Unraveling the Significance of Çalışkan's Note: An In-Depth Analysis

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Abstract—**In this paper, we undertake a comprehensive investigation of the work presented by Çalışkan (2020). Our study serves four primary objectives. Firstly, we identify that Çalışkan (2020) derived an existing inventory model using a complex approach, and subsequently propose enhancements to the methodology. Secondly, we demonstrate the variance between Çalışkan's (2020) approximated model and the model developed by Chung and Ting (1994), which went unnoticed in Çalışkan's (2020) analysis. Thirdly, we shed light on the reason behind Çalışkan (2020) arriving at the same approximated optimal solution as previously developed by Chung and Ting (1994). Fourthly, we present an alternative approach to derive the same approximated optimal solution as Çalışkan (2020) without relying on Çalışkan's (2020) proposed objective function. Through these revisions, we aim to provide valuable insights for researchers examining Çalışkan's (2020) work.**

Index Terms—**Exponential approximation, Approximated solution, Inventory model, Formulated solution**

I. Introduction

Çalışkan [1] published a paper in Production Planning & Control to consider an inventory model where the deterioration is dependent on the inventory level.

Çalışkan [1] presented a detailed derivation for the model because he mentioned that "The inventory holding cost has never been modelled exactly before,". However, we find that Widyadana et al. [2] already mentioned the exact holding cost based on Zipkin [3]. We must point out that Widyadana et al. [2] is in the references of Çalışkan [1].

Moreover, Aggarwal and Jaggi [4], Benkherouf [5], Balkhi and Benkherouf [6], Benkherouf and Balkhi [7], Chu et al. [8], Abad [9], Chu and Chen [10], Chen [11], Yang et al. [12], Avinadav and Arponen [13], and Tuan et al. [14] have written the exact holding cost and purchasing cost in their papers.

On the other hand, Mandal and Pal [15], Wu and Ouyang [16], Wu [17], Deng et al. [18], Skouri et al. [19], Cheng and Wang [20], Cheng et al. [21], Hung [22], Lin [23], Yang et al. [24], and Lin et al. [25] have studied the exact holding cost and the deteriorated items cost in their papers. Therefore, we can claim that Çalışkan [1] derived an already published inventory model such that we think relevant source papers should not have been neglected in Çalışkan [1] and recommend researchers who are interested in this study area to reference such relevant source papers for future research.

Four inventory models will be discussed in this paper: CT_k, k=1,2,3 and 4. CT_1 is the inventory model with deteriorated items cost and the exact holding cost that was mentioned in Çalışkan [1]. CT_2 is an approximation for CT_1 with $e^{\delta T} = (2+\delta T)/(2-\delta T)$ used by Çalışkan [1]. CT_3 is the inventory model with purchasing cost and an approximated holding cost proposed by Chung and Ting [26]. CT_4 is an approximation for CT_3 with $e^{\delta T} = (2+\delta T)/(2-\delta T)$ constructed by Chung and Ting [26].

$CT_2(T)$ and $CT_2(Q)$ are the same function, but they are expressed in different variables. $CT_2(T)$ used the length of one replenishment, T and $CT_2(Q)$ used the ordering quantity, Q.

Çalışkan [1] examined CT_2 in two different versions: $CT_2(T)$ and $CT_2(Q)$. Çalışkan [1] derived the optimal solution T^* of $CT_2(T)$ and the optimal solution Q^* of $CT_2(Q)$ to assert that Q^* has better expression than T^*.

We cite Çalışkan [1] concerning his further comments,

"We have just proved that the average inventory level with the approximation for $e^{\delta T}$ that we have used in this paper is indeed equal to $Q/2$. This is a priori assumed in Chung and Ting [26] without providing a proof or justification. Note that the average inventory level in the basic EQO model is also $Q/2$, which can be calculated by dividing the area of the triangle under the inventory level line by the cycle length."

We may say that the optimal solution of $CT_2(T)$ is accidentally the same as the optimal solution of $CT_4(T)$ such that Çalışkan [1] was not aware of the variation of his model of $CT_2(T)$ and $CT_4(T)$ as studied by Chung and Ting [26]. In the later section 8, we will prove that $CT_2(T)$ and $CT_4(T)$ are two different models, but they are different by a constant such that their optimal solutions are accidentally the same.

The main contribution of our paper is to explain why did $CT_2(T)$ and $CT_4(T)$ imply the same optimal solution. Moreover, we provide two different methods to obtain the results proposed by Çalışkan [1], without deriving the objective function $CT_2(Q)$ proposed by Çalışkan [1].

II. Notation and Assumptions

To be compatible with Çalışkan [1], we use the same notation and assumptions as that in his paper.

Notation:

c is the cost per deteriorated item.

D is the constant demand rate per unit time.

h is the holding cost per unit per unit of time.

Q is the ordering quantity.

S is the ordering cost per order.

T is the period for one replenishment cycle.

δ is the deterioration rate per unit per unit of time.

$I(t)$ is the inventory level, with $I(0) = Q$ and $I(T) = 0$.

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Remark. In Chung and Ting [26], a is the demand rate; θ is the deterioration rate; P is the purchasing cost; H is the holding cost.

Assumptions:
1. Deterioration is dependent on the inventory level.
2. Shortages are not allowed.
3. The lead time is neglected.
4. The purpose of Chung and Ting [26] is to derive an approximated optimal solution.
5. The purpose of Çalışkan [1] is to obtain an elegant expression for the approximated optimal solution proposed by Chung and Ting [26].
6. TC_1 is an exact inventory model with setup cost, waste cost, and inventory holding cost.
7. TC_2 is an approximated model for TC_1 proposed by Çalışkan [1].
8. TC_3 is an approximated model with setup cost, purchasing cost, and inventory holding (approximated) cost proposed by Chung and Ting [26].
9. TC_4 is an approximated model for TC_3 proposed by Chung and Ting [26].

III. Review of His Exact Inventory Model

In this section, we cite related material from Çalışkan [1] for our later discussion.
Çalışkan [1] considered an inventory model with deterioration where the deterioration is dependent on the inventory level. We cite from Çalışkan [1] some materials for our later discussions,

$$\text{"}I(0) = Q; \frac{d}{dt}I(t) = -D - \delta I(t), \text{ for } 0 \le t \le T. \quad \text{(C1)"}$$

We remind the readers that $I(T) = 0$.

Remark. (C1) indicates Equation (1) of Çalışkan [1].

To save the precious space in this journal, we only mention those necessary procedures in the development of the inventory model constructed by Çalışkan [1]. The interested readers please refer to Çalışkan [1] for the complete derivations.

We also cite from Çalışkan [1],

$$\text{"}I(t) = Qe^{-\delta t} - \frac{D}{\delta}\left(1 - e^{-\delta t}\right), \text{ for } 0 \le t \le T. \quad \text{(C3)"}$$

$$\text{"}Q = \frac{D}{\delta}\left(e^{\delta T} - 1\right). \quad \text{(C4)"}$$

"The waste cost per cycle can be calculated as follows:

$$W_c = \int_0^T c\delta I(t)dt = \int_0^T c\delta\left[Qe^{-\delta t} - \frac{D}{\delta}\left(1 - e^{-\delta t}\right)\right]dt, \quad \text{(C6)}$$

$$W_c = -cQe^{-\delta T} + cQ - cDT - \frac{cD}{\delta}e^{-\delta T} + \frac{cD}{\delta}, \quad \text{(C7)}$$

$$W_c = c\left[-\left(\frac{D}{\delta} + Q\right)e^{-\delta T} + Q - DT + \frac{D}{\delta}\right], \quad \text{(C8)}$$

$$W_c = c\left[-\left(\frac{D+\delta Q}{\delta}\right)\left(\frac{D}{D+\delta Q}\right) + Q - DT + \frac{D}{\delta}\right],$$
$$= c(Q - DT). \quad \text{(C9)}$$

Equation (C9) can also be determined intuitively. The total amount of items purchased in a cycle is Q, which includes the deteriorated items or waste and the total satisfied demand in a cycle of length T is DT; therefore, the difference is the amount of deterioration or waste in a cycle. The inventory holding cost has never been modelled exactly before, mainly for computational convenience. Chung and Ting (1994) and Widyadana, Cardenas-Barron, and Wee (2011) model it the same way as in the basic EOQ model, assuming an average inventory level of $Q/2$. Ghare and Schrader (1963) and Bahari-Kashani (1989) calculate it based on the maximum inventory level, which is an overestimation.
We calculate it exactly as follows:

$$I_c = \int_0^T hI(t)dt = \int_0^T h\left[Qe^{-\delta t} - \frac{D}{\delta}\left(1 - e^{-\delta t}\right)\right]dt,$$
$$= \frac{h}{\delta}(Q - DT). \quad \text{(C10)}$$

The last equation is due to the fact that $\int_0^T I(t)dt = (Q - DT)/\delta$ from Equations (C6) and (C9)."
and we cite Çalışkan [1] of his results,
"The total cost per inventory ordering cycle is then the sum of the ordering cost, waste cost, and inventory holding cost:

$$TC_c(Q,T) = S + c[Q - DT] + \frac{h}{\delta}(Q - DT)$$
$$= S + \frac{h+c\delta}{\delta}[Q - DT]. \quad \text{(C15)}$$

Because there are $1/T$ cycles per unit time, the average total cost per unit time will be as follows:

$$TC(Q,T) = \frac{S}{T} + \frac{h+c\delta}{\delta T}[Q - DT]. \quad \text{(C16)}$$

Substituting Equation (4) in Equation (16) results in:

$$TC_1 = TC(Q,T),$$
$$= \frac{S}{T} + \frac{h+c\delta}{\delta T}\left[\frac{D}{\delta}\left(e^{\delta T} - 1\right) - DT\right]. \quad \text{(C17)"}$$

Remark. Chung and Ting (1994) of Çalışkan [1] is Chung and Ting [26] in this paper.
Widyadana, Cardenas-Barron, and Wee (2011) of Çalışkan [1] is Widyadana et al. [2] in this paper.
Ghare and Schrader (1963) of Çalışkan [1] is Ghare and Schrader [27] in this paper.
Bahari-Kashani (1989) of Çalışkan [1] is Bahari-Kashani [28] in this paper.
"TC_1" in (C17) was added by us to help readers to distinguish four different inventory models examined in this paper.

IV. Comments on His Exact Inventory Model

In this section, we present the compactest way for the deterioration items cost and the holding cost.
The above derivations in Section 3 proposed by Çalışkan [1] are valid. However, it was presented in a lengthy process. Based on the same knowledge of Çalışkan [1] the deteriorated amount is the ordering quantity (Q) minus the demand during $[0, T]$ as DT such that researchers can directly find the deterioration cost is $W_c = c[Q - DT]$.
An alternative way to evaluate the deteriorated amount is $\int_0^T \delta I(t)dt$ such that researchers directly know that the deteriorated amount is the initial inventory level, Q, minis the demand during [0, T] such that

$$\int_0^T \delta I(t)dt = Q - DT. \quad (4.1)$$

Consequently, the holding cost is computed as $I_c = \int_0^T hI(t)dt$. Based on Equation (4.1), without lengthy integration, it follows that

$$I_c = \int_0^T hI(t)dt = \frac{h}{\delta}(Q - DT). \quad (4.2)$$

The above knowledge is already discussed in Çalışkan [1].
Our results of Equations (4.1) and (4.2) are expressed as Equations (C6-C10) in Çalışkan [1]. We would suggest readers refer to many source papers, which were not cited by Calışkan [1], to understand a historical review of Equations (4.1) and (4.2). Moreover, there seems to be a lack of rational motivation in Calışkan [1] about his lengthy derivations from Equations (C6), (C7), (C8), and (C9). To derive a

well-known result repeatly is a serious violation in academic society.

V. Literature Review for This Kind of Model

In this section, we provide a direct connection between Çalışkan [1] and the holding cost of the traditional model where the deterioration is dependent on the inventory level.
Çalışkan [1] claimed that Ghare and Schrader [27], Bahari-Kashani [28], Chung and Ting [26], and Widyadana et al. [2] all have approximated inventory models. Hence, Çalışkan [1] presented a detailed derivation for these kinds of exact inventory models.
In the literature, many papers had studied inventory models where the deterioration is dependent on the inventory level such that the exact holding cost was obtained. As we mentioned in the Introduction, Aggarwal and Jaggi [4], Benkherouf [5], Balkhi and Benkherouf [6], Benkherouf and Balkhi [7], Chu et al. [8], Abad [9], Chu and Chen [10], Chen [11], Yang et al. [12], Avinadav and Arponen [13], and Tuan et al. [14] all have computed the exact result of holding cost and purchasing cost in their papers
Moreover, as mentioned in the Introduction, we recall that Mandal and Pal [15], Wu and Ouyang [16], Wu [17], Deng et al. [18], Skouri et al. [19], Cheng and Wang [20], Cheng et al. [21], Hung [22], Lin [23], Yang et al. [24], and Lin et al. [25] had derived the exact holding cost and deterioration cost.
In recent years, inventory models of this kind are progressed to more complicated models, and deriving a closed-form expression for the optimal solution becomes too difficult. Therefore, researchers tried to develop models with approximated models with a closed-form solution. Thus, we see most researchers repeatedly reference and credit previously published findings in their paper citation, and then motivate their study problems and provide explicit derivations as research followers. However, such scholarly credits from previously published papers seem missing in Çalışkan [1]. Moreover, we cannot see a rational study motivation for the repeated computation in Çalışkan [1].

Moreover, we cite Widyadana et al. [2],
"The inventory rate can be represented as (see [27]):"
and
"The total inventory cost per unit time is equal to ordering cost plus holding cost:

$$\mathrm{TC(T)} = \frac{A}{T} + \frac{hD}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right]. \quad \text{(W3)"}$$

Remark. [27] of Widyadana et al. [2] is Zipkin [3] of this paper.
(W3) indicates that is Equation (3) of Widyadana et al. [2].

When Widyadana et al. [2] tried to develop an approximated inventory model such that they can apply the cost-difference comparison method proposed by Wee et al. [29], then they used the average inventory level $Q/2$ to simplify the exponential expression in their objective function. We agree that in the approximated inventory model, they used a roughly estimated (overestimated as claimed by Calinkan [1]) average inventory $Q/2$.

Because Widyadana et al. [2] is the Reference of Çalışkan [1], we can assume that he already knew the exact holding cost mentioned in (W3).

The above citation reveals that Çalışkan [1] almost certainly knew the results of the exact holding cost. However, Çalışkan [1] ignored the result of the exact holding cost presented in (W3) of Widyadana et al. [2].

VI. Review of His Approximated Solution

In this section, we will discuss the solution procedure presented in Çalışkan [1], and then we will provide comments in Section 7.

We cite related results from Çalışkan [1],
"Chung and Ting (1994) use the approximation $e^{\delta T} = (2+\delta T)/(2-\delta T)$. Substituting this in Equation (C17) yields:

$$\mathrm{TC_2(T)} = \mathrm{TC(T)} = \frac{S}{T} + \frac{h+c\delta}{\delta T}\left[\frac{2DT}{2-\delta T} - DT\right],$$
$$= \frac{S}{T} + (h+c\delta)\frac{DT}{2-\delta T}. \quad \text{(C18)"}$$

Remark. Chung and Ting (1994) of Çalışkan [1] is Chung and Ting [26] in this paper.
"$\mathrm{TC_2(T)}$" in (C18) was added by us to help readers.

We cite Çalışkan [1],
"Taking the derivative of Equation (C18) and setting equal to zero, we obtain:

$$\frac{-S}{T^2} + \frac{(h+c\delta)D(2-\delta T)+\delta(h+c\delta)DT}{(2-\delta T)^2} = 0. \quad \text{(C19)}$$
$$2(h+c\delta)DT^2 = 4s - 4S\delta T + S\delta^2T^2. \quad \text{(C20)}$$
$$[2(h+c\delta)D - S\delta^2]T^2 + 4S\delta T - 4S = 0. \quad \text{(C21)}$$

From the quadratic formula, the solution to Equation (C21) can be obtained as follows:

$$T^* = \frac{\sqrt{16S^2\delta^2 + 16[2(h+c\delta)D - S\delta^2]S} - 4S\delta}{2[2(h+c\delta)D - S\delta^2]}.$$
$$T^* = \frac{2\sqrt{2(h+c\delta)SD} - 2S\delta}{2(h+c\delta)D - S\delta^2}. \quad \text{(C22)}$$

It goes without saying that Equation (C22) is not very intuitive and it is too complicated to be used by practitioners in the industry. Therefore, we derive the closed form equation for Q^* instead. Substituting $e^{\delta T} = (2+\delta T)/(2-\delta T)$ in Equation (C4) yields:

$$Q = \frac{2DT}{2-\delta T}. \quad \text{(C23)}$$

Equation (C23) can also be expressed as:

$$T = \frac{2Q}{\delta Q + 2D}. \quad \text{(C24)}$$

Substituting Equations (C23) and (C24) in Equation (C16), we obtain:

$$\mathrm{TC(Q,T)} = \frac{S}{\frac{2Q}{\delta Q+2D}} + \frac{h+c\delta}{\delta T}\left[\frac{2DT}{2-\delta T} - DT\right],$$
$$= \frac{SD}{Q} + \frac{\delta S}{2} + \frac{h+c\delta}{\delta T}\frac{\delta DT^2}{2-\delta T}. \quad \text{(C25)}$$
$$\mathrm{TC_2(Q)} = \mathrm{TC(Q)} = \frac{SD}{Q} + (h+c\delta)\frac{Q}{2} + \frac{\delta S}{2}. \quad \text{(C26)"}$$

Remark. "$\mathrm{TC_2(Q)}$" in (C26) was added by us to help readers.

We cite Çalışkan [1],
"The second term in Equation (C26) is the average inventory holding plus waste cost per unit time. We have just proved that the average inventory level with the approximation for $e^{\delta T}$ that we have used in this paper is indeed equal to $Q/2$. This is a priori assumed in Chung and Ting (1994) without providing a proof or justification. Note that the average inventory level in the basic EQO model is also $Q/2$, which

can be calculated by dividing the area of the triangle under the inventory level line by the cycle length. Taking the derivative of Equation (C26) with respect to Q and setting equal to zero, we obtain the optimal order quantity equation as follows:

$$Q^* = \sqrt{\frac{2SD}{h+c\delta}}. \quad (C27)$$

Equation (C27) is much more intuitive compared to Equation (C22)."

Remark. Chung and Ting (1994) of Çalışkan [1] is Chung and Ting [26] in this paper.

VII. Review of Chung and Ting

Before we provide comments on the solution procedure of Çalışkan [1] and his criticism of Chung and Ting [26], we have to review the results of Chung and Ting [26] in advance. Because Chung and Ting [26] used different notations, we have to translate those findings of Chung and Ting [26] to the expressions to be consistent with Çalışkan [1].

We cite Chung and Ting [26], "Note that the holding cost is determined based on average inventory held, and the total inventory cost per unit time for one replenishment cycle is

$$TC_3 = c_t,$$
$$= \frac{S}{T} + \frac{a}{\theta}[\exp(\theta T) - 1]\frac{P}{T} + \frac{a}{\theta}[\exp(\theta T) - 1]\frac{H}{2}. \quad (Ch2)"$$

Remark. (Ch2) indicates that is Equation (2) in Chung and Ting [26].
"TC_3" in (Ch2) was added by us to help readers. In TC_3, the setup cost and purchasing cost are exact, but the holding cost is approximated.

We cite Chung and Ting [26],
"Applying L'Hostipital's rule, we have

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\exp(x)-1}\right) = \frac{1}{2}. \quad (Ch3)$$

When θ is small, by equation (3), we get

$$\exp(\theta T_i) \cong \frac{2+\theta T_i}{2-\theta T_i}. \quad (Ch4)$$

Using equation (Ch4), equation (Ch2) can be expressed as follows:

$$TC_4 = c_t = \frac{S}{T} + \frac{2a}{2-\theta T}P + \frac{a}{2-\theta T}HT.$$

Remark. "TC_4" in the above expression was added by us to help readers.

We cite Chung and Ting [26], "Therefore, the optimal replenishment cycle length can be obtained by:

$$T^* = \frac{2\left[\left((2aP\theta+2aH)S\right)^{1/2} - S\theta\right]}{2aP\theta+2aH-S\theta^2}. \quad (Ch5)"$$

Because Chung and Ting [26] and Çalışkan [1] used different notations such that we will rewrite the results of Chung and Ting [26] in the notation of Çalışkan [1] to help readers.

The original (approximated holding cost) inventory model studied by Chung and Ting [26] has the average cost per unit time as

$$TC_3 = c_t = \frac{S}{T} + \frac{cD}{\delta T}\left[e^{\delta T} - 1\right] + \frac{hD}{2\delta}\left[e^{\delta T} - 1\right], \quad (7.1)$$

where Chung and Ting [26] considered the approximated holding cost and the exact purchasing cost.

Applying $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$, Chung and Ting [26] further simplified their model as

$$TC_4(T) = c_t = \frac{S}{T} + \frac{2cD}{2-\delta T} + \frac{hDT}{2-\delta T}, \quad (7.2)$$

and then Chung and Ting [26] derived the closed-form approximated solution,

$$T^* = \frac{2\sqrt{2(h+c\delta)SD} - 2S\delta}{2(h+c\delta)D - S\delta^2}. \quad (7.3)$$

Our translation of Equation (7.3) which is (Ch5) mentioned in Chung and Ting [26] is the same result derived from (C22) in Çalışkan [1].

However, the two objective functions are different: the first one, $TC_2(T)$, is (C18) mentioned in Çalışkan [1], and the second one, $TC_4(T)$, is Equation (7.2), translation of (Ch2) in Chung and Ting [26].
In Section 8, we will provide a detailed explanation for the difference of $TC_2(T)$ and $TC_4(T)$.

We must point out that Çalışkan [1] used $TC_2(T)$ constructed by Çalışkan [1] to treat it as $TC_4(T)$.
Owing to the optimal solutions of $TC_2(T)$ and $TC_4(T)$ are identical such that we can say that Çalışkan [1] was not aware of the difference between $TC_2(T)$ and $TC_4(T)$.

VIII. Direction for Future Research

In the following section, we refer to a selection of recently published papers that suggest potential avenues for future research. Tang et al. [30] investigate customer behaviors in a supermarket during the Chinese New Year period. They employ customer analysis techniques to gain insights. Wan et al. [31] address the optimization of retailer warehouse operations through allocation arrangement strategies. In a study involving the breaking wave phenomenon, Unyapoti and Pochai [32] construct a binary model encompassing wave crest and shoreline evolution. Yang et al. [33] introduce a novel information system based on reciprocal accumulation generation operation and vector continued fractions. Tobar et al. [34] delve into segmentation issues, employing label enhancement and base representation methods. Purwani et al. [35] utilize the Newton-Raphson algorithm in combination with the Aitken extrapolation method to approximate stock volatility. Assis and Coelho [36] explore a remote learning and teaching project that employs temperature control as an educational tool. Considering structural dynamics, Adhitya et al. [37] analyze loads and concrete structures under earthquake conditions. Alomari and Massoun [38] utilize the Caputo fractional derivative to locate numerical solutions. Incorporating machine learning methodologies, Zhang et al. [39] develop a super-resolution image enhancement technique for morphologically sparse areas. Zhu et al. [40] investigate optimal train scheduling, taking carbon emissions into account. Mane and Lodhi [41] tackle singularly perturbed equations and provide a numerical solution using a cubic approach. These cited papers collectively offer valuable insights that can guide practitioners in aligning their research with current trends in the field.

IX. The Comparison Between $TC_2(T)$ and $TC_4(T)$

In this section, we will explain why two different models, $TC_2(T)$ and $TC_4(T)$, can imply the same optimal solution.

We recall that Çalışkan [1] and Chung and Ting [26] both used "S" as the setup cost and then for the average setup cost,

$$\frac{S}{T}, \tag{9.1}$$

appeared as the first term of (C18) and Equation (7.2).

We recall that Çalışkan [1] computed the cost of deteriorated items as

$$c(Q - DT) = c(I(0) - DT), \tag{9.2}$$

and then the average deteriorated items cost,

$$\frac{c(Q-DT)}{T} = \frac{c\delta}{\delta T}\left[\frac{D}{\delta}\left(e^{\delta T} - 1\right) - DT\right], \tag{9.3}$$

appeared in (C17), and then using $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$ to simplify Equation (9.3) to derive

$$(c\delta)\frac{DT}{2-\delta T}, \tag{9.4}$$

appeared in (C18).

In Chung and Ting [26], they evaluated the purchasing cost to find

$$cI(0), \tag{9.5}$$

to imply

$$\frac{cD}{\delta T}\left[e^{\delta T} - 1\right], \tag{9.6}$$

appeared in Equation (7.1), using $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$, to yield

$$\frac{2cD}{2-\delta T}, \tag{9.7}$$

appeared in Equation (7.2).

The results of Equations (9.4) and (9.7) are different which is the first difference between the objective functions studied by Chung and Ting [26] and Çalışkan [1].

For the holding cost, Çalışkan [1] computed the exact holding cost as

$$\frac{h}{\delta}(Q - DT) = \frac{h}{\delta}\left(\frac{D}{\delta}\left(e^{\delta T} - 1\right) - DT\right), \tag{9.8}$$

appeared in (C10), using $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$, then

$$\frac{hDT}{2-\delta T}, \tag{9.9}$$

appeared in (C18).

In Chung and Ting [26], they did not find the exact holding cost, instead, they directly used the average inventory level $I(0)h/2$ to yield

$$\frac{hD}{2\delta}\left[e^{\delta T} - 1\right], \tag{9.10}$$

appeared in Equation (7.1), applying $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$, then

$$\frac{hDT}{2-\delta T}, \tag{9.11}$$

appeared in (7.2).

The approximated average holding costs in (C18) of Çalışkan [1] and in Equation (7.2) proposed by Chung and Ting [26] are the same. However, by two different approaches, after simplification, why did the identically approximated result appear?

After our above discussions, we find that the objective functions of Chung and Ting [26] and Çalışkan [1] are two different approximated inventory models. We will begin to solve the following question:
Why two different approximated inventory models can have the same optimal solution in (C22) and (Ch5)?

We will need the following two lemmas to proceed with our discussion.

Lemma 1. The approximated holding cost of Chung and Ting [26] equals the approximated holding cost by Çalışkan [1] for the exact holding cost if and only if $e^{\delta T} = (2 + \delta T)/(2 - \delta T)$.
(Proof)
We refer to (C10) then the exact total holding cost is $\frac{hD}{\delta^2}\left(e^{\delta T} - 1 - \delta T\right)$ and then the average holding cost is $\frac{hD}{\delta^2 T}\left(e^{\delta T} - 1 - \delta T\right)$ that was proposed by Çalışkan [1].
The total approximated holding cost proposed by Chung and Ting [26] is $\frac{h}{2}I(0)T$, and then the average approximated holding cost proposed by Chung and Ting [26] is $\frac{h}{2}I(0) = \frac{h}{2}\left(\frac{D}{\delta}\right)\left(e^{\delta T} - 1\right)$.
We evaluate that

$$\frac{hD}{\delta^2 T}\left(e^{\delta T} - 1 - \delta T\right) = \frac{h}{2}\left(\frac{D}{\delta}\right)\left(e^{\delta T} - 1\right) \Leftrightarrow,$$
$$\Leftrightarrow 2\left(e^{\delta T} - 1 - \delta T\right) = \delta T\left(e^{\delta T} - 1\right),$$
$$\Leftrightarrow e^{\delta T} = \frac{2+\delta T}{2-\delta T}. \tag{9.12}$$

Lemma 2. The deteriorated items cost proposed by Çalışkan [1] is different from the purchasing cost of Chung and Ting [26] by a constant.
(Proof)
The average deteriorated items cost proposed by Çalışkan [1] is $(c\delta)\frac{DT}{2-\delta T}$, and the average purchasing cost proposed by Chung and Ting [26] is $\frac{2cD}{2-\delta T}$. We find that

$$(c\delta)\frac{DT}{2-\delta T} + cD = \frac{2cD}{2-\delta T}. \tag{9.13}$$

Remark. Intuitively, the difference between $\frac{c}{T}(I(0) - DT)$ and $\frac{c}{T}I(0)$ is a constant: cD. After simplification by $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$, the difference is still the same constant, cD.

Based on our Lemmas 1 and 2, we know that the objective functions of (C18) and Equation (7.2) (our translation of Chung and Ting [26]) are different by a constant, such that the optimal solution presented in the variable T in (C22) and (Ch5) (or our translation of Equation (7.3)) must be identical. Therefore, we present a reasonable explanation for why two different approximated inventory models proposed by Chung and Ting [26] and Çalışkan [1] can have the same optimal solution.

We can say that Çalışkan [1] was not aware of the following two things:
(A) The average purchasing cost in Chung and Ting [26] is different from his average deteriorated items cost by a constant.
(B) The twice approximated holding cost of Chung and Ting [26] equals the approximated holding cost proposed by Çalışkan [1].

X. MORE COMMENTS ON HIS SOLUTION APPROACH

We provide an application of our two lemmas to simplify the derivation of Çalışkan [1] for $CT(Q)$ as follows.

Without referring to TC_1, we can use the results concerning Chung and Ting [26], and then we substitute (C23) and (C24) into (C18) to imply that

$$TC(Q) = S\frac{\delta Q+2D}{2Q} + (h + c\delta)\frac{Q}{2},$$
$$= \frac{SD}{Q} + (h + c\delta)\frac{Q}{2} + \frac{\delta S}{2}, \qquad (10.1)$$

which is the result of (C26), to avoid the complicated computation of (C25).

In the following, we provide an alternative approach to derive the main result of Çalışkan [1]. Based on (C19), without expanding the rational function to a quadratic polynomial in T, we rewrite the expression of (C19) as $\frac{S}{T^2} = \frac{2D(h+c\delta)}{(2-\delta T)^2}$, then

$$\frac{2SD}{h+c\delta} = \frac{4D^2T^2}{(2-\delta T)^2} = Q^2, \qquad (10.2)$$

that is the finding of (C27). Hence, we present an easy derivation to replace those computations from (C20) to (C27).

We admit that the result of (C22) for the objective function of (C18) is the result of Equation (7.3) for the objective function of Equation (7.2) looks more complicated than the result of (C27). However, based on our approach to Equation (10.2), we can derive the same finding, even without constructing the objective function, TC(Q) proposed by Çalışkan [1] in (C26).

Çalışkan [1] did not provide a reasonable motivation for why did he use $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$. To the best of our knowledge, in Lemma 1 of Wan and Chu [42], they recalled that

$$xe^{-x} + 2e^{-x} + x - 2 > 0, \qquad (10.3)$$

for $x > 0$. In the proof of Lemma 1, Wan and Chu [42] mentioned that Rachamadugu [43] and Chung and Lin [44] have studied

$$e^{-x} > \frac{2-x}{2+x}. \qquad (10.4)$$

Our review of Equations (10.3) and (10.4) can be treated as sources for the substitution of $e^{\delta T} = \frac{2+\delta T}{2-\delta T}$.

Last but not least, we will show that based on (C22) and (C23) how to derive (C27).

$$\frac{D}{Q} = \frac{2-\delta T}{2T} = \frac{1}{T} - \frac{\delta}{2} = \frac{\left(2(h+c\delta)D - S\delta^2\right) - \delta\left(\sqrt{2(h+c\delta)SD} - S\delta\right)}{2\left(\sqrt{2(h+c\delta)SD} - S\delta\right)},$$
$$= \frac{\left(\sqrt{2(h+c\delta)D} - \sqrt{S}\delta\right)\sqrt{2(h+c\delta)D}}{2\sqrt{S}\left(\sqrt{2(h+c\delta)D} - \sqrt{S}\delta\right)} = \frac{\sqrt{(h+c\delta)D}}{\sqrt{2S}}. \qquad (10.5)$$

From Equation (10.5), then

$$Q = \sqrt{\frac{2S}{(h+c\delta)D}}D = \sqrt{\frac{2SD}{h+c\delta}}, \qquad (10.6)$$

which is (C27).

Hence, we provide a second approach to obtain a closed-form optimal solution without deriving the objective function of $TC_2(Q)$ of (C26).

XI. A Related Inventory Model

In this section, for Lemma 4.1 of Yang et al. [45], they tried to solve the maximum problem of the following question,

$$\max_{(q_1,q_2),q_1=q_2\geq 0} \Pi_{t_0-L1}(q_1,q_2) =$$
$$-c_1q_1 - c_2q_2 + \int_0^\infty \Delta f(a)da, \qquad (11.1)$$

where

$$\Delta = p\min\{a,q_2\} + s(q_2-a)^+ - c_u\left(a-q_2^{\ +}\right), \qquad (11.2)$$

is an abbreviation to simplify the expression.

The meaning of parameters and variables are defined as follows. q_1 is the retailer's order quantity for component 1; q_2 is the retailer's order quantity for component 2; t_0 is the starting time of the selling season; L_1 is the purchasing lead-time of the first component; c_1 is the production cost of component 1, per unit item, with $c_1 > 0$; c_2 is the production cost of component 2, per unit item, with $c_2 > 0$; p is the retail price of the final product per unit item; s is the unit salvage value of the final product, with $0 \leq s < c_1 + c_2$; c_u is the unit shortage penalty for the final product, with $c_u \geq 0$, and f(a) is the probability density function of market observation. In this section, we will provide our analytic solution procedure. In Section XII, we will discuss the findings of Yang et al. [45] to explain their questionable results. In Section XIII, we show an alternative solution technique, algebraic method, for the same optimal problem, such that those researchers who are not used to calculus still can absob this kind of inventory systems.

We begin our analytic process to rewrite Equation (11.1) as follows,

$$\Pi_{t_0-L1}(q_1) = -(c_1+c_2)q_1$$
$$+\int_0^{q_1}\left[pa + s(q_1-a)\right]f(a)da$$
$$+\int_{q_1}^{\infty}\left[pq_1 - c_u(a-q_1)\right]f(a)da. \qquad (11.3)$$

By the Leibniz rule, we derive that

$$\frac{d}{dq_1}\Pi_{t_0-L1}(q_1) = -(c_1+c_2)$$
$$+\int_0^{q_1} sf(a)da + \int_{q_1}^{\infty}(p+c_u)f(a)da. \qquad (11.4)$$

We assume the accumulated distribution of $f(a)$ as

$$F(a) = \int_0^a f(x)dx, \qquad (11.5)$$

with $F(0)=0$ and $F(\infty)=1$. Hence, we can rewrite Equation (11.4) as

$$\frac{d}{dq_1}\Pi_{t_0-L1}(q_1) = -(c_1+c_2) + sF(q_1)$$
$$+(p+c_u)(1-F(q_1)). \qquad (11.6)$$

To solve $\frac{d}{dq_1}\Pi_{t_0-L1}(q_1) = 0$, it yields that

$$p - c_1 - c_2 + c_u = (p - s + c_u)F(q_1) \qquad (11.7)$$

and then it follows that the optimal solution is derived as follows,

$$q_1 = F^{-1}\left(\frac{p-c_1-c_2+c_u}{p-s+c_u}\right). \qquad (11.8)$$

XII. Examination of Previous Result

In this section, we begin to review of the proof proposed by Yang et al. [45] for their Lemma 4.1. Yang et al. [45] mentioned that

$$\Pi_{t_0-L1}(q_1) = (p - c_1 - c_2 + c_u)q_1 - (p - s + c_u)\int_0^{q_1} F(a)da . \quad (12.1)$$

From Equation (12.1), to take the first derivation with respect to q_1, researcher can obtain the same result as Equation (11.7). However, the derivation of Equation (12.1) is questionable.

We rewrite Equation (11.3) as

$$\Pi_{t_0-L1}(q_1) = -(c_1 + c_2)q_1 + \int_0^{q_1}[pa + s(q_1 - a)]f(a)da + \int_0^{\infty}[pq_1 - c_u(a - q_1)]f(a)da - \int_0^{q_1}[pq_1 - c_u(a - q_1)]f(a)da , \quad (12.2)$$

such that we simplify Equation (12.2) to find

$$\Pi_{t_0-L1}(q_1) = (p - c_1 - c_2 + c_u)q_1 - c_u\int_0^{\infty} a\, f(a)da - (p - s + c_u)\int_0^{q_1}(q_1 - a)f(a)da . \quad (12.3)$$

Now, we compare Equations (12.1) and (12.3) to know that the following relation must be verified

$$\int_0^{q_1} F(a)da = \int_0^{q_1}(q_1 - a)f(a)da + C , \quad (12.4)$$

where C is a constant with respect to q_1.

Owing to $\frac{d}{dq_1}\int_0^{q_1} F(a)da = F(q_1)$ and based on the Leibniz rule,

$$\frac{d}{dq_1}\int_0^{q_1}(q_1 - a)f(a)da = \int_0^{q_1} f(a)da = F(q_1), \quad (12.5)$$

we know that Equation (12.4) is valid.

When we plug $q_1 = a$ into Equation (12.4), we derive that

$$C = \int_0^{a} F(a)da . \quad (12.6)$$

Consequently, we revise the findings of Yang et al. [45] from Equation (12.1) to the next result

$$\Pi_{t_0-L1}(q_1) = (p - c_1 - c_2 + c_u)q_1 - (p - s + c_u)\left[\int_0^{q_1} F(a)da - \int_0^{a} F(a)da\right]. \quad (12.7)$$

XIII. An Algebric Process

In this section, for those practitioners who did not know differential equations, we provide a second solution approach, algebraic process. We implicitly accept the Mean value theorem of integration to deal with integration problems.

Based on Equation (12.7), we compute Φ_1 and Φ_2, where

$$\Phi_1 = \Pi_{t_0-L1}(q_1) - \Pi_{t_0-L1}(q_1 - \Delta q_1) \quad (13.1)$$

and

$$\Phi_2 = \Pi_{t_0-L1}(q_1) - \Pi_{t_0-L1}(q_1 + \Delta q_1). \quad (13.2)$$

We find that

$$\Phi_1 = (p - c_1 - c_2 + c_u)\Delta q_1 - (p - s + c_u)\Delta q_1 F(q_1 - \varepsilon_1 \Delta q_1), \quad (13.3)$$

under the restriction of $\Delta q_1 > 0$. To preserve $\Phi_1 \geq 0$ that is equivalent to verify that

$$\frac{p - c_1 - c_2 + c_u}{p - s + c_u} \geq F(q_1 - \varepsilon_1 \Delta q_1). \quad (13.4)$$

with the following condition, $0 \leq \varepsilon_1 \leq 1$.

Similarly, we obtain that

$$\Phi_2 = -(p - c_1 - c_2 + c_u)\Delta q_1 + (p - s + c_u)\Delta q_1 F(q_1 + \varepsilon_2 \Delta q_1), \quad (13.5)$$

with $0 \leq \varepsilon_2 \leq 1$. Owing to $\Delta q_1 > 0$, to preserve $\Phi_2 \geq 0$ that is equivalent to show the following,

$$F(q_1 + \varepsilon_2 \Delta q_1) \geq \frac{p - c_1 - c_2 + c_u}{p - s + c_u}. \quad (13.6)$$

We combine the results of Equations (13.4) and (13.6) to derive that the optimal solution of q_1 should satisfy

$$F(q_1) = \frac{p - c_1 - c_2 + c_u}{p - s + c_u} \quad (13.7)$$

to find the same optimal solution as Equation (11.8).

XIV. Conclusion

This study has delved into the work of Çalışkan [1] and highlighted several key observations. Firstly, we have demonstrated that Çalışkan [1] derived well-established results without sufficient justification for repeatedly obtaining the published findings. Through our analysis, we have also emphasized the disparities between the two objective functions: $TC_4(T)$ of Chung and Ting [26], and $TC_2(T)$ of Çalışkan [1].

Furthermore, our investigation led us to present two essential lemmas, which elucidate how two distinct objective functions can lead to the same optimal solution. Building upon these lemmas, we introduced two alternative solution approaches that successfully yield the optimal solution derived by Çalışkan [1], all while circumventing the necessity of referencing his objective function, $TC_2(Q)$.

By addressing these aspects, our paper serves as a valuable resource for researchers, fostering open debate and enhanced comprehension of the works of Chung and Ting [26] and Çalışkan [1]. We hope that our contributions will encourage further exploration and critical analysis within the field, leading to advancements in inventory modeling and optimization techniques.

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