Extension and Generalization of Polynomial Inequalities

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***Abstract*—In this paper, we extend and generalize an improved bound on the unit circle $|z| = 1$ concerning polar derivative of a polynomial of degree n having no zero in $|z| < k$, $0 < k \leq 1$ to an inequality involving the maximum modulus of the polar derivative of the polynomial on circles of radii r, R with $1 \leq r \leq R < \infty$. As consequences of our result, we are able to extend and improve some known polynomial inequalities as well.**

***Index Terms*—polynomial inequalities, maximum modulus, polar derivative.**

I. Introduction

Let $p(z)$ be a polynomial of degree n. Then, according to a famous well-known classical result due to Bernstein [3],

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

Inequality (1) is sharp and equality holds if $p(z)$ has all its zeros at the origin. If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then Erdös conjectured and later Lax [5] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

Inequality (2) is best possible and equality holds for $p(z) = a + bz^n$, where $|a| = |b|$.

It was asked by R. P. Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z| < k, k > 0$, then how large can

$$\left\{ \max_{|z|=1} |p'(z)| / \max_{|z|=1} |p(z)| \right\} \text{ be ?}$$

A partial answer to this problem was given by Malik [6], who proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{3}$$

In an attempt to obtain inequalities analogous to (3) for the classes of polynomials having no zero in $|z| < k, k \leq 1$, Govil [2] proved the following inequality by imposing a strong restriction on the moduli of the derivatives of the polynomial and its reciprocal polynomial regarding the attainment of their maximum moduli at the same point on the unit circle.

Theorem 1. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, such that $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \tag{4}$$

where here and throughout this paper $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

By involving some coefficients of the polynomial, Singh et al. [7] refined the bound (4) of Theorem 1. In fact, they proved

Theorem 2. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, such that $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \left\{ \frac{n}{1+k^n} - \frac{k^n(|a_0| - |a_n|k^n)}{(1+k^n)(|a_0| + |a_n|k^n)} \right\} \times \max_{|z|=1} |p(z)|. \tag{5}$$

Inequality (5) *is best possible for $p(z) = z^n + k^n$.*

For a polynomial $p(z)$ of degree n, we now define the polar derivative of $p(z)$ with respect to a real or complex number β as

$$D_\beta p(z) = np(z) + (\beta - z)p'(z).$$

This polynomial $D_\beta p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\beta \to \infty} \frac{D_\beta p(z)}{\beta} = p'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Among those who first extended some of the above inequalities to polar derivative versions, Aziz [1] was one who extended inequality (3) to polar derivative by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then for any real or complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta p(z)| \leq n \left(\frac{|\beta| + k}{1+k} \right) \max_{|z|=1} |p(z)|. \tag{6}$$

During recent decades, many different authors produced a large number of results concerning the polar derivative of polynomials. More information on classical results and polar derivatives can be found in the books of Marden [11] and Milovanović et al. [12], and also see the references [8], [9], [10], [13], [14].

Further, Singh et al. [7] extended Theorem 2 to the polar derivative setting by proving the following result.

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Theorem 3. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<k$, $k\leq 1$, such that $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z|=1$, then for any real or complex number β with $|\beta|>1$,*

$$\max_{|z|=1}|D_\beta p(z)| \leq \left\{\frac{n(|\beta|+k^n)}{1+k^n}-\frac{(|\beta|-1)k^n(|a_0|-|a_n|k^n)}{(1+k^n)(|a_0|+|a_n|k^n)}\right\} \times\max_{|z|=1}|p(z)|. \tag{7}$$

II. Lemma

The following lemmas are needed for the proof of the theorem.

Lemma 4. ([4]) *If $p(z)$ is a polynomial of degree n, then on $|z|=1$,*

$$|p'(z)|+|q'(z)|\leq n\max_{|z|=1}|p(z)|. \tag{8}$$

Lemma 5. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<k$, $k\leq 1$, such that for a real or complex number α with $|\alpha|<1$, $|p'(z)|$ and $|q'(z)+n\overline{\alpha}z^{n-1}m|$, where $m=\min_{|z|=k}|p(z)|$, attain their maxima at the same point on $|z|=1$, then*

$$\max_{|z|=1}|p'(z)| \leq \frac{1}{1+k^n}\left\{n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{(|a_0+\alpha m|+|a_n|k^n)}\right\} \times\left(\max_{|z|=1}|p(z)|+m|\alpha|\right). \tag{9}$$

Proof: Since the polynomial $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ has no zero in $|z|<k$, $k\leq 1$, for any real or complex number α with $|\alpha|<1$, by Rouche's theorem, the polynomial $P(z)=p(z)+\alpha m$ has no zero in $|z|<k$, $k\leq 1$, where $m=\min_{|z|=k}|p(z)|$.

Using Theorem 2 to $P(z)=p(z)+m\alpha$, we have

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^n}-\frac{k^n\left(|a_0+\alpha m|-|a_n|k^n\right)}{(1+k^n)\left(|a_0+\alpha m|+|a_n|k^n\right)} \times\max_{|z|=1}|P(z)|,$$

which gives

$$\max_{|z|=1}|p'(z)| \leq \frac{1}{1+k^n}\left\{n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{(|a_0+\alpha m|+|a_n|k^n)}\right\} \times\max_{|z|=1}|p(z)+m\alpha|. \tag{10}$$

Suppose z_1 on $|z|=1$ is such that

$$\max_{|z|=1}|p(z)+m\alpha|=|p(z_1)+m\alpha|. \tag{11}$$

Now, we choose the argument of α such that

$$\begin{aligned}|p(z_1)+m\alpha| &= |p(z_1)|+m|\alpha| \\ &\leq \max_{|z|=1}|p(z)|+m|\alpha|.\end{aligned} \tag{12}$$

Using (11) and (12) to (10), we get

$$\max_{|z|=1}|p'(z)| \leq \frac{1}{1+k^n}\left\{n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{(|a_0+\alpha m|+|a_n|k^n)}\right\} \times\left(\max_{|z|=1}|p(z)|+m|\alpha|\right).$$

This completes the proof of Lemma 5. ■

Lemma 6. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<k$, $k\leq 1$ and $R\geq 1$ such that $|p'(Rz)|$ and $|q'(\frac{z}{R})|$ attain their maxima at the same point on $|z|=1$, then for every real or complex number α with $|\alpha|<1$*

$$\max_{|z|=R}|p'(z)| \leq \frac{R^{n-1}}{R^n+k^n}\left\{n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{R^n(|a_0+\alpha m|+|a_n|k^n)}\right\} \times\left(\max_{|z|=R}|p(z)|+m|\alpha|\right), \tag{13}$$

where $m=\min_{|z|=k}|p(z)|$.

Proof: Since $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ has no zero in $|z|<k$, $k\leq 1$, the polynomial $P(z)=p(Rz)$ has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\leq 1$.

Applying Lemma 5 to $P(z)=p(Rz)$, we have

$$\max_{|z|=1}|P'(z)| \leq \frac{1}{1+\left(\frac{k}{R}\right)^n} \times\left[n-\frac{\left(\frac{k}{R}\right)^n\left\{|a_0+\alpha m|-|a_n|R^n\left(\frac{k}{R}\right)^n\right\}}{\left\{|a_0+\alpha m|+|a_n|R^n\left(\frac{k}{R}\right)^n\right\}}\right] \times\left(\max_{|z|=1}|P(z)|+m|\alpha|\right),$$

where

$$m=\min_{|z|=\frac{k}{R}}|P(z)|=\min_{|z|=\frac{k}{R}}|p(Rz)|=\min_{|z|=k}|p(z)|,$$

which on simplification gives

$$\max_{|z|=R}|p'(z)| \leq \frac{R^{n-1}}{R^n+k^n}\left\{n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{R^n(|a_0+\alpha m|+|a_n|k^n)}\right\} \times\left(\max_{|z|=R}|p(z)|+m|\alpha|\right).$$

This ends the proof of Lemma 6. ■

Lemma 7. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<k$, $k\leq 1$ and $1\leq r\leq t\leq R<\infty$ such that $|p'(tz)|$ and $|q'(\frac{z}{t})|$ attain their maxima at the same point on $|z|=1$, then for every real or complex number α*

with $|\alpha| < 1$

$$\begin{aligned}&\max_{|z|=R}|p(z)|\\ &\leq \max_{|z|=r}|p(z)| + \left(\max_{|z|=r}|p(z)| + m|\alpha|\right)\\ &\times\left[\exp\left\{\int_r^R \frac{t^{n-1}}{t^n+k^n}\right.\right.\\ &\left.\left.\times\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{t^n(|a_0+\alpha m| + |a_n|k^n)}\right)dt\right\} - 1\right],\end{aligned} \tag{14}$$

where $m = \min_{|z|=k}|p(z)|$.

Proof: If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ has no zero in $|z| < k$, $k \leq 1$, then for $t \geq 1$, $P(z) = p(tz)$ has no zero in $|z| < \frac{k}{t}$, $\frac{k}{t} \leq 1$.
Applying Lemma 5 to $P(z) = p(tz)$, we have

$$\begin{aligned}&\max_{|z|=1}|P'(z)|\\ &\leq \frac{1}{1+\left(\frac{k}{t}\right)^n}\left[n - \frac{\left(\frac{k}{t}\right)^n\left\{|a_0+\alpha m| - |a_n|t^n\left(\frac{k}{t}\right)^n\right\}}{\left\{|a_0+\alpha m| + |a_n|t^n\left(\frac{k}{t}\right)^n\right\}}\right]\\ &\times\left(\max_{|z|=1}|P(z)| + m|\alpha|\right),\end{aligned}$$

where

$$m = \min_{|z|=\frac{k}{t}}|P(z)| = \min_{|z|=\frac{k}{t}}|p(tz)| = \min_{|z|=k}|p(z)|,$$

which is equivalent to

$$\begin{aligned}&\max_{|z|=t}|p'(z)|\\ &\leq \frac{t^{n-1}}{t^n+k^n}\left\{n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{t^n(|a_0+\alpha m| + |a_n|k^n)}\right\}\\ &\times\left(\max_{|z|=t}|p(z)| + m|\alpha|\right).\end{aligned} \tag{15}$$

Now, for $1 \leq r \leq t \leq R < \infty$ and $0 \leq \theta < 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| \leq \int_r^R |p'(te^{i\theta})|dt,$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})|dt. \tag{16}$$

Since

$$\int_r^R |p'(te^{i\theta})|dt \leq \int_r^R \max_{|z|=t}|p'(z)|dt,$$

using inequality (15) in (16), we get

$$\begin{aligned}&|p(Re^{i\theta})|\\ &\leq |p(re^{i\theta})|\\ &+\int_r^R\left\{\frac{t^{n-1}}{t^n+k^n}\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{t^n(|a_0+\alpha m| + |a_n|k^n)}\right)\right.\\ &\left.\times\left(\max_{|z|=t}|p(z)| + m|\alpha|\right)\right\}dt.\end{aligned}$$

Taking maximum over θ on both sides of the above inequality, we get

$$\begin{aligned}&\max_{|z|=R}|p(z)|\\ &\leq \max_{|z|=r}|p(z)|\\ &+\int_r^R\left\{\frac{t^{n-1}}{t^n+k^n}\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{t^n(|a_0+\alpha m| + |a_n|k^n)}\right)\right.\\ &\left.\times\left(\max_{|z|=t}|p(z)| + m|\alpha|\right)\right\}dt.\end{aligned} \tag{17}$$

Let $\phi(R)$ denote the right hand side of (17). Then

$$\begin{aligned}\phi'(R) &= \frac{R^{n-1}}{R^n+k^n}\\ &\times\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{R^n(|a_0+\alpha m| + |a_n|k^n)}\right)\\ &\times\left(\max_{|z|=R}|p(z)| + m|\alpha|\right).\end{aligned} \tag{18}$$

Since

$$\max_{|z|=R}|p(z)| \leq \phi(R),$$

inequality (18) can be written as

$$\begin{aligned}\phi'(R) - \frac{R^{n-1}}{R^n+k^n}&\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{R^n(|a_0+\alpha m| + |a_n|k^n)}\right)\\ &\times\{\phi(R) + m|\alpha|\} \leq 0.\end{aligned} \tag{19}$$

Multiplying both sides of (19) by

$$\begin{aligned}&\exp\left\{-\int\frac{R^{n-1}}{R^n+k^n}\right.\\ &\left.\times\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{R^n(|a_0+\alpha m| + |a_n|k^n)}\right)dR\right\},\end{aligned}$$

we have

$$\begin{aligned}&\frac{d}{dR}\left[\{\phi(R)+m|\alpha|\}\exp\left\{-\int\frac{R^{n-1}}{R^n+k^n}\right.\right.\\ &\left.\left.\times\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{R^n(|a_0+\alpha m| + |a_n|k^n)}\right)dR\right\}\right]\\ &\leq 0.\end{aligned} \tag{20}$$

Inequality (20) implies that the function

$$\begin{aligned}&\{\phi(R)+m|\alpha|\}\exp\left\{-\int\frac{R^{n-1}}{R^n+k^n}\right.\\ &\left.\times\left(n - \frac{k^n(|a_0+\alpha m| - |a_n|k^n)}{R^n(|a_0+\alpha m| + |a_n|k^n)}\right)dR\right\}\end{aligned}$$

is a non-increasing function of R in $[1, \infty)$ and hence for

$1 \leq r \leq R < \infty$

$$\begin{aligned}&\{\phi(R)+m|\alpha|\}\exp\bigg\{-\int\frac{R^{n-1}}{R^n+k^n}\\&\times\left(n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{R^n(|a_0+\alpha m|+|a_n|k^n)}\right)dR\bigg\}\\&\leq\{\phi(r)+m|\alpha|\}\exp\bigg\{-\int\frac{r^{n-1}}{r^n+k^n}\\&\times\left(n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{r^n(|a_0+\alpha m|+|a_n|k^n)}\right)dr\bigg\},\end{aligned}$$

which simplifies to

$$\begin{aligned}&\{\phi(R)+m|\alpha|\}\\&\leq\{\phi(r)+m|\alpha|\}\\&\times\exp\bigg\{\int_r^R\frac{t^{n-1}}{t^n+k^n}\\&\times\left(n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{t^n(|a_0+\alpha m|+|a_n|k^n)}\right)dt\bigg\}.\end{aligned}$$

Using the value of $\phi(R)$ which is the right hand side of (17), putting $\phi(r)=\max\limits_{|z|=r}|p(z)|$ and simplifying, the above inequality becomes

$$\begin{aligned}&\int_r^R\bigg\{\frac{t^{n-1}}{t^n+k^n}\left(n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{t^n(|a_0+\alpha m|+|a_n|k^n)}\right)\\&\times\left(\max_{|z|=t}|p(z)|+m|\alpha|\right)\bigg\}dt\\&\leq\left(\max_{|z|=r}|p(z)|+m|\alpha|\right)\\&\times\bigg[\exp\bigg\{\int_r^R\frac{t^{n-1}}{t^n+k^n}\\&\times\left(n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{t^n(|a_0+\alpha m|+|a_n|k^n)}\right)dt\bigg\}-1\bigg].\quad(21)\end{aligned}$$

Using (21) in (17), we are with the desired result and the proof of Lemma 7 is complete. ∎

III. Main Result

In this paper, we obtain an extension and generalization of Theorem 3 which improves and extends Theorem 1 by involving some of the coefficients of the polynomial $p(z)$ and $\min\limits_{|z|=k}|p(z)|$. In fact, we obtain

Theorem 8. *If $p(z)=\sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<k$, $k\leq 1$ and $1\leq r\leq R<\infty$ such that $|p'(Rz)|$ and $|q'(\frac{z}{R})|$ attain their maxima at the same point on $|z|=1$, then for every real or complex numbers α, β with $|\alpha|<1$, $|\beta|>1$*

$$\begin{aligned}&\max_{|z|=R}|D_\beta p(z)|\\&\leq n\bigg\{1+\frac{(|\beta|-1)R^n}{(R^n+k^n)}\\&\times\left(1-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right)\bigg\}\\&\times\bigg[\max_{|z|=r}|p(z)|+\left(\max_{|z|=r}|p(z)|+m|\alpha|\right)\\&\times\bigg\{\exp\bigg(\int_r^R\frac{t^{n-1}}{(t^n+k^n)}\\&\times\left(n-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{t^n(|a_0+\alpha m|+|a_n|k^n)}\right)dt\bigg)-1\bigg\}\bigg]\\&+\frac{mn|\alpha|(|\beta|-1)R^n}{(R^n+k^n)}\\&\times\left\{1-\frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right\},\end{aligned}\quad(22)$$

where $m=\min\limits_{|z|=k}|p(z)|$.

Proof: For $R\geq 1$, $\frac{k}{R}\leq 1$. Since $p(z)$ has no zero in $|z|<k$, $k\leq 1$, the polynomial $P(z)=p(Rz)$ has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\leq 1$. If $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then it can be easily verified that

$$nP(z)-zP'(z)=z^{n-1}Q'(z).\quad(23)$$

For any complex number β with $|\beta|>1$ and $0\leq\theta<2\pi$, we have

$$D_\beta P(z)=nP(z)+(\beta-z)P'(z),$$

which on using (23) implies for $|z|=1$

$$\begin{aligned}|D_\beta P(z)|&\leq|nP(z)-zP'(z)|+|\beta||P'(z)|\\&=|Q'(z)|+|P'(z)|-|P'(z)|+|\beta||P'(z)|\\&\leq n\max_{|z|=1}|P(z)|+(|\beta|-1)|P'(z)|,\end{aligned}\quad(24)$$

where the last inequality is given by applying Lemma 4 to $P(z)$. Putting $P(z)=p(Rz)$, (24) gives

$$|D_\beta p(Rz)|\leq n\max_{|z|=1}|p(Rz)|+(|\beta|-1)R|p'(Rz)|,$$

which is equivalent to

$$\begin{aligned}&\max_{|z|=R}|D_\beta p(z)|\\&\leq n\max_{|z|=1}|p(Rz)|+(|\beta|-1)R\max_{|z|=1}|p'(Rz)|\\&=n\max_{|z|=R}|p(z)|+(|\beta|-1)R\max_{|z|=R}|p'(z)|.\end{aligned}\quad(25)$$

Applying Lemma 6 to (25), we get

$$\max_{|z|=R}|D_\beta p(z)| \leq n\max_{|z|=R}|p(z)| + \frac{(|\beta|-1)R^n}{(R^n+k^n)} \times \left\{n - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{R^n(|a_0+\alpha m|+|a_n|k^n)}\right\} \times \left(\max_{|z|=R}|p(z)| + m|\alpha|\right),$$

which implies

$$\max_{|z|=R}|D_\beta p(z)| \leq n\left\{1 + \frac{(|\beta|-1)R^n}{(R^n+k^n)} \times \left(1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right)\right\}\max_{|z|=R}|p(z)| + \frac{mn|\alpha|(|\beta|-1)R^n}{(R^n+k^n)} \times \left\{1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right\}.$$

Now, using Lemma 7, the last inequality gives

$$\max_{|z|=R}|D_\beta p(z)| \leq n\left\{1 + \frac{(|\beta|-1)R^n}{(R^n+k^n)} \times \left(1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right)\right\} \times \left[\max_{|z|=r}|p(z)| + \left(\max_{|z|=r}|p(z)| + m|\alpha|\right) \times \left\{\exp\left(\int_r^R \frac{t^{n-1}}{(t^n+k^n)} \times \left(n - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{t^n(|a_0+\alpha m|+|a_n|k^n)}\right)dt\right) - 1\right\}\right] + \frac{mn|\alpha|(|\beta|-1)R^n}{(R^n+k^n)} \times \left\{1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right\}.$$

This concludes the proof of Theorem 8. ■

Remark 9. *Dividing both sides of* (22) *of Theorem* 8 *by* $|\beta|$ *and letting* $|\beta| \to \infty$, *the following corollary is obtained.*

Corollary 10. *If* $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \leq 1$ *and* $1 \leq r \leq R < \infty$ *such that* $|p'(Rz)|$ *and* $|q'(\frac{z}{R})|$ *attain their maxima at the same point on* $|z| = 1$, *then for every real or complex number* α *with* $|\alpha| < 1$

$$\max_{|z|=R}|p'(z)| \leq \frac{nR^{n-1}}{R^n+k^n}\left(1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right) \times \left[\max_{|z|=r}|p(z)| + \left(\max_{|z|=r}|p(z)| + m|\alpha|\right) \times \left\{\exp\left(\int_r^R \frac{t^{n-1}}{(t^n+k^n)} \times \left(n - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{t^n(|a_0+\alpha m|+|a_n|k^n)}\right)dt\right) - 1\right\}\right] + \frac{mn|\alpha|R^{n-1}}{(R^n+k^n)} \times \left\{1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right\}, \tag{26}$$

where $m = \min_{|z|=k}|p(z)|$.

Remark 11. *Putting* $r = R$, *Theorem* 8 *reduces to the following result which further reduces to Theorem* 3 *when* $\alpha = 0$ *and* $R = 1$.

Corollary 12. *If* $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \leq 1$ *and* $1 \leq R < \infty$ *such that* $|p'(Rz)|$ *and* $|q'(\frac{z}{R})|$ *attain their maxima at the same point on* $|z| = 1$, *then for every real or complex number* α, β *with* $|\alpha| < 1$, $|\beta| > 1$

$$\max_{|z|=R}|D_\beta p(z)| \leq n\left\{1 + \frac{(|\beta|-1)R^n}{(R^n+k^n)} \times \left(1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right)\right\} \times \max_{|z|=R}|p(z)| + \frac{mn|\alpha|(|\beta|-1)R^n}{(R^n+k^n)} \times \left\{1 - \frac{k^n(|a_0+\alpha m|-|a_n|k^n)}{nR^n(|a_0+\alpha m|+|a_n|k^n)}\right\}, \tag{27}$$

where $m = \min_{|z|=k}|p(z)|$.

Remark 13. *Dividing both sides of* (27) *of Corollary* 12 *by* $|\beta|$ *and letting* $|\beta| \to \infty$, *we have the following interesting corollary which further reduces to Theorem* 2 *when* $\alpha = 0$ *and* $R = 1$.

Corollary 14. *If* $p(z) = \sum_{\nu=0}^n a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \leq 1$ *and* $R \geq 1$ *such that* $|p'(Rz)|$ *and* $|q'(\frac{z}{R})|$ *attain their maxima at the same point on* $|z| = 1$, *then for every real or complex number* α *with* $|\alpha| < 1$

$$
\begin{aligned}
&\max_{|z|=R} |p'(z)| \\
&\le \frac{nR^{n-1}}{R^n + k^n} \\
&\times \left(1 - \frac{k^n(|a_0 + \alpha m| - |a_n|k^n)}{nR^n(|a_0 + \alpha m| + |a_n|k^n)}\right) \max_{|z|=R} |p(z)| \\
&+ \frac{mn|\alpha|R^{n-1}}{R^n + k^n} \left(1 - \frac{k^n(|a_0 + \alpha m| - |a_n|k^n)}{nR^n(|a_0 + \alpha m| + |a_n|k^n)}\right),
\end{aligned}
\tag{28}
$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 15. *When* $\alpha = 0$*, Corollary* 14 *yields an improvement and generalization of Theorem* 1.

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