Stochastic Vendor-buyer Inventory System with Defective Items

Shusheng Wu, Jinyuan Liu, Pin-Yi Lo

Abstract—**With a rational criterion, we developed an analytical method that proves the uniqueness for the optimal solution for stochastic vendor-buyer inventory model with defective items. Our method approach implies a vigorous approach to locate the optimal solution. The doubtful results that derived by the iterative method are demonstrated by the numerical example of the published paper is examined to indicate the virtue of our approach. Our findings offer an analytical basis for stochastic vendor-buyer inventory models with defective items that will facilitate decision-makers to realize the solution mechanism to find the minimum average cost.**

Index Terms—**Integratedvendor-buyer inventory model, Minimax distribution free approach, Inventory systems, Analytical solution procedure**

I. INTRODUCTION

THERE are many inventory models had been developed by researchers. For example, with a short period of time price discount, Hsu and Yu [1] studied economic ordering quantity systems with imperfective items. For stochastic lead time demand and defective goods, Ho [2] examined integrated inventory systems under a minimax distribution free procedure. Under finite-range stochastic lead-time environment, Nasri et al. [3] considered quality-adjusted inventory systems to reduce the lead time variability for investment. With imperfect quality, Maddah and Jaber [4] solved the economic order quantity.Under the consideration of stocklevel related demand, warrantyperiod, inspection planning, and imperfect production, Chung and Wee [5] developed integrated deteriorating and production inventory systems. Under procedure unreliability, Chung [6] studied buyer and vendor integrated inventory systems to find necessary and sufficient criteria for the existence of the optimal solution. Under procurement cost with variable leadtime, Chandra and Grabis [7] examined inventory policies. Under shortage backlogging and imperfect quality, Wee et al. [8] considered inventory systems for optimal solution of ordering product. With heuristic solution procedure and crashable lead time, Hoque [9] developed integrated buyer and vendor inventory systems. Wang and Hill [10] solved lead time uncertainty effect for recursive behavior under safety stock reduction. With production imperfect quality, Papachristos and Konstantaras [11] constructed economic production quantity systems. For flexible inventory models, Wadhwa et al. [12] examined flexibility to reduce lead time. Under process unreliability, Huang [13] studied buyer and vendor integrated production inventory systems to locate the optimal solution. Under product with imperfect quality, Huang [14] developed inventory systems to integrate buyer and vendor cooperation. Under remanufacturing and stochastic inventory control, Inderfurth and Van Der Laan [15] considered policy improvement and leadtime effects for inventory systems. Under variable lead time, Ouyang and Wu [16] solved mixed inventory systems through a minimax distribution free process. With lost sales for variable lead and backorders, Ouyang et al. [17] studied mixture inventory systems. Paknejad et al. [18] examined a continuous review system with defective units. For several inventory systems, Moon and Gallego [19] constructed solution process by distribution free procedures. Ben-Daya and Raouf [20] derived inventory systems where the decision variable is lead time. Under controllable lead time, Liao and Shyu [21] solved stochastic inventory systems. Under randomly and long lead time, Foote et al. [22] developed inventory ordering problems for heuristic process. Under imperfect production processes, Rosenblatt and Lee [23] considered economic production cycles. Porteus [24] studied setup cost reduction, process quality improvement, and optimal lot sizing for inventory systems. Gupta [25] examined inventory models for the effect of lead time. With Gamma distribution demand during lead time, Das [26] obtained approximated solution for inventory systems. We study the integrated vendor-buyer inventory models with defective items, controllable lead time, and sublot sampling inspection policy that was considered by Wu and Ouyang [27]. Following this research trend, Luo et al. [28] developed a new solution approach to locate the optimal solution for inventory systems with defective units and stochastic demand. For stochastic distribution free inventory systems, Lin [29] constructed a new iterative algorithm to find the minimum solution. For inventory systems with defective items, lead time, fulfill rate, and ordering quantity, Tung et al. [30] examined previous published inventory models. Under the conditions of crashable lead time and sublot sampling examination process, Wu et al. [31] extended Wu and Ouyang [27] to build integrated vendor-buyer inventory models. With defective items, Ouyang et al. [32] generalized Wu and Ouyang [27] to develop integrated vendor-buyer inventory systems. Those mentioned five papers had extended this kind of inventory models to a more general and practical assumptions. However, the above

Manuscript received August 8, 2023; revised January 18, 2024.
Shusheng Wu is an instructor in the School of Mathematics and Science Teaching Center, Weifang University of Science and Technology, Weifang 262799, China (e-mail: wushusheng@wfust.edu.cn).
Jinyuan Liu is an Associate Professor in the School of General Studies, Weifang University of Science and Technology, Weifang 262799, China (e-mail: shgljy@126.com).
Pin-Yi Lo is retired from the Central Police University, Taoyuan 33304, Taiwan (email: pin-yi@yahoo.com.tw)

mentioned five papers are too eager to develop new inventory models and then they did not provide a deeply examination for the solution procedure of Wu and Ouyang [27]. We will follow this research tendency to develop our analytical solution approach to find the optimal solution. Our results will help practitioners to construct new models to meet the challenge in the real world application.

II. Assumptions and Notation

To explain our method, for examples, we consider the integrated vendor-buyer inventory model of Wu and Ouyang [27], so that we use the following assumptions and notation.

Assumptions:

(1) There is a single-vendor and single-buyer for a single product in this model.

(2) The vendor's production rate of expected non-defective items is finite and greater than the buyer's demand, i.e., $(1-M_\rho)P > D$.

(3) The reorder point r satisfies $r = \mu L + k\sigma\sqrt{L}$, where k is the safety factor that is a decision variable.

(4) If we let

$$L_0 = \sum_{j=1}^{n} b_j, \tag{2.1}$$

and

$$L_i = \sum_{i+1}^{n} b_j + \sum_{j=1}^{i} a_j, \tag{2.2}$$

so the lead time crashing cost $R(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$, is given by

$$R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j). \tag{2.3}$$

(5) The components of lead time are crashed one at a time starting with component 1 (because it has the minimum unit crashing cost), and then component 2, etc.

(6) The i-th component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Further, for convenience, we rearrange c_i such that $c_1 \le c_2 \le \cdots \le c_n$.

(7) The lead time L has n mutually independent components.

(8) Replenishments are made whenever the inventory level (based on the number of non-defective items) falls to the reorder point r.

(9) Inventory is continuously reviewed.

(10) Upon arrival of an order, all the items are inspected and defective items in each lot will be returned to the vendor at the time of delivery of the next lot.

(11) We assume that the number of defective items in an arriving order of size Q is a binomial random variable with parameters Q and p, where p $(0 \le p \le 1)$ represents the defective rate in an order lot.

(12) An arriving order may contain some defective items.

Notations:

δ is the buyer's proportion of quantity inspected per shipment, $0 < \delta \le 1$.

Q is the order quantity of the buyer (decision variable).

ρ is the defective rate in an order lot (independent of lot size) which is a random variable and has a probability density function (p.d.f.) $g(\rho)$, $0 < \rho < 1$, with finite mean M_ρ.

β is the fraction of the demand during the stock-out period will be backordered, $\beta \in [0,1]$.

w is the buyer's unit treatment cost for uninspected defective items.

y is the buyer's unit inspection cost.

π_0 is the buyer's profit per unit.

π is the buyer's shortage cost per unit short.

h_b is the buyer's non-defective (including uninspected defective items) holding cost per item per unit time.

h_v is the vendor's holding cost per item per unit time.

F is the transportation cost per delivery.

A_v is the vendor's set-up cost per set-up.

A_b is the buyer's ordering cost per order.

P is the production rate of the vendor.

D is the expected demand per unit time on the buyer.

Z is the number of defective items among the inspected $\frac{\delta Q}{m}$ units, a random variable.

m is the number of shipments delivered from the vendor to the buyer in one production cycle, a positive integer (decision variable).

r is the reorder point of the buyer (decision variable).

L is the length of lead time for the buyer (decision variable).

X is the lead time demand which has a p.d.f. $f(x)$ with finite mean DL and standard deviation $\sigma\sqrt{L}$, where σ denotes the standard derivation of the demand per unit time, for lead time L.

$E(\cdot)$ is the mathematical expectation.

x^+ is the maximum value of x and 0, $x^+ = \max\{x,0\}$.

III. Review of Previous Results

For distribution-free model, we directly quote the objective function of Wu and Ouyang [27],

$$\begin{aligned} EAC^u(Q,k,L) &= \frac{AD}{Q(1-E(p))} + (Q-1)h'\frac{E(p-p^2)}{1-E(p)}, \\ &+ \frac{h}{2}\left\{Q(1-E(p)) + Q\frac{E(p^2)-E^2(p)}{1-E(p)} + \frac{E(p-p^2)}{1-E(p)}\right\}, \\ &+ h\sqrt{L}\sigma\left\{k + \frac{1-\beta}{2}\left(\sqrt{1+k^2}-k\right)\right\} + \frac{Dv}{1-E(p)}, \\ &+ \frac{D(\pi+\pi_0(1-\beta))}{2Q(1-E(p))}\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right), \\ &+ \frac{D}{Q(1-E(p))}\left(c_i(L_{i-1}-L) + \sum_{j=1}^{i-1} c_j(b_j-a_j)\right), \end{aligned} \tag{3.1}$$

for $L \in [L_i, L_{i-1}]$, where $EAC^u(Q,k,L)$ is an estimated least upper bound for the exact model, $EAC(Q,k,L)$. Wu and

Ouyang [27] claimed that $EAC^u(Q,k,L)$ is a concave (convex down) function for the restricted sub-domain $L\in[L_i,L_{i-1}]$ such that the minimum values will be attained on the two boundary point, the left boundary L_i or the right boundary L_{i-1}. Without lose of generality, we will adopt L to represent L_i or L_{i-1} to simplify the expressions. For the approximated model of $EAC^u(Q,k,L)$, they evaluated the first partial derivative for two variables, Q and k. Based on the results of $\frac{\partial}{\partial k}EAC^u(Q,k,L)=0$, and $\frac{\partial}{\partial Q}EAC^u(Q,k,L)=0$, Wu and Ouyang [27] implied that

$$Q=\sqrt{\frac{2D}{h\delta}}\left[A+c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(b_j-a_j),\right.$$
$$\left.+\frac{\pi+\pi_0(1-\beta)}{2}\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right)\right]^{1/2}, \quad (3.2)$$

and

$$\frac{2\sqrt{1+k^2}}{\sqrt{1+k^2}-k}=1-\beta+\frac{D(\pi+\pi_0(1-\beta))}{hQ(1-E(p))}, \quad (3.3)$$

where

$$\delta=1-2E(p)+E(p^2)+2\frac{h'}{h}E(p(1-p)), \quad (3.4)$$

is an abbreviation.

Wu and Ouyang [27] mentioned that through iterative algorithms, their optimal solution can be derived. However, we may assert that their two sequences, iterated term by term by Equations (3.1) and (3.2) repeatedly, are lack of support to verify their converges. Moreover, even if their two sequence both are convergent, why their limitsare the optimal solutions for ordering quantity and safety factor which were not examined by Wu and Ouyang [27]. In the next section, we will construct our solution procedure to illustrate that there is a pair of an optimal order quantity and a safety factor under a reasonable condition to derive our feasible domain with a lower bound and an upper bound.

IV. Our Revision

We simplify the expression in Equations (3.2) and (3.3) as follows,

$$Q^2=\alpha_1+\alpha_2\left(\sqrt{1+k^2}-k\right), \quad (4.1)$$

and

$$\frac{\sqrt{1+k^2}-k}{\sqrt{1+k^2}}=\frac{2\alpha_3 Q}{(1-\beta)\alpha_3 Q+\alpha_4}, \quad (4.2)$$

where we assume the following four abbreviations to simplify the expression,

$$\alpha_1=\frac{2D}{h\delta}\left(A+c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(b_j-a_j)\right), \quad (4.3)$$

$$\alpha_2=\frac{D}{h\delta}(\pi+\pi_0(1-\beta))\sigma\sqrt{L}, \quad (4.4)$$

$$\alpha_3=h(1-E(p)), \quad (4.5)$$

and

$$\alpha_4=D(\pi+\pi_0(1-\beta)). \quad (4.6)$$

Owing to Equation (4.2), if we compute

$$1-\frac{\sqrt{1+k^2}-k}{\sqrt{1+k^2}}=1-\frac{2\alpha_3 Q}{(1-\beta)\alpha_3 Q+\alpha_4}, \quad (4.7)$$

to imply that

$$\frac{k}{\sqrt{1+k^2}}=\frac{\alpha_4-(1+\beta)\alpha_3 Q}{(1-\beta)\alpha_3 Q+\alpha_4}. \quad (4.8)$$

From Equation (4.8), with $k\ge 0$, we recall that k is the safe factor, and then we derive an upper bound for the ordering quantity, Q, as following,

$$\alpha_4\ge(1+\beta)\alpha_3 Q. \quad (4.9)$$

If we take square on both side of Equation (4.8), and then minus one, it yields that

$$\frac{1}{k^2}=\frac{4\alpha_3 Q(\alpha_4-\beta\alpha_3 Q)}{(\alpha_4-(1+\beta)\alpha_3 Q)^2}. \quad (4.10)$$

According to the restriction of Equation (4.9), we know that $\alpha_4-\beta\alpha_3 Q>0$ such that the numerator of Equation (4.10) is positive and then the right-hand side of Equation (4.10) is positive which is consistent with the left-hand side of Equation (4.10). Consequently, we derive a relationship to denote the safety factor k as a function in the ordering quantity Q in the following,

$$k=\frac{\alpha_4-(1+\beta)\alpha_3 Q}{2\sqrt{\alpha_3 Q(\alpha_4-\beta\alpha_3 Q)}}. \quad (4.11)$$

Based on Equation (4.11), we derive that

$$\sqrt{1+k^2}=\left(1+\frac{(\alpha_4-(1+\beta)\alpha_3 Q)^2}{4\alpha_3 Q(\alpha_4-\beta\alpha_3 Q)}\right)^{1/2},$$
$$=\frac{\alpha_4+(1-\beta)\alpha_3 Q}{2\sqrt{\alpha_3 Q(\alpha_4-\beta\alpha_3 Q)}}. \quad (4.12)$$

If we substitute Equations (4.11) and (4.12) into Equation (4.1), then it implies that

$$Q^2=\alpha_1+\alpha_2\sqrt{\frac{\alpha_3 Q}{\alpha_4-\beta\alpha_3 Q}}. \quad (4.13)$$

From Equation (4.13), we have a lower bound for Q so that

$$Q>\sqrt{\alpha_1}. \quad (4.14)$$

By Equation (4.9) again, it yields that

$$\alpha_4-\beta\alpha_3 Q\ge\alpha_3 Q, \quad (4.15)$$

such that we obtain that

$$\sqrt{\frac{\alpha_3 Q}{\alpha_4-\beta\alpha_3 Q}}\le 1, \quad (4.16)$$

and then owing to Equation (4.13), we derive the second upper bound for Q,

$$Q\le\sqrt{\alpha_1+\alpha_2}. \quad (4.17)$$

In the following, we will begin to consider these two upper bounds: $\frac{\alpha_4}{(1+\beta)\alpha_3}$ from Equation (4.9) and $\sqrt{\alpha_1+\alpha_2}$ from Equation (4.17). From the numerical example in Wu and Ouyang [27], the data we have collected include: $v = \$1.6$, $h' = \$12$, $h = \$20$, $A = \$200$ per order, $D = 600$ units/year, $\pi = \$50$, $\pi_0 = \$150$, $\sigma = 7$, there are three components in the lead-time, where $L_0 = 8$, $R(L_0) = 0$, $L_1 = 6$, $R(L_1) = 5.6$, $L_2 = 4$, $R(L_2) = 22.4$, $L_3 = 3$, and $R(L_3) = 57.4$. There are four values, $0, 0.5, 0.8,$ and 1, for the fraction of the backordered demand, β. The defective rate, denoted by p, follows a Beta distribution, and probability density function is expressed as

$$g(p) = 4(1-p)^3. \quad (4.18)$$

Consequently, researchers know the expected value, $E(p) = 1/5$, and the expected value of $E(p^2) = 1/15$ to derive the variance.

In the following Table 1, we first obtain our two upper bounds, $\frac{\alpha_4}{(1+\beta)\alpha_3}$ and $\sqrt{\alpha_1 + \alpha_2}$, under variously proposed values of L_i and β, and then the ratio of $\frac{\alpha_4}{(1+\beta)\alpha_3}$ over $\sqrt{\alpha_1 + \alpha_2}$ is expressed to point out that

$$\frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1 + \alpha_2}. \quad (4.19)$$

From Table 1, all results are greater than one to reveal that $\frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1 + \alpha_2}$. Hence, by Equations (4.14) and (4.17), and our findings in Table 1, we derive a pair of lower bound and upper bound for the ordering quantity Q,

$$\sqrt{\alpha_1} < Q \le \sqrt{\alpha_1 + \alpha_2}. \quad (4.19)$$

From $Q \le \sqrt{\alpha_1 + \alpha_2}$ and Table 1, it derives that $\alpha_4 > (1+\beta)\alpha_3 Q$. Owing to Equation (4.11), it yields that $k > 0$ to show that

$$\sqrt{1+k^2} - k = \frac{1}{\sqrt{1+k^2} + k} < 1, \quad (4.20)$$

such that by Equation (4.1), we find that $Q < \sqrt{\alpha_1 + \alpha_2}$. Hence, we improve Equation (4.19) as follows,

$$\sqrt{\alpha_1} < Q < \sqrt{\alpha_1 + \alpha_2}. \quad (4.21)$$

We now solve Equation (4.13) under the restriction of Equation (4.21) by setting the next supplementary function,

$$f(Q) = (Q^2 - \alpha_1)\sqrt{\alpha_4 - \beta\alpha_3 Q} - \alpha_2\sqrt{\alpha_3 Q}. \quad (4.22)$$

It shows that

$$f''(Q) = 2(\alpha_4 - \beta\alpha_3 Q)^{-0.5}(\alpha_4 - 2\beta\alpha_3 Q),$$
$$+\left[\alpha_2\sqrt{\alpha_3}(\alpha_4 - \beta\alpha_3 Q)^{1.5} - Q^{1.5}(\beta\alpha_3)^2(Q^2 - \alpha_1)\right],$$
$$\times(\alpha_4 - \beta\alpha_3 Q)^{-1.5} Q^{-1.5}. \quad (4.23)$$

From Equation (4.11), we havethat

$$\alpha_4 \ge (1+\beta)\alpha_3 Q \ge 2\beta\alpha_3 Q, \quad (4.24)$$

and again by Equation (4.11),

$$\sqrt{\alpha_3}(\alpha_4 - \beta\alpha_3 Q)^{1.5} \ge \alpha_3^2 Q^{1.5} \ge Q^{1.5}(\beta\alpha_3)^2, \quad (4.25)$$

together with Equation (4.21), it yields that

$$\alpha_2 > (Q^2 - \alpha_1). \quad (4.26)$$

By combining results of Equation (4.23) through Equation (4.26), the inequality below holds,

$$f''(Q) > 0. \quad (4.27)$$

Consequently, the supplementary function $f(Q)$ is convex up. The supplementary function can be simplified as follows

$$f(Q) = \sqrt{\alpha_4 - \beta\alpha_3 Q} \times \left[Q^2 - \alpha_1 - \alpha_2\sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}}\right]. \quad (4.28)$$

Next, we will show that $f\left(\sqrt{\alpha_1 + \alpha_2}\right) > 0$.

If $Q = \sqrt{\alpha_1 + \alpha_2}$, we show that

$$Q = \sqrt{\alpha_1 + \alpha_2} < \frac{\alpha_4}{(1+\beta)\alpha_3}, \quad (4.29)$$

Hence, if $Q = \sqrt{\alpha_1 + \alpha_2}$, then

$$\alpha_3 Q < \alpha_4 - \beta\alpha_3 Q, \quad (4.30)$$

and then we obtain that

$$f\left(\sqrt{\alpha_1 + \alpha_2}\right) >,$$
$$\sqrt{\alpha_4 - \beta\alpha_3\sqrt{\alpha_1 + \alpha_2}} \times \left[\left(\sqrt{\alpha_1 + \alpha_2}\right)^2 - \alpha_1 - \alpha_2\right] = 0. \quad (4.31)$$

Based on the following three things: (i) From $f\left(\sqrt{\alpha_1}\right) = -\alpha_2\sqrt{\alpha_1\alpha_3} < 0$, (ii) $f\left(\sqrt{\alpha_1 + \alpha_2}\right) > 0$, and (iii) $f(Q)$ is a convex up function, we partition the figure of $f(Q)$ into two sections. On the left-hand side, $f(Q)$ decreases to its minimum, and then on the right-hand side, $f(Q)$ increases from its minimum. Hence, $f(Q) = 0$ has a unique solution. for later discussion, we denote it as Q^*, where satisfies $f(Q^*) = 0$.

Based on our above discussion, we have proved that Equation (4.13) under the restriction of Equation (4.21) have a unique solution, which is the only solution that satisfies the first partial derivative system. Hence, we show the minimum problem has a unique solution.

V. Numerical Examples

We recall those data from Wu and Ouyang [27] to assume the following numerical example: $v = \$1.6$, $h' = \$12$, $h = \$20$, $A = \$200$ per order, $D = 600$ units/year, $\pi = \$50$, $\pi_0 = \$150$, $\sigma = 7$, and there are three components in the lead-time, with the following lead-time and lead-time crashable cost: $L_0 = 8$, $R(L_0) = 0$, $L_1 = 6$, $R(L_1) = 5.6$, $L_2 = 4$, $R(L_2) = 22.4$, $L_3 = 3$, and $R(L_2) = 57.4$, Four values of $0, 0.5, 0.8,$ and 1, are the fraction of the backordered demand, β. The defective rate, expressed as p, satisfies the Beta distribution, where the probability density function is denoted as

$$g(p) = 4(1-p)^3. \quad (5.1)$$

Hence, we derive the expected value, $E(p)=1/5$, and the expected value of $E(p^2)=1/15$ for the variance.The numerical findings are expressed in the next table 2.

Based on the above table, we compare our findings with that of Wu and Ouyang [27] to express the results in the following table 3. Based on the first and the third column of the above table, we point out that our results are superior to that of Wu and Ouyang [27].

VI. A Detailed Examination of Their Iterative Algorithm

Next, we present a detailed examination of the solution algorithm of Wu and Ouyang [27]. Based on Equation (3.2), the ordering quantity Q is expressed as a mapping of the safety factor, k. Consequently, for a given value of k, then we can obtain a corresponding value of the ordering quantity, Q(k).

We check their results that was cited as Equation (3.3), then the safety factor, k, was denoted as not as an explicit function of the ordering quantity, Q such that researchers only obtain a value for the quantity of $\frac{\sqrt{1+k^2}}{\sqrt{1+k^2}-k}$. Therefore, we cannot directly use Equation (3.3). On the other hand, we use an equivalent relationship of Equation (4.11) for our investigation. Based on the above discussion, we adopt Equations (3.2) and (4.11) to execute an iterative procedure proposed by Wu and Ouyang [27]. For example, with $k_0=0$, $i=3$, $L_2-L_3=b_3-a_3$, we derive that

$$Q_{n+1}=\left[\theta_1+\theta_2\left(\sqrt{1+k_n^2}-k_n\right)\right]^{1/2}, \tag{6.1}$$

and

$$k_{n+1}=\frac{1-\theta_3 Q_n}{2\sqrt{\theta_3 Q_n}}, \tag{6.2}$$

with three abbreviations,

$$\theta_1=\frac{2D}{h\delta}\left(A+\sum_{j=1}^{3}c_j\left(b_j-a_j\right)\right), \tag{6.3}$$

$$\theta_2=\frac{D\sigma}{h\delta}\left(\pi+\pi_0\right)\sqrt{L_3}, \tag{6.4}$$

and

$$\theta_3=\frac{h\left(1-E(p)\right)}{D\left(\pi+\pi_0\right)}. \tag{6.5}$$

In the following table 4, the iterative findings of (k_n) and (Q_n) are presented.

Based on the table 4, under the conditions of (i) the lead-time, L_3, and (ii) the fraction of the backordered demand, $\beta=0$, we find that Wu and Ouyang [27] should obtain

$$Q^*=179.736088, \tag{6.6}$$

and

$$k^*=3.152452. \tag{6.7}$$

Nevertheless, Wu and Ouyang [27] claimed that their optimal ordering quantity is $Q^*=183$. Consequently, we can assert that their iterative algorithm is too complicated such that even Wu and Ouyang [27] cannot operative their iterative algorithm.

VII. Direction for Future Research

To help researchers to locate the possible directions for future studies, we cited several recently published papers to indicate those important research trends.

In the following, we provide a brief for our cited articles. Based on block partition strategy, Wang et al. [33] studied the oblique QR decomposition with respect to block landweber scheme. Kusuma and Dirgantara [34] examined a new metaheuristic with run-catch optimizer and then applied it for addressing outsourcing optimization problems.

Referring to fractional-order backstepping strategy and input saturation, Tian et al. [35] developed a class of engineering system for finite-time control. According to conditional value at risk and copula-based value at risk, Ismail et al. [36] constructed currency exchange portfolio risk estimation.

Using an equiangular spiral bubble net predation, Liu et al. [37] derived an improved whale optimization algorithm. Referring to weighted loss and transfer learning, Oktavian et al. [38] studied aconvolutional neural network to examine a classification of Alzheimer's disease. Based on image segmentation, Nie and Zhang [39] examined a relative J-divergence under a threshold selection.

Hao and Zhu [40] developedan improved adaptive LASSO method to apply it for an autoregressive model. With spatial constraint, Chen et al. [41] constructeda fixed-time trajectory tracking control of nonholonomic issues. Huang et al. [42] deriveda Leslie-Gower predator-prey model to check the wind effect. Referring to Covid-19 pandemic, Chaerani et al. [43] studied electricity strategy business by Optimization Modeling through asystematic literature review.

Owing to error analysis and comparing edge detection methods, Khairudin et al. [44] examined the selection quality for two dimensional objects. Based on the interrelationship between happiness level in Indonesia and age groups, Sholihah et al. [45] developeda correlation approach for a hybrid singly ordered correspondence.

According to deep learning, Chuanget al. [46] constructeda facial feature classification for drug addicts. Referring to an evidential reasoning approach, Huang et al. [47] derived two-dimensional frameworks to decide the weights for experts' scores. Based on our above discussion, we provide several hot spots for the future studies.

VIII. Conclusion

For the inventory model with defective items, we present a complete analysis to prove that there is a unique optimal solution for order quantity with respect to a nonnegative safety factor. Our solution method with an implicit expression formula sets up a theoretical development for the existence and uniqueness of the optimal order quantity. In this paper, we also examined the algorithm of Wu and Ouyang [27] to present revisions such our iterative algorithm also derived the same optimal solution that is the same result as that obtained by our analytic approach.

Table 1. The ratio of $\frac{\alpha_4}{(1+\beta)\alpha_3}$ over $\sqrt{\alpha_1+\alpha_2}$

	$\beta=0$	$\beta=0.5$	$\beta=0.8$	$\beta=1$
$L=L_0$	18.86	9.68	6.21	4.17
$L=L_1$	20.09	10.26	6.55	4.37
$L=L_2$	21.86	11.07	6.99	4.60
$L=L_3$	22.96	11.51	7.18	4.65

Table 2. For $i=0,1,2,3$, the local optimal solutions.

β	$EAC(Q_i,k_i,L_i)$	k_i	Q_i	L_i
0	6439.6128	3.0286	193.9755	8
	6128.2948	3.0983	185.7649	6
	5793.5899	3.1652	178.3587	4
	5697.4107	3.1525	179.7361	3
0.5	5894.2264	2.4307	179.6360	8
	5647.1663	2.4796	173.2518	6
	5394.2303	2.5204	168.1674	4
	5353.5645	2.4964	171.1390	3
0.8	5463.9046	1.9391	168.7155	8
	5268.8470	1.9734	163.7541	6
	5081.7755	1.9969	160.4683	4
	5085.6258	1.9670	164.6704	3
1.0	5086.9094	1.4867	159.6076	8
	4938.4861	1.5101	155.8536	6
	4810.1428	1.5213	154.0888	4
	4853.4265	1.4884	159.3316	3

Table 3. The comparison between our findings and that of Wu and Ouyang [27].

	Our findings		Wu and Ouyang [27]	
β	$EAC(Q_i,k_i,L_i)$	Q_i	$EAC(Q_i,k_i,L_i)$	Q_i
0	5697.410674	179.736103	5697.95	183
0.5	5353.564526	171.139009	5354.01	174
0.8	5081.775547	160.468314	5082.14	163
1.0	4810.142818	154.088812	4810.65	157

Table 4. An iterative procedure proposed by Wu and Ouyang [27]

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
Q_n	315.621514	194.448259	181.250214	179.895302	179.752870
k_n	2.291610	3.024756	3.138608	3.150988	3.152298
	$n=6$	$n=7$	$n=8$	$n=9$	
Q_n	179.737853	179.736272	179.736111	179.736088	
k_n	3.152436	3.152450	3.152452	3.152452	

References

[1] W. K. Hsu, H. F. Yu, "EOQ model for imperfective items under a one-time-only discount," *Omega*, vol. 37, no. 5, 2009, pp. 1018-1026.

[2] C. H. Ho, "A minimax distribution free procedure for an integrated inventory model with defective goods and stochastic lead time demand," *International Journal of Information and Management Sciences*, vol. 20, 2009, pp. 161-171.

[3] F. Nasri, J. Paknejad, J. Affisco, "Investing in lead-time variability reduction in a quality-adjusted inventory model with finite-range stochastic lead-time," *Journal of Applied Mathematics and Decision Sciences*, 2008, Article number 795869.

[4] B. Maddah, M. Y. Jaber, "Economic order quantity for items with imperfect quality: revisited," *International of Production Economics*, vol. 112, no. 2, 2008, pp. 808-815.

[5] C. J. Chung, H. M. Wee, "An integrated production-inventory deteriorating model for pricing policy considering imperfect production, inspection planning and warranty-period- and stock-level-dependant demand," *International Journal of Systems Science*, vol. 39, no. 8, 2008, pp. 823-837.

[6] K. J. Chung, "A necessary and sufficient condition for the existence of the optimal solution of a single-vendor single-buyer integrated production-inventory model with process unreliability consideration," *International Journal of Production Economics*, vol. 113, no. 1, 2008, pp. 269-274.

[7] C. Chandra, J. Grabis, "Inventory management with variable lead-time dependent procurement cost,"*Omega*, vol. 36, no. 5, 2008, pp. 877-887.

[8] H. M. Wee, J. Yu, M. C. Chen, "Optimal inventory model for goods with imperfect quality and shortage backordering," *Omega*, vol. 35, 2007, pp. 7-11.

[9] M. A. Hoque, "An alternative model for integrated vendor-buyer inventory under controllable lead time and its heuristic solution," *International Journal of Systems Science*, vol. 38, no. 6, 2007, pp. 501-509.

[10] P. Wang, J. A. Hill, "Recursive behavior of safety stock reduction: The effect of lead-time uncertainty," *Decision Sciences*, vol. 37, no. 2, 2006, pp. 285-290.

[11] S. Papachristos, I. Konstantaras, "Economic ordering quantity models for goods with imperfect Quality," *International Journal of Production Economics*, vol. 100, 2006, pp. 148-154.

[12] S. Wadhwa, K. S. Rao, F. T. S. Chan, "Flexibility-enabled lead-time reduction in flexible systems," *International Journal of Production Research*, vol. 43, no. 15, 2005, pp. 3131-3162.

[13] C. K. Huang, "An optimal policy for a single-vendor single-buyer integrated production-inventory problem with process unreliability consideration," *International Journal of Production Economics*, vol. 91, 2004, pp. 91-98.

[14] C. K. Huang, "An integrated vendor-buyer cooperative inventory model for goods with imperfect quality,"*Production Planning & Control*, vol. 13, no. 4, 2002, 355-361.

[15] K. Inderfurth, E. Van Der Laan, "Leadtime effects and policy improvement for stochastic inventory control with remanufacturing," *International Journal of Production Economics*, vol. 71, 2001, pp. 381-390.

[16] L. Y. Ouyang, K. S. Wu, "A minimax distribution free procedure for mixed inventory model with variable lead time," *International Journal of Production Economics*, vol. 56-57, 1998, pp. 511-516.

[17] L. Y. Ouyang, N. C. Yeh, K. S. Wu, "Mixture inventory model with backorders and lost sales for variable lead time," *Journal of the Operational Research Society*, vol. 47, 1996, pp. 829-832.

[18] M. J. Paknejad, F. Nasri, J. F. Affisco, "Defective units in a continuous review (s, Q) system," *International Journal of Production Research*, 33, 1995, pp. 2767-2777.

[19] I. Moon, G. Gallego, "Distribution free procedures for some inventory models," *Journal of Operations Research Society*, vol. 45, no. 6, pp. 651-658.

[20] M. Ben-Daya, A. Raouf, "Inventory models involving lead time as a decision variable," *Journal of the Operational Research Society*, vol. 45, no. 5, 1994, pp. 579-582.

[21] C. J. Liao, C. H. Shyu, "Stochastic inventory model with controllable lead time," *International Journal of Systems Science*, vol. 22, no. 11, 1991, pp. 2347-2354.

[22] B. Foote, N. Kebriaei, H. Kumin, "Heuristic policies for inventory ordering problemswith long and randomly varying lead time," *Journal of Operations Management*, vol. 7, 1988, pp. 115-124.

[23] M. J. Rosenblatt, H. L. Lee, "Economic production cycles with imperfect production processes," *IIE Transactions*, vol. 17, 1986, pp. 48-54.

[24] E. L. Porteus, "Optimal Lot Sizing, Process Quality Improvement and Setup Cost Reduction," *Operations Research*, vol. 34, no. 1, 1986, pp. 137-144.

[25] N. K. Gupta, "Effect of Lead Time on Inventory - a Working Result," *Operations Research Society*, vol. 30, 1982, pp. 477-481.

[26] C. Das, "Approximate Solution to the Inventory Model for Gamma Lead Time Demand," *Management Science*, vol. 22, 1975, pp. 273-282.

[27] K. S. Wu, L. Y. Ouyang, "(Q, r, L) Inventory Model with Defective Items," *Computers and Industrial Engineering*, vol. 39, no. 1-2, 2001, pp. 173-185.

[28] X. R. Luo, C. H. Chu, H. C. J. Chao, "Novel Solution Method for Inventory Models with Stochastic Demand and Defective Units," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3528706, 2020, 13 pages.

[29] J. Lin, "A Study on Iterative Algorithm for Stochastic Distribution Free Inventory Models," *Abstract and Applied Analysis*, vol. 2013, Article ID 251694, 2013, 3 pages.

[30] C. T. Tung, Y. W. Wou, S. W. Lin, P. Deng, "Technical Note on (Q, r, L) Inventory Model with Defective Items," *Abstract and Applied Analysis*, vol. 2010, Article ID 878645, 2010, 8 pages.

[31] K. S. Wu, L. Y. Ouyang, C. H. Ho, "Integrated Vendor - buyer Inventory System with Sublot Sampling Inspection Policy and Controllable Lead Time," *International Journal of Systems Science*, vol. 38, no. 4, 2007, pp. 339-350.

[32] L. Y. Ouyang, K. S. Wu, C. H. Ho, "Analysis of Optimal Vendor-buyer Integrated Inventory Policy Involving Defective Items," *International Journal of Advanced Manufacturing Technology*, vol. 29, 2006, pp. 1232-1245.

[33] C. Wang, C. Che, X. Xia, "Oblique QR Decomposition Based Block Partition Strategy for Block Landweber Scheme," *IAENG International Journal of Applied Mathematics*, vol. 53, no.3, 2023, pp. 884-891.

[34] P. D. Kusuma, F. M. Dirgantara, "Run-Catch Optimizer: A New Metaheuristic and Its Application to Address Outsourcing Optimization Problem," *Engineering Letters*, vol. 31, no.3, 2023, pp. 1045-1053.

[35] X. Tian, X. Hu, J. Gu, C. Man, S. Fei, "Finite-time Control of a Class of Engineering System with Input Saturation via Fractional-order Backstepping Strategy," *IAENG International Journal of Computer Science*, vol. 50, no.3, 2023, pp. 941-946.

[36] I. Ismail, G. Y. Yee, A. Zhang, H. Zhou, "Currency Exchange Portfolio Risk Estimation Using Copula-Based Value at Risk and Conditional Value at Risk," *IAENG International Journal of Applied Mathematics*, vol. 53, no.3, 2023, pp. 892-898.

[37] K. Liu, L. Xue, S. Liu, "Improved Whale Optimization Algorithm Combined with the Equiangular Spiral Bubble Net Predation," *Engineering Letters*, vol. 31, no.3, 2023, pp. 1054-1069.

[38] M. W. Oktavian, N. Yudistira, A. Ridok, "Classification of Alzheimer's Disease Using the Convolutional Neural Network (CNN) with Transfer Learning and Weighted Loss," *IAENG International Journal of Computer Science*, vol. 50, no.3, 2023, pp. 947-953.

[39] F. Nie, P. Zhang, "Threshold Selection with Relative J-Divergence for Image Segmentation," *IAENG International Journal of Applied Mathematics*, vol. 53, no.3, 2023, pp. 899-906.

[40] W. Hao, Y. Zhu, "The Improved Adaptive LASSO Method and Applications in Autoregressive Model," *Engineering Letters*, vol. 31, no.3, 2023, pp. 1070-1075.

[41] T. Chen, Z. Zhao, Q. Zhao, Y. Yao, R. Zhou, Y. Shang, "Fixed-Time Trajectory Tracking Control of Nonholonomic WMRs Subject to Spatial Constraint," *IAENG International Journal of Computer Science*, vol. 50, no.3, 2023, pp. 954-959.

[42] C. Huang, F. Chen, Q. Zhu, Q. Li, "How the Wind Changes the Leslie-Gower Predator-prey System?," *IAENG International Journal of Applied Mathematics*, vol. 53, no.3, 2023, pp. 907-915.

[43] D. Chaerani, H. Napitupulu, A. Z. Irmansyah, N. W. Priambodo, M. I. Felani, H. B. Tambunan, J. Hartono, "A Systematic Literature Review on Optimization Modeling to Electricity Strategy Business during Covid-19 Pandemic," *Engineering Letters*, vol. 31, no.3, 2023, pp. 1076-1095.

[44] M. Khairudin, R. Mahaputra, M. L. Hakim, A. Widowati, B Rahmatullah, A. A. M. Faudzi, "Choosing the Quality of Two Dimension Objects by Comparing Edge Detection Methods and Error Analysis," *IAENG International Journal of Computer Science*, vol. 50, no.3, 2023, pp. 960-969.

[45] F. Sholihah, I. Ginanjar, Y. Hidayat, "A Hybrid Singly Ordered Correspondence with Correlation Approach to Analyzing the Relationship between Age Groups and Happiness Level in Indonesia," *IAENG International Journal of Applied Mathematics*, vol. 53, no.3, 2023, pp. 916-923.

[46] C. H. Chuang, C. T. Tung, Y. S. Chang, E. Lin, C. P. Yen, "Facial Feature Classification of Drug Addicts Using Deep Learning," *Engineering Letters*, vol. 31, no.3, 2023, pp. 1096-1103.

[47] X. Huang, P. Gui, J. Yao, W. Zhu, C. Zhou, X. Li, S. Li, "An Applied View to Determine the Weights of Experts' Scores Based on an Evidential Reasoning Approach Under Two-Dimensional Frameworks," *IAENG International Journal of Computer Science*, vol. 50, no.3, 2023, pp. 970-979.

Shusheng Wu received his Master degreein 2010 from the Department of Computational Mathematics, Ocean University of China, and currently is an instructor in the School of Mathematics and Science Teaching Center, Weifang University of Science and Technology. The main research directions are Inventory Models, Analytic Hierarchy Process, Fuzzy Set Theorem, Algebraic Methods in Operational Research, and Pattern Recognition.

Jinyuan Liu received his Master degree from School of Mathematics and Systems Science, Shandong University of Science and Technology in 2009 and currently is an Associate Professorat the School of General Studies of Weifang University of Science and Technology. The main research directions are Pattern Recognition, Lanchester's Model, Fuzzy Set Theorem, Isolate Points, Analytic Hierarchy Process, and Inventory Models.

Pin-Yi Lo received his Master degree from the Graduate Program of National Development and Cross-Strait Relations, College of Social Sciences, Ming Chuan University, in 2007 and currently is retired from the Central Police University. His major research topics are Inventory Systems, Fuzzy Set Theorem, Pattern Recognition, and Analytic Hierarchy Process.