-point, width (i.e.) $\langle m(\tilde{a}), w(\tilde{a})\rangle$ form,

$$\textbf{Max } \tilde{\phi} = \frac{\langle 4,1\rangle\tilde{x}_1+\langle 2.5,1.5\rangle\tilde{x}_2+\langle 9,2\rangle}{\langle 1.25,0.75\rangle\tilde{x}_1+\langle 1.5,0.5\rangle\tilde{x}_2+\langle 5,1\rangle}$$

Subject to,

$$\begin{gathered}\tilde{x}_1 + 3\tilde{x}_2 \preceq 30\\ -\tilde{x}_1 + 2\tilde{x}_2 \preceq 5\\ m(\tilde{x}_1), m(\tilde{x}_2) \geq 0\end{gathered}$$

Solution by Charnes and cooper method:

Using the Charnes and Cooper transformation, the ILFPP above is converted into its equivalent ILPP:

$$\text{Maximize } \tilde{\zeta} = \langle 4,1\rangle\tilde{y}_1 + \langle 2.5,1.5\rangle\tilde{y}_2 + \langle 9,2\rangle\tilde{y}_0$$

Subject to

$$\begin{gathered}\langle 1.25,0.75\rangle\tilde{y}_1 + \langle 1.5,0.5\rangle\tilde{y}_2 + \langle 5,1\rangle\tilde{y}_0 = 1\\ \tilde{y}_1 + 3\tilde{y}_2 - 30\tilde{y}_0 \preceq 0\\ -\tilde{y}_1 + 2\tilde{y}_2 - 5\tilde{y}_0 \preceq 0\\ m(\tilde{y}_1), m(\tilde{y}_2) \geq 0,\ m(\tilde{y}_0) > 0\end{gathered}$$

TABLE I
INITIAL TABLE

		$\tilde{c}_j$	0	0	0	0	0	-1	
B.V	$\tilde{c}_B$	$\tilde{Y}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{t}\downarrow$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{A}_1$	θ
$\leftarrow \tilde{A}_1$	-1	1	$\langle 1.25, 0.75\rangle$	$\langle 1.5, 0.5\rangle$	$\langle 5, 1\rangle$	0	0	1	$\langle 0.2, 1\rangle \rightarrow$
$\tilde{s}_1$	0	0	1	3	-30	1	0	0	
$\tilde{s}_2$	0	0	-1	2	-5	0	1	0	
		$\tilde{\zeta}_j$	$\langle -1.25, 0.75\rangle$	$\langle -1.5, 0.5\rangle$	$\langle -5, 1\rangle$	0	0	-1	
		$\tilde{\zeta}_j - \tilde{c}_j$	$\langle -1.25, 0.75\rangle$	$\langle -1.5, 0.5\rangle$	$\langle -5, 1\rangle \uparrow$	0	0	0	

the final simplex table of the problem in two-phase method involving interval parameters:

TABLE II
FINAL TABLE

		$\tilde{c}_j$	$\langle 4, 1\rangle$	$\langle 2.5, 1.5\rangle$	$\langle 9, 2\rangle$	0	0	
B.V	$\tilde{c}_B$	$\tilde{Y}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_0$	$\tilde{s}_1$	$\tilde{s}_2$	θ
$\tilde{y}_0$	$\langle 9, 2\rangle$	0.00235	$\langle 0, 0.75\rangle$	$\langle -0.0529, 0.5\rangle$	$\langle 1, 1\rangle$	-0.0294	0	
$\tilde{y}_1$	$\langle 4, 1\rangle$	0.7059	$\langle 1, 0.75\rangle$	$\langle 1.4118, 0.5\rangle$	$\langle 0, 1\rangle$	0.1176	0	
$\tilde{s}_2$	0	0.8235	$\langle 0, 0.75\rangle$	$\langle 3.1471, 0.5\rangle$	$\langle 0, 1\rangle$	-0.0294	1	
		$\tilde{\zeta}_j$	$\langle 4, 1\rangle$	$\langle 5.1711, 2\rangle$	$\langle 9, 2\rangle$	$\langle 0.2058, 2\rangle$	0	
		$\tilde{\zeta}_j - \tilde{c}_j$	$\langle 0, 1\rangle$	$\langle 2.6711, 2\rangle$	$\langle 0, 2\rangle$	$\langle 0.2058, 2\rangle$	0	

Since $\tilde{\zeta}_j - \tilde{c}_j \succeq \tilde{0}$ for all j, the current basic feasible solution is optimal. The optimal solution is $\tilde{y}_1 = 0.7059$, $\tilde{y}_2 = 0$, $\tilde{y}_0 = 0.0235$ and $\tilde{\zeta} = [\mathbf{1.0351}, \mathbf{5.0351}]$.

Solution by Kanti Swarup's simplex-like algorithm: The standard form of the given problem is,

$$\textbf{Max } \tilde{\phi} = \frac{\langle 4,1\rangle\tilde{x}_1+\langle 2.5,1.5\rangle\tilde{x}_2+\langle 9,2\rangle}{\langle 1.25,0.75\rangle\tilde{x}_1+\langle 1.5,0.5\rangle\tilde{x}_2+\langle 5,1\rangle}$$

Subject to,

$$\begin{gathered}\tilde{x}_1 + 3\tilde{x}_2 + \tilde{s}_1 \approx 30\\ -\tilde{x}_1 + 2\tilde{x}_2 + \tilde{s}_2 \approx 5\\ m(\tilde{x}_1), m(\tilde{x}_2) > 0\end{gathered}$$

By using the algorithm described in section 4C, the above problem is solved and the final iteration is given as

TABLE III
INITIAL TABLE

			$\tilde{c}_j$	$\langle 4, 1\rangle$	$\langle 2.5, 1.5\rangle$	0	0	
			$\tilde{d}_j$	$\langle 1.25, 0.75\rangle$	$\langle 1.5, 0.5\rangle$	0	0	
B.V	$\tilde{p}_B$	$\tilde{q}_B$	$\tilde{X}_B$	$\tilde{x}_1\downarrow$	$\tilde{x}_2$	$\tilde{s}_1$	$\tilde{s}_2$	θ
$\leftarrow \tilde{s}_1$	0	0	30	1	3	1	0	$30 \rightarrow$
$\tilde{s}_2$	0	0	5	-1	2	0	1	-5
$\tilde{\phi}_1 = \langle 9, 2\rangle$		$\tilde{\Delta}_1$		$\langle 4, 1\rangle$	$\langle 2.5, 1.5\rangle$	0	0	
$\tilde{\phi}_2 = \langle 5, 1\rangle$		$\tilde{\Delta}_2$		$\langle 1.25, 0.75\rangle$	$\langle 1.5, 0.5\rangle$	0	0	
$\tilde{\phi} = \langle 1.8, 2\rangle$		$\tilde{\Delta}$		$\langle 8.75, 2\rangle\uparrow$	$\langle -1, 2\rangle$	$\langle 0, 2\rangle$	$\langle 0, 2\rangle$	

TABLE IV
FINAL TABLE

			$\tilde{c}_j$	$\langle 4, 1\rangle$	$\langle 2.5, 1.5\rangle$	0	0	
			$\tilde{d}_j$	$\langle 1.25, 0.75\rangle$	$\langle 1.5, 0.5\rangle$	0	0	
B.V	$\tilde{p}_B$	$\tilde{q}_B$	$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{s}_1$	$\tilde{s}_2$	θ
$\tilde{x}_1$	$\langle 4, 1\rangle$	$\langle 1.25, 0.75\rangle$	30	1	3	1	0	
$\tilde{s}_2$	0	0	35	0	5	1	1	
$\tilde{\phi}_1 = \langle 129, 2\rangle$		$\tilde{\Delta}_1$		$\langle 0, 1\rangle$	$\langle -9.5, 1.5\rangle$	$\langle -4, 1\rangle$	$\langle 0, 1\rangle$	
$\tilde{\phi}_2 = \langle 5, 1\rangle$		$\tilde{\Delta}_2$		$\langle 0, 0.75\rangle$	$\langle -2.25, 0.75\rangle$	$\langle -1.25, 0.75\rangle$	$\langle 0, 0.75\rangle$	
$\tilde{\phi} = \langle 1.8, 2\rangle$		$\tilde{\Delta}$		$\langle 0, 2\rangle$	$\langle -113.5, 2\rangle$	$\langle -8.75, 2\rangle$	$\langle 0, 2\rangle$	

Since $\tilde{\phi}_j \succeq \tilde{0}$ for all j, the current basic feasible solution is optimal. The optimal solution is $\tilde{x}_1 = 30,\ \tilde{x}_2 = 0$ and where $\tilde{\phi} = [\mathbf{1.0353}, \mathbf{5.0353}]$.

Solution by denominator objective restriction method

(C) $\textbf{Max } \tilde{P} = \langle 4,1\rangle\tilde{x}_1 + \langle 2.5,1.5\rangle\tilde{x}_2 + \langle 9,2\rangle$

Subject to

$$\begin{gathered}\tilde{x}_1 + 3\tilde{x}_2 \preceq 30\\ -\tilde{x}_1 + 2\tilde{x}_2 \preceq 5\\ m(\tilde{x}_1), m(\tilde{x}_2) > 0\end{gathered}$$

(D) $\textbf{Min } \tilde{Q} = \langle 1.25,0.75\rangle\tilde{x}_1 + \langle 1.5,0.5\rangle\tilde{x}_2 + \langle 5,1\rangle$

Subject to

$$\begin{gathered}\tilde{x}_1 + 3\tilde{x}_2 \preceq 30\\ -\tilde{x}_1 + 2\tilde{x}_2 \preceq 5\\ m(\tilde{x}_1), m(\tilde{x}_2) > 0\end{gathered}$$

TABLE V
INITIAL TABLE

		$\tilde{p}_j$	$\langle 4, 1\rangle$	$\langle 2.5, 1.5\rangle$	$\tilde{0}$	$\tilde{0}$	
$\tilde{c}_B$	$\tilde{X}_B$	$\tilde{b}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{s}_1$	$\tilde{s}_2$	θ
$\tilde{0}$	$\tilde{s}_1$	$\tilde{30}$	1	3	1	0	$\frac{30}{1} \rightarrow$
$\tilde{0}$	$\tilde{s}_2$	$\tilde{5}$	-1	2	0	1	$\frac{-5}{1}$
		$\tilde{z}_j$	0	0	0	0	
		$\tilde{z}_j - \tilde{p}_j$	$\langle -4, 1\rangle\uparrow$	$\langle -2.5, 1.5\rangle$	$\tilde{0}$	$\tilde{0}$	

On solving the numerator the solution obtained is $\tilde{x}_1 = \tilde{30},\ \tilde{x}_2 = \tilde{0}$ and the numerator,

(C) $\textbf{Max } \tilde{P} = \langle 4,1\rangle\tilde{x}_1 + \langle 2.5,1.5\rangle\tilde{x}_2 + \langle 9,2\rangle$

becomes, $\tilde{P}(\tilde{x}) = \langle 129, 2\rangle \Rightarrow [129, 131]$. Substituting the values in the objective function, the solution is $\langle 3.0352, 2\rangle \Rightarrow [\mathbf{1.0352}, \mathbf{5.0352}]$.
Considering, the solution of (C) as the initial feasible solution of the denominator (D) and solving in the same procedure we get,

TABLE VI
FINAL TABLE

$\tilde{c}_B$	$\tilde{X}_B$	$\tilde{b}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{s}_1$	$\tilde{s}_2$	θ
		$\tilde{q}_j$	$\langle -1.25, 0.75\rangle$	$\langle -1.5, 0.5\rangle$	$\tilde{0}$	$\tilde{0}$	
$\tilde{0}$	$\tilde{x}_1$	$\tilde{25}$	1	-2	0	-1	
$\tilde{0}$	$\tilde{s}_1$	$\tilde{5}$	0	5	1	1	
		$\tilde{z}_j$	$\langle -1.25, 0.75\rangle$	$\langle 2.5, 0.75\rangle$	$\langle 0, 0.75\rangle$	$\langle 1.25, 0.75\rangle$	
		$\tilde{z}_j - \tilde{q}_j$	$\langle 0, 0.75\rangle$	$\langle 1, 0.75\rangle$	$\langle 0, 0.75\rangle$	$\langle 1.25, 0.75\rangle$	

On solving the denominator, the solution obtained is $\tilde{x}_1 = \tilde{25},\ \tilde{x}_2 = \tilde{0}$ and the denominator,

$$\text{(D) } \mathbf{Min}\ \tilde{Q} = \langle 1.25, 0.75\rangle \tilde{x}_1 + \langle 1.5, 0.5\rangle \tilde{x}_2 + \langle 5, 1\rangle$$

becomes, $\tilde{Q}(\tilde{x}) = \langle 36.25, 1\rangle \Rightarrow [35.25, 37.25]$. Substituting the values in the objective function, the solution is $\langle 3.00, 2\rangle \Rightarrow [\mathbf{1}, \mathbf{5}]$.
Hence with the step(4) of algorithm the optimal solution is $\tilde{\phi}(\tilde{X}_n) \succeq \tilde{\phi}(\tilde{X}_{n+1}) \Rightarrow [\mathbf{1.0352}, \mathbf{5.0352}]$

Solution by graphical method

$$\mathbf{Max}\ \tilde{\phi} = \frac{\langle 4,1\rangle \tilde{x}_1 + \langle 2.5,1.5\rangle \tilde{x}_2 + \langle 9,2\rangle}{\langle 1.25,0.75\rangle \tilde{x}_1 + \langle 1.5,0.5\rangle \tilde{x}_2 + \langle 5,1\rangle}$$

Subject to,

$$\tilde{x}_1 + 3\tilde{x}_2 \preceq 30$$
$$-\tilde{x}_1 + 2\tilde{x}_2 \preceq 5$$
$$m(\tilde{x}_1), m(\tilde{x}_2) \succ 0$$

Solving the numerator $\tilde{P}(\tilde{x})$ and denominator $\tilde{Q}(\tilde{x})$ to find the focus point:

$$\tilde{P}(\tilde{x}) = 4\tilde{x}_1 + 2.5\tilde{x}_2 = -9$$

$$\tilde{Q}(\tilde{x}) = 1.25\tilde{x}_1 + 1.5\tilde{x}_2 = -5$$

For (C) $\tilde{P}(\tilde{x})$ the points are $(-2.25, 0)$ and $(0, -3.6)$ and for (D) $\tilde{Q}(\tilde{x})$ the points are $(0, -3.33)$ and $(-4, 0)$. Solve the constraints for calculating the extreme points:

$$\tilde{x}_1 + 3\tilde{x}_2 = 30$$
$$-\tilde{x}_1 + 2\tilde{x}_2 = 5$$

The obtained extreme points are $(30, 0), (0, 10), (-5, 0)$ and $(0, 2.5)$.

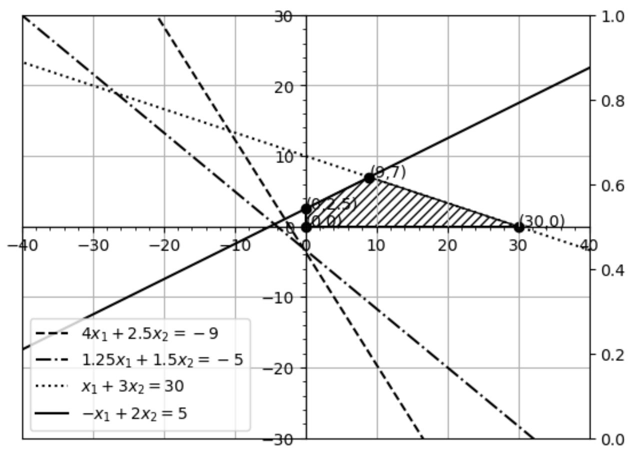

Fig. 1. Graphical method

From the above figure the shaded region $(\mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{2.5}), (\mathbf{9}, \mathbf{7})$ and $(\mathbf{30}, \mathbf{0})$ is the feasible region with the solution of $[\mathbf{1.0352}, \mathbf{5.0352}]$

VI. STRATEGIC DECISION-MAKING WITH INTERVAL LINEAR FRACTIONAL PROGRAMMING

ILFP empowers the corporation to strategically allocate resources to $\tilde{x}_1 = \tilde{30}$ and $\tilde{x}_2 = \tilde{0}$ based on the uncertain profit margins and production costs. This flexibility enables the corporation to adapt swiftly to evolving market circumstances. The ILFP model assists in optimizing production levels to make the best use of available machine hours and maintain steady supply levels with a total profit of $[1.0351, 5.0351]$. By encompassing uncertainty through intervals, ILFP aids in risk reduction and robust decision-making in a dynamic business environment.

VII. RESULT ANALYSIS

By employing any of the approaches mentioned, comprehensive solutions to test problems with interval numbers can be determined. The optimal solution offers the best allocation of resources ($\tilde{x}_1$ and $\tilde{x}_2$) to maximize profit under the given uncertainty in production costs and profit contributions. However, since Charnes and Cooper's method transformed an ILFPP into an interval linear programming problem, it is preferable to another way from the perspective of calculation and time consumption. A close-approach strategy is a graphical approach. Also, this might provide a better precise approximation.

VIII. CONCLUSION AND FUTURE WORK

In this article, we used different methods such as Charnes and Cooper, Kanti Swarup, Denominator objective restriction method and Graphical method to solve interval linear fractional programming problem, showing the importance and elementary of the processes in interval case. The results were compared with the graphical method to show the efficient approach of these methods. All the strategies mentioned can be employed to solve a wide range of practical problems, including the efficient allocation of fixed resources among various activities based on their relative importance.

Further research endeavors might investigate various approaches based on this study. To examine and develop more effective optimization techniques formulated for ILFP to improve solution accuracy and efficiency. It is promising to extend the proposed approach to multi-objective ILFP problems which involves balancing multiple competing objectives and refining solutions that adhere to diverse criteria.

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