Distributed Adaptive Bipartite Synchronization of Networks with Competitive Relationships and Disturbances

Chaoyang Li, Shidong Zhai, Tianhong Zhou, Hao Peng

Abstract—**This paper studies the problem of under what conditions the general nonlinear networks with directional competition and perturbations to achieve fully distributed BS (bipartite synchronization). For the norm bounded disturbances with unknown upper bound, under the assumption that the signed digraph is structurally balanced and contains a directed spanning tree, a fully distributed observer-based adaptive protocols is designed for BS problem of general nonlinear networks with no leader and neural networks approximation. For the disturbances generated by exosystems, based on a disturbance observer, a fully distributed disturbance observer-based adaptive protocol is proposed to make network achieve bounded BS. In order to reduce the chattering phenomenon, some adaptive gains and adaptive parameter vector of neural networks approximation are proposed. Finally, the theoretical results are verified by two numerical simulation examples.**

Index Terms—**Bipartite synchronization, competitive relationships, disturbances, nonlinear networks.**

I. Introduction

SINCE competition and cooperation are ubiquitous in social, biological, physical and other fields, in the last decade, the research on the impact of competition and cooperation on network synchronization (consensus) behavior has attracted more and more attention from scholars in various related fields [1], [2], [3], [4], [5], [6], [7], [8]. A network with competition and cooperation can be represented by a signed digraph, where the positive edge represents cooperation and the negative edge represents competition. Thus, the consensus problem in a network with competitive and cooperative relationships becomes the problem of studying how to design consensus agreements based on signed graphs [9]. Since Altafini's pioneering work, there has been many literature studies on the problem of consensus on signed graphs, such as BS [1], [10], [11], [12], [13], [14], [15], [16], modulus synchronization [17], interval BS [3] and so on.

Depending on whether there is a leader, existing work on BS can be divided into two categories, that is, leaderless BS [18], [19], [20], [21], [22] and leader-following BS [23], [11], [24], [12], [25], [26]. In the absence of a leader, if the system has its own dynamics, it is generally necessary to obtain the coupling strength condition for all nodes to achieve BS. Since these conditions generally depend on the Laplacian matrix of the signed graph, the resulting conditions are not fully distributed. In order to overcome this problem, few fully distributed BS algorithms observer-based were proposed in [19], [21], [22]. Under the undirected signed graph, a adaptive BS strategy has been proposed to guarantee BS for leaderless networks [19]. The finite-time BS problem of first-order networks has been studied in [20]. Some adaptive BS strategies have been designed for general networks with input saturation [21], and general nonlinear networks[22]. Leader-follower BS of linear uncertain networks has been studied in [11]. The BS tracking for networks with leader's unknown input was addressed in [24], and for higher-order heterogeneous nonlinear uncertain networks, the same problem was studied in [12], where agent nonlinear dynamics (including leader dynamics) are general and unknown. When the topology is switching, [25] addressed the leader-following BS for the nonlinear networks subject to exogenous disturbances.

Due to the fact that actual systems can be affected by various disturbances, it is very necessary to study the impact of disturbances on the system. There are generally two types of disturbance, one is the bounded disturbance, and the other is the disturbance generated by external systems. For the BS problem of networks, [20], [23], [11] addressed the condition that the system is affected by bounded disturbance. Norm bounded disturbances may come from inputs [20], [23], [11] or from the system itself, such as in the case of nonlinear uncertainties [19], [24], [22]. Under the Lipschitz-like nonlinear condition, [25] addressed the BS problem of networks with exogenous disturbances, and a leader-following BS method based on disturbance observer is proposed. Based on the above literature review, in the absence of leadership, there is little literature on the impact of various disturbances on the BS, especially the fully distributed protocol design.

Motivated by the above literature review, we will research the leaderless BS problem of general nonlinear networks with directed competitive relationships and disturbances. In this paper, the situation that network nodes are subject to the norm bounded disturbance and the external system disturbance is considered respectively. The main contributions of this paper are the following three areas:

- First, unlike the works in [19], [24], [22], we consider the bounded disturbances and disturbances generated by exosystems, and obtain some fully distributed distur-

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bance observer-based adaptive protocols for the leaderless BS problem.

- Second, compared with the Lipschitz-like nonlinear condition [25], the general nonlinear systems with neural networks approximation are considered in this paper and a modified adaptive gain vector is proposed to estimate the unknown constant vector of approximation.
- Third, some adaptive gains and adaptive parameter vectors of neural networks approximation are designed for reducing the chattering phenomenon. For the disturbances generated by exosystems, the bounded BS is achieved by a fully distributed disturbance observer-based adaptive protocol.

The paper is organized as follows. Section II presents notations, some properties for digraphs, and the problem formulation. Section III provides a fully distributed observer-based adaptive protocol for the BS of general nonlinear networks with bounded disturbances. Section IV gives a fully distributed disturbance observer-based adaptive protocol for the bounded BS of general nonlinear networks with disturbances produced from the external system. Section V gives two numerical examples to verify the theorems deduced in Sections III and IV. Section VI summarizes the conclusions of this paper and describes the direction of future work.

II. Preliminaries

A. Notations

The symbol $\otimes$ represents the Kronecker product. $\mathrm{diag}(r_1, \cdots, r_N)$ represents a diagonal matrix consisting of $r_1, \cdots, r_N$. $\|\cdot\|_1$, $\|\cdot\|$ and $\|x\|_\infty$ represent the 1-norm, 2-norm and ∞-norm respectively. Let $\mathbf{1}_N$ denote an n-dimensional column vector with each element equal to 1. $\mathbf{0}$ represents a matrix with elements all 0. $X \succ (\prec) 0$ represents X is positive (negative) definite matrix, $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ represent the minimum and maximum eigenvalues for the symmetric matrix X respectively. I_n represents the identity matrix of dimension n. The $\mathrm{sgn}(\cdot)$ function is defined as: when $x > 0$, $\mathrm{sgn}(x) = 1$; $x < 0$, $\mathrm{sgn}(x) = -1$; $x = 0$, $\mathrm{sgn}(x) = 0$.

B. Directed interaction graphs

In this subsection, we introduce some concepts of signed digraph and some lemmas that will be used in the sequel.

Let $\mathcal{V} = \{v_1, \cdots, v_N\}$ be the node set. The interactions among nodes can be represented by the signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents the edge set and $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ represents the weighted adjacency matrix. The edge $\mathcal{E}_{ij} \in \mathcal{E}$ represents a node set $(v_i,\ v_j)$ means v_i can receive information from v_j. The coupling configuration information for each edge in $\mathcal{G}$ is recorded in $\mathcal{A}$. v_i and v_j is cooperative if the weight $a_{ij} > 0$, v_i and v_j is competitive if the weight $a_{ij} < 0$, and $a_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{E}$. For any pair of nodes $v_i, v_j \in \mathcal{V}$ that are not identical, there exists a directed path from v_j to v_i in $\mathcal{V}$ which consists of a set of directed edges, that is, $(v_i, v_{i1}), \cdots, (v_{i,k-1}, v_{ik}), (v_{ik}, v_j) \in \mathcal{E}$. The digraph is strongly connected, if there always exists directed path from v_j to v_i for any two different agents in $\mathcal{V}$. For a node $v_j \in \mathcal{V}$, there always exists directed path for any not identical node $v_i \in \mathcal{V}$,the topology is called containing a directed spanning tree. Let $\mathcal{L}_c = (l_{ij}) \in \mathbb{R}^{N\times N}$ define the Laplacian matrix of the signed digraph $\mathcal{G}$, where $l_{ij} = -a_{ij}$, $i \neq j$ and $l_{ii} = \sum_{\mathrm{j}=1}^{N} |a_{ij}|$.

Definition 1. *[1] The signed digraph $\mathcal{G}$ is structurally balanced if the node set can be separated into $\mathcal{V}_a$ and $\mathcal{V}_b$, satisfying $\mathcal{V}_a \cup \mathcal{V}_b = \mathcal{V}$ and $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$ such that $\forall v_i, v_j \in \mathcal{V}_h (h \in \{a, b\})$, $a_{ij} \geq 0$ and $\forall v_i \in \mathcal{V}_{k_1}, v_j \in \mathcal{V}_{k_2} (\{k_1, k_2\} = \{a, b\})$, $a_{ij} \leq 0$.*

The gauge transformation matrices set is expressed as

$$\mathcal{P} = \{P = \mathrm{diag}(p_1, \cdots, p_n), p_i \in \{\pm 1\}\}.$$

Lemma 1. *[1] The signed digraph $\mathcal{G}$ is structurally balanced if and only if $\exists P \in \mathcal{P}$ such that PAP has all nonnegative elements.*

Assumption 1. *The signed digraph $\mathcal{G}$ is structurally balanced and contains a directed spanning tree.*

Lemma 2. *[27] With Assumption 1, the node set $\bar{\mathcal{V}}$ can be separated into $\bar{\mathcal{V}}_a = \{\bar{v}_1, \cdots, \bar{v}_k\}$ and $\bar{\mathcal{V}}_b = \{\bar{v}_{k+1}, \cdots, \bar{v}_N\}$, such that*
1) The subdigraph $\bar{\mathcal{V}}_a$ is strongly connected.
2) The node in $\bar{\mathcal{V}}_a$ has no neighbors in $\bar{\mathcal{V}}_b$.

Without losing the generality, this paper assumes that $\bar{\mathcal{V}}_a = \{v_1, \cdots, v_k\}$ and $\bar{\mathcal{V}}_b = \{v_{k+1}, \cdots, v_N\}$. Based on Lemma 2, the Laplacian matrix $\mathcal{L}_c$ can be written as

$$\mathcal{L}_c = \begin{bmatrix} \mathcal{L}_{c1} & \mathbf{0} \\ \mathcal{L}_{c2} & \mathcal{L}_{c3} \end{bmatrix},$$

where $\mathcal{L}_{c1} \in \mathbb{R}^{k\times k}$ is the Laplacian matrix of $\bar{\mathcal{V}}_a$, $\mathcal{L}_{c3} \in \mathbb{R}^{(N-k)\times(N-k)}$.

Lemma 3. *[21], [24] The signed digraph $\mathcal{G}_1$ with the Laplacian matrix $\mathcal{L}_{c1}$ is strongly connected, there exists a matrix $\bar{R} = \mathrm{diag}(r_1, \cdots, r_k) \succ 0$, such that*

$$\min_{z^T x=0} x^T \hat{\mathcal{L}}_{c1} x \geq \frac{\lambda_2(\hat{\mathcal{L}}_{c1})}{k} x^T x,$$

where $\hat{\mathcal{L}}_{c1} = \bar{R}\mathcal{L}_{c1} + \mathcal{L}_{c1}^{\mathrm{T}}\bar{R}$ with $r = [r_1, \cdots, r_k]^T$ as the left zero eigenvector for $P_1\mathcal{L}_{c1}P_1$, $\lambda_2(\hat{\mathcal{L}}_{c1})$ is the second minimum eigenvalue of $\hat{\mathcal{L}}_{c1}$, $P_1 z$ represents vector with all positive elements.

Lemma 4. *[28] For the Laplacian matrix $\mathcal{L}_{c3}$, there exists a diagonal matrix $\bar{G} = \mathrm{diag}(g_1, \cdots, g_{N-k}) \succ 0$ such that $\bar{G}\mathcal{L}_{c3} + \mathcal{L}_{c3}^T\bar{G} \succ 0$.*

C. Problem formulation

Consider the following general nonlinear network with N agents defined on a signed digraph $\mathcal{G}$

$$\dot{x}_i = Ax_i(t) + B[u_i(t) + f_i(x_i, t) + d_i(t)], \qquad (1)$$

where $i = 1, \cdots, N$, $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ is the control input, and $d_i \in \mathbb{R}^m$ is the disturbance. $f_i(x_i, t)$ is the smooth function, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices.

Assumption 2. *[29], [19] The function $f_i(x_i, t)$ satisfies*

$$f_i(x_i, t) = \phi_i(x_i, t)w_i + \varsigma_i(t), \qquad (2)$$

where $w_i \in \mathbb{R}^q$ is an unknown constant parameter vector, and $\phi_i(x_i, t) \in \mathbb{R}^{m\times q}$ is a bounded continuous function *matrices, $\varsigma_i(t) \in \mathbb{R}^m$ is a bounded approximation error satisfying $\|\varsigma_i(t)\|_\infty < \upsilon$ with υ being a positive constant.*

Assumption 3. *The pair (A, B) is controllable.*

Remark 1. *Note that Assumption 2 holds, many well known practical physical systems satisfy this condition, such as robot manipulator dynamics [30], and yaw dynamics of ship [31]. In fact, any smoothed nonlinear function can be expressed as an approximate form of a neural network. With Assumption 3, for the following Riccati inequality equation, there exists a solution with $Q \succ 0$.*

$$QA + A^TQ - 2QBB^TQ + I_n \prec 0. \tag{3}$$

Definition 2. *If $\lim_{t\to\infty}\|x_i(t) - p_ip_jx_j(t)\| = 0, \forall i, j \in \mathcal{V}$, then the general nonlinear networks (1) can achieve BS.*

Definition 3. *The general nonlinear networks (1) can achieve bounded BS if there exists a control input such that $\|e_{ij}(t)\| \leq \beta(\|e_{ij}(0)\|, t)+\varepsilon, \forall i, j \in \mathcal{V}$, where ε is a positive constant, $e_{ij}(t) = x_i(t)-p_ip_jx_j(t)$, and $\beta(\cdot, \cdot)$ is a class $\mathcal{KL}$ function.*

III. Bounded disturbances with unknown upper bound

In this section, we will focus on designing a fully distributed observer-based adaptive protocol for the BS problem of a general nonlinear networks (1) with no leader and neural network approximation, while consider the bounded disturbances $d_i(t)$ with an unknown upper bound.

Assumption 4. *Each agent is subject to a bounded disturbance, such that*

$$\|d_i(t)\|_\infty \leq \omega, \ i = 1, \cdots, N,$$

where $\omega > 0$ is an unknown bounded constant.

To solve the BS problem, we first design the state observer as follows

$$\begin{aligned}\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + (z_i(t) + \delta_i(t))BK\eta_i(t),\\ \dot{z}_i(t) =& \eta_i^T(t)\Gamma\eta_i(t),\end{aligned} \tag{4}$$

where $\hat{x}_i \in \mathbb{R}^n$ is the observer state, $\eta_i = \sum_{j=1}^N |a_{ij}|(\hat{x}_i - \text{sgn}(a_{ij})\hat{x}_j)$ is the synchronization error of the observer state, $z_i(t)$ denotes the adaptive gain with $z_i(0) > 0$, $\delta_i(t) = \eta_i^T(t)Q\eta_i(t)$, $K = -B^TQ$ with $Q \succ 0$ being the solution of (3) and $\Gamma = QBB^TQ$.

The following fully distributed observer-based adaptive protocol is proposed to solve the BS problem

$$\begin{aligned}u_i(t) =& -\phi_i(x_i, t)\hat{w}_i(t) + (z_i(t) + \delta_i(t))BK\eta_i(t)\\ &+ K(x_i(t) - \hat{x}_i(t)) + \gamma_i(t)\text{sgn}(K(x_i(t) - \hat{x}_i(t))),\end{aligned} \tag{5}$$

where $\hat{w}_i(t)$ is the estimation of w_i, $K = -B^TQ$ with $Q \succ 0$ is the solution of (3), $\hat{w}_i(t)$ is the solution of following equation

$$\begin{aligned}\dot{\hat{w}}_i(t) =& -m_i\phi_i^T(x_i, t)K(x_i(t) - \hat{x}_i(t)) - m_is_i(\hat{w}_i(t)\\ &- \bar{w}_i(t)),\\ \dot{\bar{w}}_i(t) =& n_is_i(\hat{w}_i(t) - \bar{w}_i(t)),\end{aligned} \tag{6}$$

where m_i and s_i are positive constants, $\gamma_i(t)$ is the adaptive gain defined by

$$\begin{aligned}\dot{\gamma}_i(t) =& \|K(x_i(t) - \hat{x}_i(t))\|_1 - \bar{s}_i(\gamma_i(t) - \hat{\gamma}_i(t)),\\ \dot{\hat{\gamma}}_i(t) =& \bar{s}_i(\gamma_i(t) - \hat{\gamma}_i(t)),\end{aligned} \tag{7}$$

where $\bar{s}_i$ is a positive constant.

Let $\tilde{x}_i = x_i - \hat{x}_i$ be the observer error, $\tilde{w}_i = w_i - \hat{w}_i$, $\tilde{\bar{w}} = w_i - \bar{w}_i$, the closed-loop dynamics can be expressed as

$$\begin{aligned}\dot{\tilde{x}}_i =& (A + BK)\tilde{x}_i + B[\bar{d}_i + \gamma_i\text{sgn}(K\tilde{x}_i) + \phi_i\tilde{w}_i],\\ \dot{\hat{x}}_i =& A\hat{x}_i + (z_i + \delta_i)BK\eta_i,\\ \dot{\tilde{w}}_i =& m_i\phi_i^TK\tilde{x}_i + m_is_i(\hat{w}_i - \bar{w}_i),\\ \dot{\tilde{\bar{w}}}_i =& -n_is_i(\hat{w}_i - \bar{w}_i),\end{aligned} \tag{8}$$

where $\bar{d}_i = \varsigma_i + d_i$, and $\|\bar{d}_i\|_\infty \leq \omega + \upsilon$.

Let $\bar{\hat{x}}_1 = [\hat{x}_1^T, \cdots, \hat{x}_k^T]^T$, $\bar{\hat{x}}_2 = [\hat{x}_{k+1}^T, \cdots, \hat{x}_N^T]^T$ with $\hat{x} = [\bar{\hat{x}}_1^T, \bar{\hat{x}}_2^T]^T$, $\bar{\eta}_1 = [\eta_1^T, \cdots, \eta_k^T]^T$, $\bar{\eta}_2 = [\eta_{k+1}^T, \cdots, \eta_N^T]^T$ with $\eta = [\bar{\eta}_1^T, \bar{\eta}_2^T]^T$, we have

$$\begin{aligned}\bar{\eta}_1 =& (\mathcal{L}_{c1} \otimes I_n)\bar{\hat{x}}_1,\\ \bar{\eta}_2 =& (\mathcal{L}_{c3} \otimes I_n)\bar{\hat{x}}_2 + (\mathcal{L}_{c2} \otimes I_n)\bar{\hat{x}}_1.\end{aligned} \tag{9}$$

Let $\tilde{x} = [\tilde{x}_1^T, \cdots, \tilde{x}_N^T]^T$, and $\gamma = \text{diag}(\gamma_1, \cdots, \gamma_N)$, then the closed-loop dynamics can be expressed as

$$\begin{aligned}\dot{\tilde{x}} =& [I_N \otimes (A + BK)]\tilde{x} + (I_N \otimes B)[\bar{d}\\ &+ (\gamma \otimes I_n)\text{sgn}(K\tilde{x}) + \Phi\tilde{W}],\\ \dot{\bar{\eta}}_1 =& [I_N \otimes A + \mathcal{L}_{c1}(Z_1 + \bar{\delta}_1) \otimes BK]\bar{\eta}_1,\\ \dot{\bar{\eta}}_2 =& [I_N \otimes A + \mathcal{L}_{c3}(Z_2 + \bar{\delta}_2) \otimes BK]\bar{\eta}_2\\ &+ [\mathcal{L}_{c2}(Z_1 + \bar{\delta}_1) \otimes BK]\bar{\eta}_1,\\ \dot{\tilde{W}} =& m_is_i\Phi^T(I_N \otimes K)\tilde{x} + m_i(\hat{W} - \bar{W}),\\ \dot{\tilde{\bar{W}}} =& -n_is_i(\hat{W} - \bar{W}),\end{aligned} \tag{10}$$

where $\text{sgn}(K\tilde{x}) = [\text{sgn}(K\tilde{x}_1)^T, \cdots, \text{sgn}(K\tilde{x}_N)^T]^T$, $\hat{W} = [\hat{w}_1^T, \cdots, \hat{w}_N^T]^T$, $\bar{W} = [\bar{w}_1^T, \cdots, \bar{w}_N^T]^T$, $\tilde{W} = [\tilde{w}_1^T, \cdots, \tilde{w}_N^T]^T$, $\tilde{\bar{W}} = [\tilde{\bar{w}}_1^T, \cdots, \tilde{\bar{w}}_N^T]^T$, $\bar{d} = [\bar{d}_1^T, \cdots, \bar{d}_N^T]^T$, $\Phi = \text{diag}(\phi_1, \cdots, \phi_N)$. $Z_1 = \text{diag}(z_1, \cdots, z_k)$, $Z_2 = \text{diag}(z_{k+1}, \cdots, z_N)$ with $Z = \text{diag}(Z_1, Z_2)$, and $\bar{\delta}_1 = \text{diag}(\delta_1, \cdots, \delta_k)$, $\bar{\delta}_2 = \text{diag}(\delta_{k+1}, \cdots, \delta_N)$ with $\delta = \text{diag}(\bar{\delta}_1, \bar{\delta}_2)$.

Theorem 1. *Suppose that Assumptions 1–4 hold. Under the fully distributed observer-based adaptive protocol (5), the general nonlinear networks (1) on the antagonistic digraph can achieve BS.*

Proof: Considering the following Lyapunov function

$$V_1 = \mu V_{11} + V_{12} \tag{11}$$

with

$$\begin{aligned}V_{11} =& \sum_{i=1}^k \frac{1}{2}r_i(2z_i + \delta_i)\delta_i + \sum_{i=1}^k \frac{1}{2}r_i(z_i - \beta_1)^2,\\ V_{12} =& \sum_{i=k+1}^N \frac{1}{2}g_i(2z_i + \delta_i)\delta_i + \sum_{i=k+1}^N \frac{1}{2}g_i(z_i - \beta_2)^2,\end{aligned} \tag{12}$$

where μ, β_1, β_2 are positive constants, r_i and g_i are given by Lemma 3 and Lemma 4.

Take the time-derivative of V_{11} along closed-loop system

(10)

$$\begin{aligned}\dot{V}_{11}=&\sum_{i=1}^{k}[r_i(z_i+\delta_i)\dot{\delta}_i+r_i\delta_i\dot{z}_i]+\sum_{i=1}^{k}r_i(z_i-\beta_1)\dot{z}_i\\=&\sum_{i=1}^{k}r_i(z_i+\delta_i)\dot{\delta}_i+\sum_{i=1}^{k}r_i(z_i+\delta_i-\beta_1)\dot{z}_i\\=&2\bar{\eta}_1^T\left[\left(Z_1+\bar{\delta}_1\right)R\otimes Q\right]\dot{\bar{\eta}}_1\\&+\sum_{i=1}^{k}r_i(z_i+\delta_i-\beta_1)\eta_i^T\Gamma\eta_i\\=&\bar{\eta}_1^T[\left(Z_1+\bar{\delta}_1\right)R\otimes(QA+A^TQ)\\&-\left(Z_1+\bar{\delta}_1\right)\hat{\mathcal{L}}_{c1}\left(Z_1+\bar{\delta}_1\right)\otimes\Gamma]\bar{\eta}_1\\&+\bar{\eta}_1^T\left[\left(Z_1+\bar{\delta}_1-\beta_1I_k\right)R\otimes\Gamma\right]\bar{\eta}_1,\end{aligned} \tag{13}$$

where $\hat{\mathcal{L}}_{c1}=R\mathcal{L}_{c1}+\mathcal{L}_{c1}^{\mathrm{T}}R$. Define $\Pi=[(Z_1+\bar{\delta}_1)\otimes I_n]\bar{\eta}_1$, we have

$$\begin{aligned}\Pi^T[(Z_1+\bar{\delta}_1)^{-1}P_1r\otimes\mathbf{1}_N]=&\bar{\eta}_1^T(P_1r\otimes\mathbf{1}_N)\\=&\bar{\tilde{x}}_1^T(P_1^{-1}(P_1\mathcal{L}_{c1}^TP_1)r\otimes\mathbf{1}_N)\\=&0,\end{aligned} \tag{14}$$

where we use the fact that $r^TP_1\mathcal{L}_{c1}P_1=0$, with $P_1=\mathrm{diag}(p_1,\cdots,p_k)$. Because each element of P_1r is positive, we can get each element of $P_1(Z_1+\bar{\delta}_1)^{-1}P_1r\otimes\mathbf{1}_N$ is also positive. By Lemma 3, we have

$$\begin{aligned}\Pi^T(\hat{\mathcal{L}}_{c1}\otimes I_n)\Pi\geq&\frac{\lambda_2(\hat{\mathcal{L}}_{c1})}{k}\Pi^T\Pi\\=&\frac{\lambda_2(\hat{\mathcal{L}}_{c1})}{k}\bar{\eta}_1^T[\left(Z_1+\bar{\delta}_1\right)^2\otimes I_n]\bar{\eta}_1.\end{aligned} \tag{15}$$

Substituting (15) into (13), we have

$$\begin{aligned}\dot{V}_{11}\leq&\bar{\eta}_1^T[\left(Z_1+\bar{\delta}_1\right)R\otimes(QA+A^TQ+\Gamma)\\&-\left(\frac{\lambda_2(\hat{\mathcal{L}}_{c1})}{k}(Z_1+\bar{\delta}_1)^2+\beta_1R\right)\otimes\Gamma]\bar{\eta}_1.\end{aligned} \tag{16}$$

Taking the derivative of V_{12} along closed-loop system (10)

$$\begin{aligned}\dot{V}_{12}=&\sum_{i=k+1}^{N}[g_i(z_i+\delta_i)\dot{\delta}_i+g_i\delta_i\dot{z}_i]+\sum_{i=k+1}^{N}g_i(z_i-\beta_2)\dot{z}_i\\=&\sum_{i=k+1}^{N}g_i(z_i+\delta_i)\dot{\delta}_i+\sum_{i=k+1}^{N}g_i(z_i+\delta_i-\beta_2)\dot{z}_i\\=&2\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)G\otimes Q\right]\dot{\bar{\eta}}_2\\&+\sum_{i=k+1}^{N}g_i(z_i+\delta_i-\beta_2)\eta_i^T\Gamma\eta_i\\=&\bar{\eta}_2^T[\left(Z_2+\bar{\delta}_2\right)G\otimes\left(QA+A^TQ\right)\\&-\left(Z_2+\bar{\delta}_2\right)\hat{\mathcal{L}}_{c3}\left(Z_2+\bar{\delta}_2\right)\otimes\Gamma]\bar{\eta}_2\\&-2\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)G\mathcal{L}_{c2}\left(Z_1+\bar{\delta}_1\right)\otimes\Gamma\right]\bar{\eta}_1\\&+\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2-\beta_2I_{N-k}\right)G\otimes\Gamma\right]\bar{\eta}_2,\end{aligned} \tag{17}$$

where $\hat{\mathcal{L}}_{c3}=G\mathcal{L}_{c3}+\mathcal{L}_{c3}^TG$ with λ_0 as the smallest eigenvalue.

By Lemma 4, we have

$$\begin{aligned}&-\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)\hat{\mathcal{L}}_{c3}\left(Z_2+\bar{\delta}_2\right)\otimes\Gamma\right]\bar{\eta}_2\\\leq&-\lambda_0\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)^2\otimes\Gamma\right]\bar{\eta}_2.\end{aligned} \tag{18}$$

By Young's Inequality, we have

$$\begin{aligned}&-2\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)G\mathcal{L}_{c2}\left(Z_1+\bar{\delta}_1\right)\otimes\Gamma\right]\bar{\eta}_1\\\leq&\frac{\lambda_0}{2}\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)^2\otimes\Gamma\right]\bar{\eta}_2\\&+\frac{2\rho_{\max}^2(G\mathcal{L}_{c2})}{\lambda_0}\bar{\eta}_1^T\left[\left(Z_1+\bar{\delta}_1\right)^2\otimes\Gamma\right]\bar{\eta}_1.\end{aligned} \tag{19}$$

Choosing $\mu=\frac{\lambda_2(\hat{\mathcal{L}}_{c1})}{k}(\mu_1+\frac{2\rho_{\max}^2(G\mathcal{L}_{c2})}{\lambda_0})$ and combining (16)-(19), we have

$$\begin{aligned}\dot{V}_1\leq&\mu\bar{\eta}_1^T\left[\left(Z_1+\bar{\delta}_1\right)R\otimes\left(QA+A^TQ+\Gamma\right)\right]\bar{\eta}_1\\&-\bar{\eta}_1^T\left[\left(\mu_1\left(Z_1+\bar{\delta}_1\right)^2+\mu\beta_1R\right)\otimes\Gamma\right]\bar{\eta}_1\\&+\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)G\otimes\left(QA+A^TQ+\Gamma\right)\right]\bar{\eta}_2\\&-\bar{\eta}_2^T\left[\left(\frac{\lambda_0}{2}\left(Z_2+\bar{\delta}_2\right)^2-\beta_2G\right)\otimes\Gamma\right]\bar{\eta}_2.\end{aligned} \tag{20}$$

Choosing $\beta_1\geq\frac{9\mu\lambda_{\max}(R)}{4\mu_1}$ and $\beta_2\geq\frac{9\lambda_{\max}(G)}{2\lambda_0}$. By Young's Inequality, we have

$$\begin{aligned}\dot{V}_1\leq&\mu\bar{\eta}_1^T\left[\left(Z_1+\bar{\delta}_1\right)R\otimes\left(QA+A^TQ-2\Gamma\right)\right]\bar{\eta}_1\\&+\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)G\otimes\left(QA+A^TQ-2\Gamma\right)\right]\bar{\eta}_2\\\leq&0.\end{aligned} \tag{21}$$

Considering the following Lyapunov function

$$\begin{aligned}V_2=&\tilde{x}^T\left(I_N\otimes Q\right)\tilde{x}+\frac{1}{m_i}\tilde{W}^T\tilde{W}+\frac{1}{n_i}\tilde{\bar{W}}^T\tilde{\bar{W}}\\&+\sum_{i=1}^{N}(\gamma_i-\beta_3)^2+\sum_{i=1}^{N}(\hat{\gamma}_i-\beta_3)^2,\end{aligned} \tag{22}$$

where β_3 being a positive constant that need to determine.

The derivative of V_2 along closed-loop system (10) as follows

$$\begin{aligned}\dot{V}_2=&2\tilde{x}^T\left(I_N\otimes Q\right)\dot{\tilde{x}}+\frac{2}{m_i}\tilde{W}^T\dot{\tilde{W}}+\frac{2}{n_i}\tilde{\bar{W}}^T\dot{\tilde{\bar{W}}}\\&+2\sum_{i=1}^{N}(\gamma_i-\beta_3)\dot{\gamma}+2\sum_{i=1}^{N}(\hat{\gamma}_i-\beta_3)\dot{\hat{\gamma}}.\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}&2\tilde{x}^T\left(I_N\otimes Q\right)\dot{\tilde{x}}\\=&\tilde{x}^T\left[I_N\otimes\left(QA+A^TQ-2\Gamma\right)\right]\tilde{x}+2\tilde{x}^T\left(I_N\otimes QB\right)\bar{d}\\&+2\tilde{x}^T\left(I_N\otimes QB\right)\Phi\tilde{W}+2\tilde{x}^T(\gamma\otimes QB)\mathrm{sgn}(K\tilde{x}).\end{aligned} \tag{24}$$

On the one hand,

$$\begin{aligned}&\frac{2}{m_i}\tilde{W}^T\dot{\tilde{W}}+\frac{2}{n_i}\tilde{\bar{W}}^T\dot{\tilde{\bar{W}}}\\=&2\tilde{W}^T\Phi^T\left(I_N\otimes K\right)\tilde{x}+2s_i\tilde{W}^T(\hat{W}-\bar{W})\\&-2s_i\tilde{\bar{W}}^T(\hat{W}-\bar{W})\\=&2\tilde{W}^T\Phi^T\left(I_N\otimes K\right)\tilde{x}+2\sum_{i=1}^{N}s_i\tilde{w}_i^T(\hat{w}_i-\bar{w}_i)\\&-2\sum_{i=1}^{N}s_i\tilde{\bar{w}}_i^T(\hat{w}_i-\bar{w}_i)\\=&2\tilde{W}^T\Phi^T\left(I_N\otimes K\right)\tilde{x}-2\sum_{i=1}^{N}s_i(\hat{w}_i-\bar{w}_i)^2.\end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned}
&2\sum_{i=1}^{N}\left(\gamma_i-\beta_3\right)\dot{\gamma}+2\sum_{i=1}^{N}\left(\hat{\gamma}_i-\beta_3\right)\dot{\hat{\gamma}}\\
=&2\sum_{i=1}^{N}\left(\gamma_i-\beta_3\right)\left[\|K\tilde{x}_i\|_1-\bar{s}_i(\gamma_i-\hat{\gamma}_i)\right]\\
&+2\sum_{i=1}^{N}\bar{s}_i\left(\hat{\gamma}_i-\beta_3\right)\left(\gamma_i-\hat{\gamma}_i\right)\\
=&-2\sum_{i=1}^{N}\beta_3\|K\tilde{x}_i\|_1+2\sum_{i=1}^{N}\gamma_i\|K\tilde{x}_i\|_1\\
&-2\sum_{i=1}^{N}\bar{s}_i(\gamma_i-\hat{\gamma}_i)^2.
\end{aligned}\tag{26}$$

Substituting (24)-(26) to (23), we have

$$\begin{aligned}
\dot{V}_2=&\tilde{x}^T\left[I_N\otimes\left(QA+A^TQ-2\Gamma\right)\right]\tilde{x}\\
&+2\tilde{x}^T(\gamma\otimes QB)\mathrm{sgn}(K\tilde{x})+2\tilde{x}^T\left(I_N\otimes QB\right)\bar{d}\\
&-2\sum_{i=1}^{N}\beta_3\|K\tilde{x}_i\|_1+2\sum_{i=1}^{N}\gamma_i\|K\tilde{x}_i\|_1\\
&-2\sum_{i=1}^{N}\bar{s}_i(\gamma_i-\hat{\gamma}_i)^2-2\sum_{i=1}^{N}s_i(\hat{w}_i-\bar{w}_i)^2.
\end{aligned}\tag{27}$$

By choosing $\beta_3>\theta$, we can get

$$\dot{V}_2\leq\tilde{x}^T[I_N\otimes(QA+A^TQ-2\Gamma)]\tilde{x},\tag{28}$$

where $\theta=\omega+\upsilon$, $\tilde{x}_i^TQB\mathrm{sgn}(K\tilde{x}_i)=-\|K\tilde{x}_i\|_1$. Thus, we can derive that

$$\begin{aligned}
\dot{V}_1+\dot{V}_2\leq&\mu\bar{\eta}_1^T\left[\left(Z_1+\bar{\delta}_1\right)R\otimes\left(QA+A^TQ-2\Gamma\right)\right]\bar{\eta}_1\\
&+\bar{\eta}_2^T\left[\left(Z_2+\bar{\delta}_2\right)G\otimes\left(QA+A^TQ-2\Gamma\right)\right]\bar{\eta}_2\\
&+\tilde{x}^T[I_N\otimes(QA+A^TQ-2\Gamma)]\tilde{x}\\
\leq&0.
\end{aligned}\tag{29}$$

Therefore, we can get V_1 is bounded, $\bar{\eta}_1$, $\bar{\eta}_2$ and z_i are also bounded. We know that $\dot{V}_1\equiv0$ represents $\bar{\eta}_1\equiv0$ and $\bar{\eta}_2\equiv0$, so $\eta\equiv0$. Through LaSalle Invariance principle, we have that η asymptotically converges to zero. In addition, we can get V_2 is bounded and $\int_0^\infty\tilde{x}^T\tilde{x}dt$ exists and is bounded, $\tilde{x}$, $\tilde{W}$, γ_i are also bounded, that $\tilde{x}^T\dot{\tilde{x}}$ is finite. By Barbalat's lemma [32], we can get that $\tilde{x}$ asymptotically converges to zero. Therefore, the problem is solved. ■

Remark 2. *Note that the observer-based adaptive protocol (5) is fully distributed. Compared to literature [19], we consider directed graphs and bounded disturbances. In this paper, we provide the adaptation laws (6) and (7) which can regard as adaptive form for the σ-modification [33]. The adaptation laws (6) and (7) can be seen as extensions of laws in [24], [22], which avoid the overlarge gains and reduce the chattering phenomenon.*

IV. Disturbances from external systems

In this section, we consider the bounded BS problem for a general nonlinear networks described by equation (1), with no leader and neural network approximation. We assume that the disturbances $d_i(t)$ experienced by each agent are produced by an external system described by the following dynamics

$$\begin{cases}\dot{\xi}_i=S\xi_i,\\ d_i=D\xi_i,\end{cases}\tag{30}$$

where $\xi_i\in\mathbb{R}^n$ is the state of the external system (30), $S\in\mathbb{R}^{n\times n}$ and $D\in\mathbb{R}^{m\times n}$ are constant matrices.

Assumption 5. *Suppose that (BD,S) is observable.*

Remark 3. *Note that in Assumption 2 there exist nonlinear small residual errors. However, in this section we do not consider the nonlinear small residual error, that is*

$$\|\varsigma_i(t)\|_\infty=0,i=1,\cdots,N,$$

To solve the external disturbances, we design a disturbance observer as shown below

$$\begin{aligned}
\dot{c}_i=&(S+FBD)(c_i-Fx_i)+FAx_i+FBu_i\\
&+FB\phi_i(x_i,t)\hat{w}_i,\\
\hat{\xi}_i=&c_i-Fx_i,\\
\hat{d}_i=&D\hat{\xi}_i,
\end{aligned}\tag{31}$$

where $c_i\in\mathbb{R}^n$ is the state and $\hat{d}_i\in\mathbb{R}^m$ is the output, $F\in\mathbb{R}^{n\times n}$ is needed to be designed such that $S+FBD$ is Hurwitz. $\hat{x}_i$, δ_i, z_i and η_i are the same as they were defined before. Assumption 5 ensures that there exists a matrix F.

To solve the bounded BS problem, the following fully distributed disturbance observer-based adaptive protocol is proposed

$$\begin{aligned}
u_i(t)=&-\phi_i(x_i,t)\hat{w}_i(t)+(z_i(t)+\delta_i(t))BK\eta_i(t)\\
&+K\left(x_i(t)-\hat{x}_i(t)\right)-\hat{d}_i(t),
\end{aligned}\tag{32}$$

where $\hat{d}_i(t)$ is the estimation of $d_i(t)$, $K=-B^TQ$ with $Q\succ0$ being the solution of (3), $\hat{w}_i(t)$ is the estimation of $w_i(t)$ as follows

$$\dot{\hat{w}}_i(t)=-\bar{m}_i\phi_i^T(x_i,t)K\left(x_i(t)-\hat{x}_i(t)\right)-\bar{m}_i\kappa_i\hat{w}_i(t),\tag{33}$$

where $\bar{m}_i$ and κ_i are positive constants.

Let $\tilde{x}_i=x_i-\hat{x}_i$, $e_i=\xi_i-\hat{\xi}_i$ and $\tilde{w}_i=w_i-\hat{w}_i$, then one can get the closed-loop system as follows

$$\begin{aligned}
\dot{\tilde{x}}_i=&(A+BK)\tilde{x}_i+B[\phi_i(x_i,t)\tilde{w}_i+De_i],\\
\dot{e}_i=&(S+FBD)e_i+FB\phi_i(x_i,t)\tilde{w}_i,\\
\dot{\tilde{w}}_i=&\bar{m}_i{\phi_i}^T(x_i,t)K\tilde{x}_i+\bar{m}_i\kappa_i\hat{w}_i.
\end{aligned}\tag{34}$$

Let $\tilde{x}=[\tilde{x}_1^T,\cdots,\tilde{x}_N^T]^T$, $e=[e_1^T,\cdots,e_N^T]^T$. The one can get the following form of closed-loop system

$$\begin{aligned}
\dot{\tilde{x}}=&[I_N\otimes(A+BK)]\tilde{x}+(I_N\otimes B)\Phi\tilde{W}+(I_N\otimes BD)e,\\
\dot{e}=&[I_N\otimes(S+FBD)]e+(I_N\otimes FB)\Phi\tilde{W},\\
\dot{\tilde{W}}=&\bar{m}_i\Phi^T(I_N\otimes K)\tilde{x}+\bar{m}_i\kappa_i\hat{W},
\end{aligned}\tag{35}$$

where $\tilde{W}=[\tilde{w}_1^T,\cdots,\tilde{w}_N^T]^T$, $\Phi=\mathrm{diag}(\phi_1,\cdots,\phi_N)$, $\hat{W}=[\hat{w}_1^T,\cdots,\hat{w}_N^T]^T$.

Theorem 2. *Suppose that Assumptions 1-3 and Assumption 5 are satisfied. Under the fully distributed disturbance observer-based adaptive protocol (32), bounded BS of the general nonlinear networks (1) on the antagonistic digraph can be achieved.*

Proof: Considering the Lyapunov function

$$V_3 = \tilde{x}^T\left(I_N \otimes Q\right)\tilde{x} + \frac{1}{\bar{m}_i}\tilde{W}^T\tilde{W} + \bar{\mu}e^T(I_N \otimes \bar{Q})e, \quad (36)$$

where $\bar{\mu}$ is a positive constant to be determined, $\bar{Q} \succ 0$ is the solution of $(S+FBD)^T\bar{Q} + \bar{Q}(S+FBD) \prec 0$.

Taking the derivative of V_3 along closed-loop system (35), we have

$$\begin{aligned}\dot{V}_3 =& 2\tilde{x}^T\left(I_N \otimes Q\right)\dot{\tilde{x}} + \frac{2}{\bar{m}_i}\tilde{W}^T\dot{\tilde{W}} + 2\bar{\mu}e^T(I_N \otimes \bar{Q})\dot{e}, \\ =& \tilde{x}^T\left[I_N \otimes \left(QA + A^TQ - 2\Gamma\right)\right]\tilde{x} + 2\kappa_i\tilde{W}^T\hat{W} \\ &+ 2\bar{\mu}e^T[I_N \otimes \bar{Q}(S+FBD)]e + 2\tilde{x}^T\left(I_N \otimes QBD\right)e \\ &+ 2\bar{\mu}e^T(I_N \otimes \bar{Q}FB)\Phi\tilde{W}.\end{aligned} \quad (37)$$

Note that

$$\begin{aligned}2\tilde{x}^T(I_N \otimes QBD)e \le& \sum_{i=1}^N \tilde{x}_i^T\tilde{x}_i + \sum_{i=1}^N e_i^TD^TB^TQ^2BDe_i \\ \le& \frac{1}{2}\sum_{i=1}^N \|\tilde{x}_i\|^2 + 2\tau_1\sum_{i=1}^N \|e_i\|^2,\end{aligned} \quad (38)$$

where $\tau_1 = \lambda_{\max}(D^TB^TQ^2BD)$.

$$\begin{aligned}2\kappa_i\tilde{W}^T\hat{W} =& 2\kappa_i\tilde{W}^TW - 2\kappa_i\tilde{W}^T\tilde{W} \\ =& 2\kappa_i\sum_{i=1}^N \tilde{w}_i^Tw_i - 2\kappa_i\sum_{i=1}^N \tilde{w}_i^T\tilde{w}_i \\ \le& \kappa_i\sum_{i=1}^N \|w_i\|^2 - \kappa_i\sum_{i=1}^N \|\tilde{w}_i\|^2,\end{aligned} \quad (39)$$

where $W = [w_1^T, \cdots, w_N^T]^T$ and κ_i is to be determined later.

$$\begin{aligned}&2e^T[I_N \otimes \bar{Q}(S+FBD)]e \\ =& 2\sum_{i=1}^N e_i^T\bar{Q}(S+FBD)e_i \\ \le& -\tau_2\sum_{i=1}^N {e_i}^Te_i = -\tau_2\sum_{i=1}^N \|e_i\|^2,\end{aligned} \quad (40)$$

where $\tau_2 = \lambda_{\min}(-2\bar{Q}(S+FBD))$.

$$\begin{aligned}2e^T(I_N \otimes \bar{Q}FB)\Phi\tilde{W} =& 2\sum_{i=1}^N e_i^T\bar{Q}FB\phi_i\tilde{w}_i \\ \le& \frac{\tau_2}{2}\sum_{i=1}^N e_i^Te_i + \frac{2}{\tau_2}\sum_{i=1}^N \|\phi_i\|^2\|\bar{Q}FB\|^2\|\tilde{w}_i\|^2 \\ \le& \frac{\tau_2}{2}\sum_{i=1}^N \|e_i\|^2 + \tau_3\sum_{i=1}^N \|\tilde{w}_i\|^2,\end{aligned} \quad (41)$$

where $\tau_3 = \frac{2}{\tau_2}\|\bar{Q}FB\|^2\phi_{\max}$, and $\phi_{\max}$ is the upper bound of $\|\phi_i\|^2$.

By choosing $\bar{\mu} = \frac{(4\tau_1+2\alpha_1)}{\tau_2}$, $\kappa_i = \bar{\mu}\tau_3 + \alpha_2$, then we have

$$\begin{aligned}\dot{V}_3 \le& \tilde{x}^T[I_N \otimes (QA + A^TQ - 2\Gamma)]\tilde{x} + \frac{1}{2}\sum_{i=1}^N \|\tilde{x}_i\|^2 \\ &- \alpha_1\sum_{i=1}^N \|e_i\|^2 - \alpha_2\sum_{i=1}^N \|\tilde{w}_i\|^2 + \kappa_i\sum_{i=1}^N \|w_i\|^2 \\ \le& -0.5\sum_{i=1}^N \|\tilde{x}_i\|^2 - \alpha_1\sum_{i=1}^N \|e_i\|^2 - \alpha_2\sum_{i=1}^N \|\tilde{w}_i\|^2 \\ &+ \kappa_i\sum_{i=1}^N \|w_i\|^2 \\ \le& -\alpha_3V_3 + \kappa_i\sum_{i=1}^N \|w_i\|^2,\end{aligned} \quad (42)$$

where α_1, α_2 are positive numbers, and $\alpha_3 = \frac{\min\{1/2,\alpha_1,\alpha_2\}}{\max\{\lambda_{\max}(Q),1/\bar{m}_i,\bar{\mu}\cdot\lambda_{\max}(\bar{Q})\}}$. Let $\alpha_4 = \kappa_i\sum_{i=1}^N \|w_i\|^2$.

Then the following inequality holds

$$\dot{V}_3 \le -\alpha_3\left(V_3 - \frac{\alpha_4}{\alpha_3}\right),$$

which implies

$$V_3 \le e^{-\alpha_3 t}\left(V_3(0) - \frac{\alpha_4}{\alpha_3}\right) + \frac{\alpha_4}{\alpha_3}.$$

Hence, one has

$$\begin{aligned}\lambda_{\min}(Q)\|\tilde{x}\|^2 \le& \tilde{x}^T\left(I_N \otimes Q\right)\tilde{x} \\ \le& V_3 \\ \le& e^{-\alpha_3 t}\left(V_3(0) - \frac{\alpha_4}{\alpha_3}\right) + \frac{\alpha_4}{\alpha_3},\end{aligned}$$

and bounded BS of the general nonlinear networks (1) is achieved. Therefore, the problem is solved. ■

Remark 4. *When the disturbances are generated by exosystems (30), BS is difficult to achieve, and bounded BS can be achieved by the fully distributed disturbance observer-based adaptive protocol (32). The references [24], [22] did not consider this situation and only addressed the BS problem.*

V. Two numerical examples

This section will provide two numerical simulation examples to demonstrate the effectiveness of the theoretical results obtained. We consider a third-order nonlinear network consisting of six agents. The system matrix for the network is expressed as follows

$$A = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix},\ B = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

and nonlinear functions $f_i(x_i,t)$ are assumed to be

$$f_i(x_i,t) = x_{i3}\sin(x_{i2}) + x_{i1}\cos(x_{i3}).$$

Then, we choose Q as the solution to the Riccati equation (3)

$$Q = \begin{bmatrix}5.1201 & 5.7017 & 2.1169\\5.7017 & 10.4176 & 4.3302\\2.1169 & 4.3302 & 3.0545\end{bmatrix}.$$

It follows that $K = [-2.1169, -4.3302, -3.0545]$.

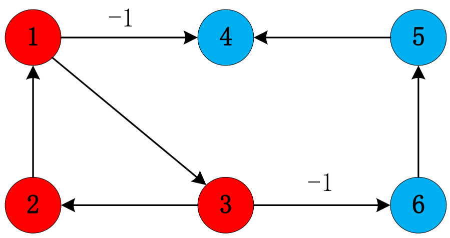

Fig. 1. The signed digraph with six agents, the digraph contains a directed spanning tree and a strongly connected subdigraph. Edges without markers all have a weight of 1.

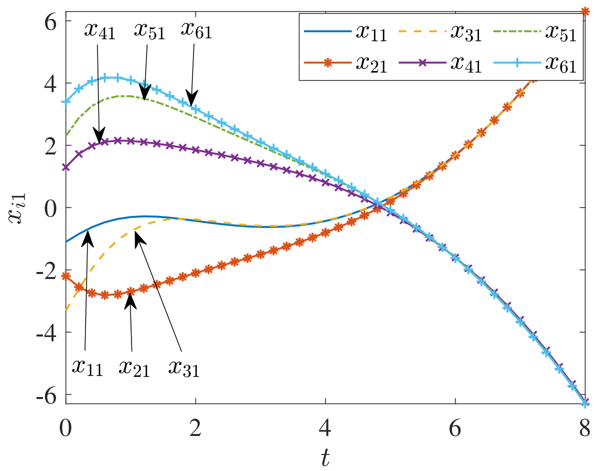

Fig. 2. Simulation results for the states x_{i1} of the network (1) under the fully distributed observer-based adaptive protocol (5).

Example 1. *This example considers the case where each node is affected by bounded disturbances, which is expressed as*

$$d_i(t) = \cos(x_{i2} + 1) + \sin(x_{i1} + x_{i2}).$$

Through the distributed state observer (4) and the adaptive controller (5), the adaptive gain z_i is mentioned in (4) with $z_i(0) > 0$, the adaptive control gain γ_i is mentioned in (7) with $\bar{s}_i = 1$. In addition, the initial values of z_i and γ_i are random. The basic function is

$$\phi_i = [\sin(x_{i1} + x_{i2}), \sin(x_{i1} - x_{i2}), \cos(1 + x_{i2})],$$

and $w_i = [-1, 1, 1]^T$ with $s_i = 1$. The interaction digraph is shown in Fig. 1. According to Theorem 1, the network (1) with the fully distributed observer-based adaptive protocol (5) achieves the BS.

The state trajectories of x_i are shown in Fig. 2, Fig. 3 and Fig. 4 that achieve BS under the fully distributed observer-based adaptive protocol (5). The states of agents diverge to two sides. The tracking errors $\tilde{x}_i$ are shown in Fig. 5, Fig. 6 and Fig. 7. We can see that under the bounded disturbance condition, the tracking error of $\tilde{x}_i$ asymptotically approaches zero, which is consistent with our theoretical analysis. The adaptive gain γ_i is shown in Fig. 8.

Example 2. *In this example, we consider a scenario where each agent in the third-order nonlinear network is affected*

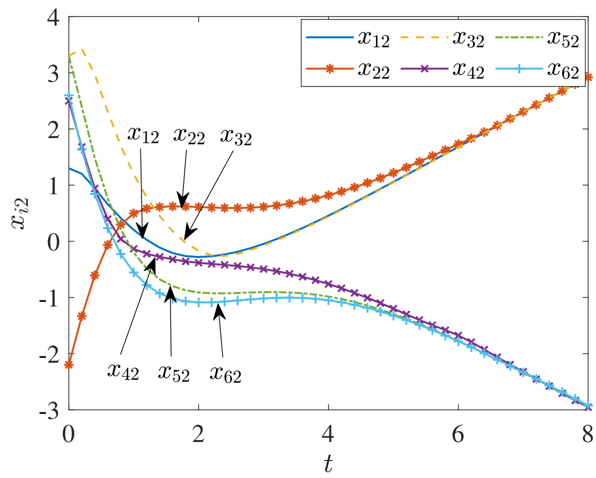

Fig. 3. Simulation results for the states x_{i2} of the network (1) under the fully distributed observer-based adaptive protocol (5).

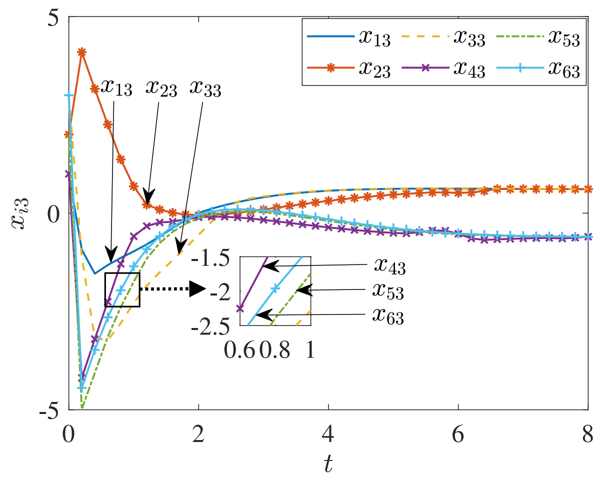

Fig. 4. Simulation results for the states x_{i3} of the network (1) under the fully distributed observer-based adaptive protocol (5).

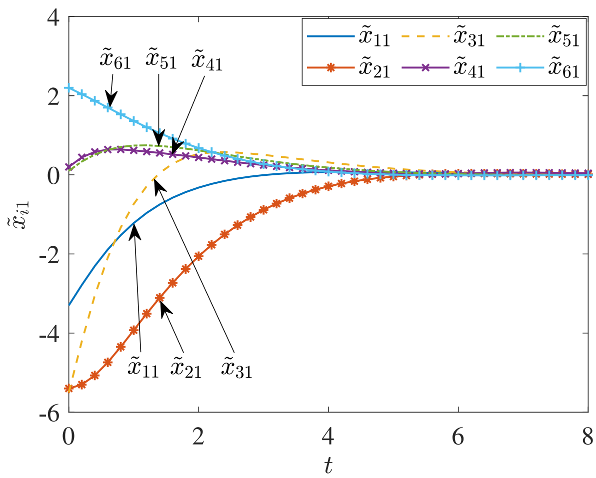

Fig. 5. The tracking errors $\tilde{x}_{i1}$ under the fully distributed adaptive protocol (5).

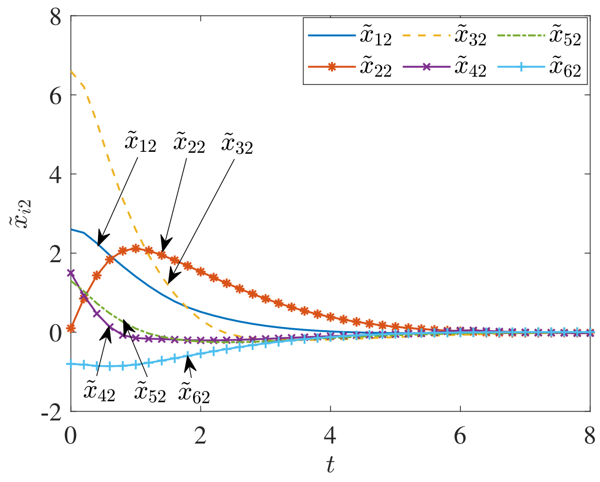

Fig. 6. The tracking errors $\tilde{x}_{i2}$ under the fully distributed adaptive protocol (5).

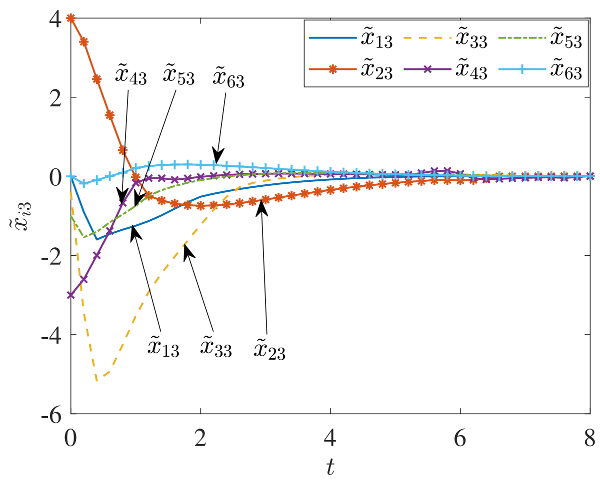

Fig. 7. The tracking errors $\tilde{x}_{i3}$ under the fully distributed adaptive protocol (5).

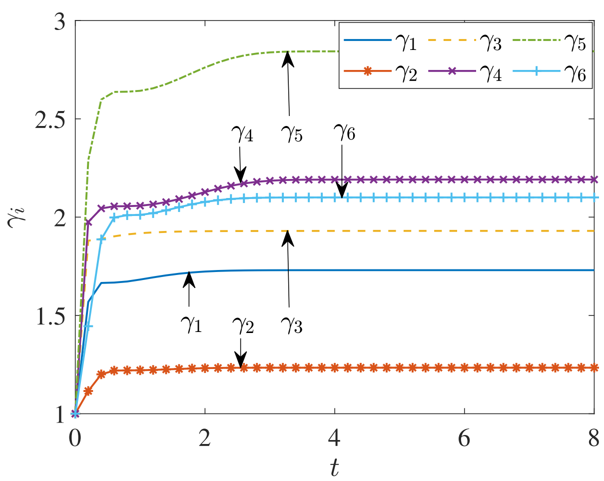

Fig. 8. The daptive gain γ_i under the adaptive protocol (7).

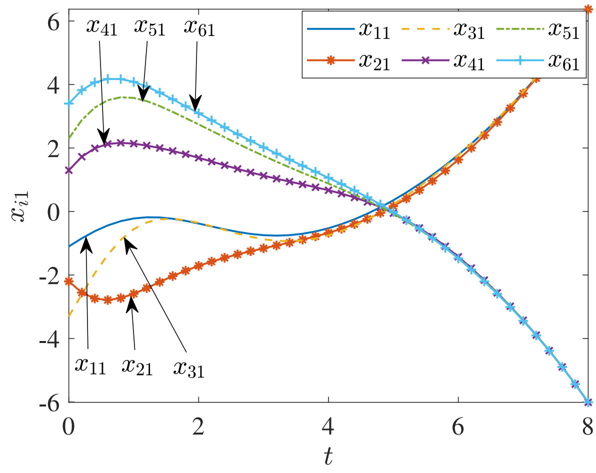

Fig. 9. Simulation results for the states x_{i1} of the network (1) under the fully distributed disturbance observer-based adaptive protocol (32).

by external disturbances described by equation (30), where

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, D = [0, 0, 1].$$

The basic function is

$$\phi_i = [\sin(x_{i1}x_{i3}), \sin(x_{i1} - x_{i3}), \cos(x_{i3}) + 1],$$

and $w_i = [0.5i, i - 0.5, 2]^T$. *Choosing*

$$F = \begin{bmatrix} 0 & 0 & -19 \\ 0 & 0 & -24 \\ 0 & 0 & -7 \end{bmatrix}$$

to solve the linear matrix inequality $(S + FBD)^T\bar{Q} + \bar{Q}(S + FBD) \prec 0$. *According to Theorem 2, the state trajectories of* x_i *are shown in Fig. 9, Fig. 10 and Fig. 11 that achieve bounded BS under the fully distributed observer-based adaptive protocol (32). The states of agents diverge to two sides. The tracking errors* $\tilde{x}_i$ *are shown in Fig. 12, Fig. 13 and Fig. 14. We can see that under the condition of external perturbation, the absolute value of the tracking errors* $\tilde{x}_i$ *is less than a bounded constant close to zero, the bounded BS is achieved, which is consistent with our theoretical analysis.*

VI. Concluding Remarks

This paper addressed the BS problem for general nonlinear networks with neural networks approximation and external disturbances in signed digraphs. For the agents subject to bounded disturbances with an unknown upper bound, when the signed digraph contains a directed spanning tree, designed the fully distributed observer-based adaptive protocol to make network achieve leadless BS. When the agents are affected by disturbances generated by a external system, the fully distributed disturbance observer-based adaptive protocol is designed to the general nonlinear nonlinear networks with no leader and neural networks approximation, and the closed-loop network achieves bounded BS. Finally, the theoretical results are verified by two numerical simulation examples. Future research can be carried out on switching topologies or cluster BS.

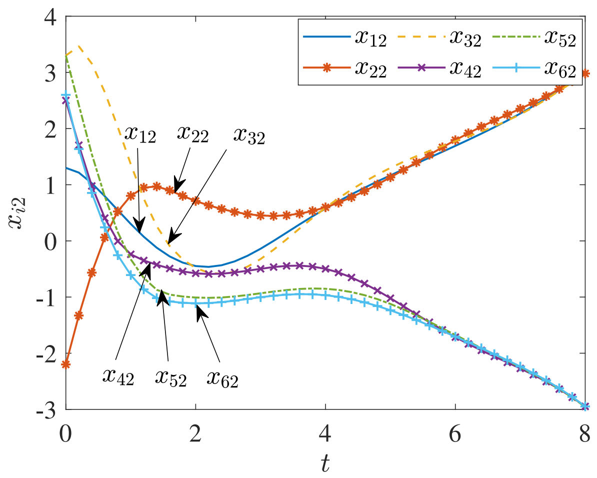

Fig. 10. Simulation results for the states x_{i2} of the network (1) under the fully distributed disturbance observer-based adaptive protocol (32).

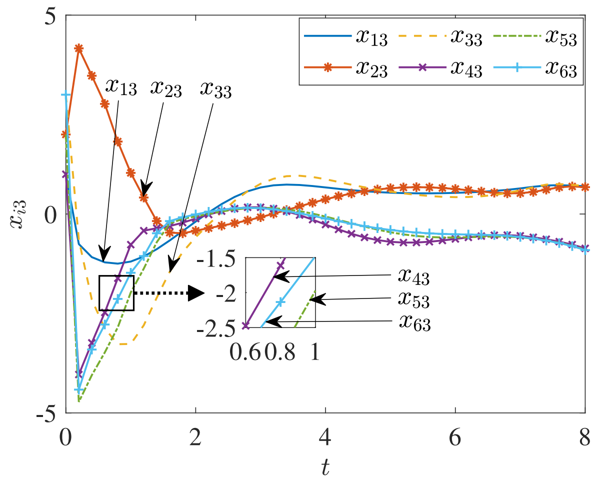

Fig. 11. Simulation results for the states x_{i3} of the network (1) under the fully distributed disturbance observer-based adaptive protocol (32).

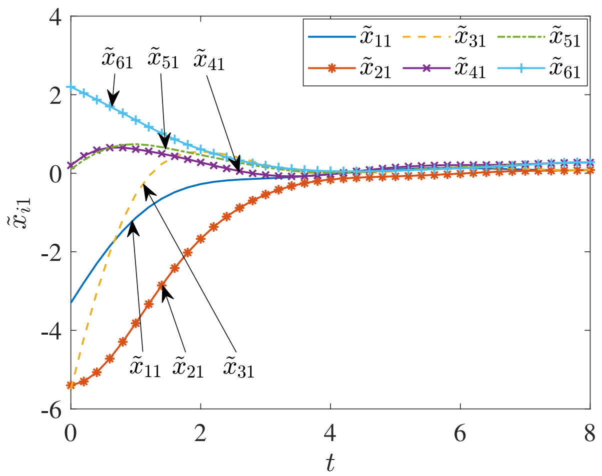

Fig. 12. The tracking errors $\tilde{x}_{i1}$ under the fully distributed disturbance observer-based adaptive protocol (32).

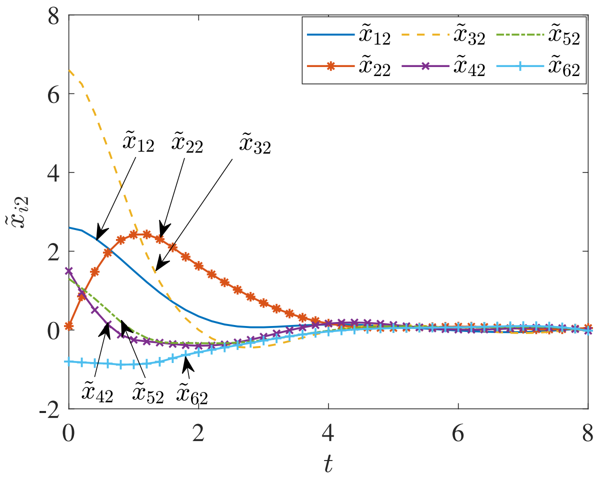

Fig. 13. The tracking errors $\tilde{x}_{i2}$ under the fully distributed disturbance observer-based adaptive protocol (32).

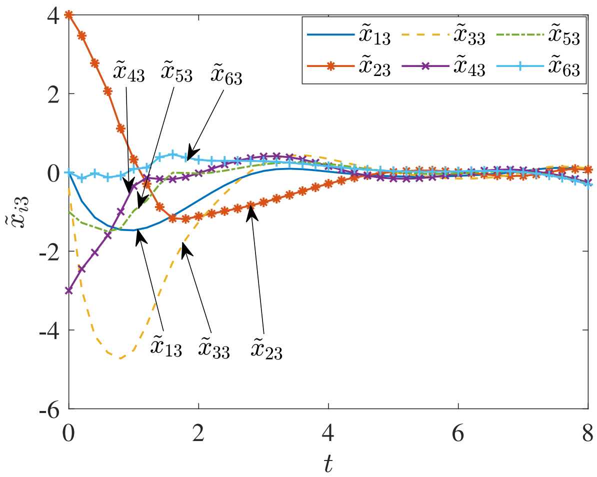

Fig. 14. The tracking errors $\tilde{x}_{i3}$ under the fully distributed disturbance observer-based adaptive protocol (32).

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