

# An Algorithm to Find the Optimal Matching in Halin Graphs

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**Abstract**—This paper deals with the problem of searching the maximum-weighted matching in Halin graphs. We give an  $O(|V|)$  algorithm.

**Keywords**—maximum-weighted, matching, Halin graph, algorithm

## I. INTRODUCTION

Finding a matching with the maximum total weight is the wellknown assignment problem of assigning people to jobs and maximize the profits.

We consider the maximum-weighted matching problem in graphs. That is, given an undirected graph  $G=(V,E)$  and let  $w$  be a nonnegative real-valued function on its edges; we wish to find a matching with the maximum total weight. This is the wellknown assignment problem of assigning people to jobs and maximum the profits. Let  $M$  be a matching of graph  $G$  and  $v \in V(G)$ . If  $v$  is incident with an edge in  $M$ , then  $v$  is called saturated, otherwise  $v$  is said to be unsaturated.

A Halin graph  $H=T \cup C$  is obtained by embedding a tree  $T$  having no nodes of degree 2 in the plane, and then adding a cycle  $C$  to join the leaves of  $T$  in such a way that the resulting graph is planar. If  $T$  is a star, that is, a single vertex joining to the other vertices, then  $H$  is called a wheel. Suppose  $T$  has at least two nonleaves. Let  $w$  be a nonleaf of  $T$  which is adjacent to only one nonleaf of  $T$ . Then the set of leaves of  $T$  adjacent to  $w$ , which we denote by  $C(w)$ , comprises a consecutive subsequence of the cycle  $C$ . We call the subgraph of  $H$  induced by  $\{w\} \cup C(w)$  a fan and call  $w$  the centre of the fan. In Fig. 1, the black vertices are the centers of the fans.

A solution of this problem can be computed in  $O(|V|^3)$  time by Edmonds' method [4] with the modifications proposed by Gabow [5] and Lawler [7]. Gail, Micali and Gabow developed an  $O(|V||E|\log|V|)$  algorithm in [6]. We restrict the problem to Halin graphs and present a linear time algorithm. For the terminology and notation not defined in this paper, reader can refer to [1].

Let  $G=(V,E)$  be a graph. For any  $S \subseteq V(G)$ , let  $\delta(S)$  denote the set of edges with one end-vertex in  $S$  and the other in  $V(G)-S$ ; that is  $\delta(S)=\{uv \in E: u \in S, v \notin S\}$ . For  $v \in V$ , we abbreviate  $\delta(\{v\})$  by  $\delta(v)$ . The edges of  $G$  having both ends in  $S \subseteq V$  are denoted by  $\chi(S)=\{uv \in E: u \in S \text{ and } v \in S\}$ .

Let  $F$  be a fan. We use  $H \times F$  to denote the graph obtained from  $H$  by shrinking  $F$  to a new 'pseudo-vertex', say  $v_F$ . That is  $V(H \times F)=\{v_F\} \cup \{V(H) \setminus V(F)\}$  and the edges of  $H \times F$  are defined as follows:

1. An edge with both ends in  $F$  is deleted;
2. An edge with both ends in  $H-F$  remains unchanged;
3. An edge of  $\delta(F)$  now joins the incident vertex of  $H-F$  and  $v_F$

Denote the closure of  $H$  under the operation of "Shrinking Fan" by  $H^*$ .

A graph  $G$  is called a wheel if  $G$  consists of a cycle every vertex of which is joined to a single common vertex by an edge.

An edge cutset of a connected graph  $G=(V,E)$  is a set of edges whose removal leaves a disconnected graph. If it consists of exactly  $k$  edges, then we call it a  $k$ -edge cutset. The shores of an edge cutset  $J$  are the vertices of the (two) components of  $G-J$ . Given a fan  $F$  of  $H$ , the three edges connecting  $V(F)$  to  $V(H-F)$  compose a 3-edge cutset of  $H$ . We denote the 3-edge cutset by  $EC_3(F)$ .

This paper is organized as follows: in section 2) we review the lemmas which are basic for the method of "Shrinking Fan" in Halin graphs, in section 3) we reveal the feasibility of finding the maximum-weighted matching in Halin graphs by means of "Shrinking Fans", in section 4)-5) we present the algorithm to find the maximum-weighted matching in Halin graphs and in section 6) extensions of the algorithm are provided.

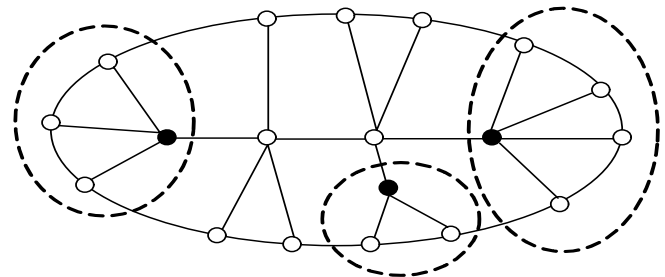


Figure.1 A Halin graph

## II. PRELIMINARY RESULTS

**Lemma 1** (Cornuejols, Naddef and Pulleyblank[2]). A Halin graph  $H=T \cup C$  which is not a wheel has at least two fans.

**Lemma 2** (Cornuejols, Naddef and Pulleyblank [2]). If  $F$  is a fan in a Halin graph  $H$ , then  $H \times F$  is a Halin graph.

The  $O(n)$  algorithm for the TSP developed in [2] uses Lemma 1 and Lemma 2 recursively to reduce the Halin graph into a wheel. After each reduction step, the edge weights are modified so that the optimal objective function value of the TSP on the original graph is the same as that on the reduced graph.

### III. ANALYSIS

Let  $G=(V,E)$  be a general graph with edge weights  $w: E \rightarrow R$  and  $M$  is a matching of  $G$ . An incidence vector  $x$  is associated with the edges of  $G$ :  $x_e$  is set to 1 if  $e \in M$ ; otherwise,  $x_e$  is set to 0. For any subset  $S \subseteq E$ , we define  $x(S) = \sum x_e (e \in S)$ . Moreover, let  $O$  consist of all non-singleton odd cardinality subsets of  $V$ :  $O = \{B \subseteq V: |B| \text{ is odd and } |B| \geq 3\}$ .

The maximum-weighted matching problem of  $G$  with weighted function  $w$  can be formulated as :

(WM) maximum  $w^T x$ .

Subject to

1.  $x(\chi(B)) \leq \lfloor |B|/2 \rfloor$  for all  $B \in O$ ,
2.  $x_e \in \{0,1\}$  for all  $e \in E$ .

(the edges which are set to 1 are not adjacent and there don't exit two adjacent vertices which are both unsaturated).

Now we investigate the method of "Shrinking Fan". It is a variant usage to the idea of shrinking a fan in [2]. Cornuejols, Naddef and Pulleyblank described a linear system for Travelling Salesman Polytope  $P(H)$  for a Halin graph  $H$ . Moreover, they proved the following lemma:

**Lemma 3** (Cornuejols, Naddef and Pulleyblank[2]). Let  $K$  be a 3-edge cut in a hamiltonian graph  $G=(V,E)$  and let  $S$  be a shore of the cut. Then a linear system for  $P(G)$  can be obtained by taking the union of linear system for  $P(G \times S)$  and  $P(G \times (V - S))$ .

It is an application of "divide and conquer". In the light of this, we provide a system to solve the WM problem in Halin graphs. Firstly, we introduce the basic function of our system.

Let  $F$  be a fan of  $H$ ,  $M$  be a matching of  $H$  and  $EC_3(F) = \{i,j,k\}$ . Let  $i=u_i v_i, j=u_j v_j$  and  $k=u_k v_k$  such that  $u_i, u_j, u_k \in V(F)$ ,  $v_i, v_j, v_k \in V(H-F)$ ,  $j, k \in C$  and the direction  $v_j, u_j, u_k, v_k$  is anti-clockwise in the cycle  $C$ . The vertex  $u_i$  has three situations: 1)  $u_i$  is unsaturated; 2) If  $i \in M$ , we say the vertex  $u_i$  is externally saturated in  $F$ ; 3) otherwise, internally. The denotation is the same to  $u_j$  and  $u_k$ . Only the situation of  $\{X(u_i), X(u_j), X(u_k)\}$  affects the situation of  $M \cap (H-F)$ . Hence, we use a function  $W$  to store the total weight of maximum-weighted matching to a certain fan  $F$ , corresponding

to each chosen situation of  $EC_3(F)$ . We use  $X(v)$  to denote the state of the vertex  $v$ .  $X(v)=1$  means  $v$  is externally saturated.  $X(v)=0$  means  $v$  is internally saturated.  $X(v)=-1$  means  $v$  is unsaturated. Let  $W(F, X(u_i), X(u_j), X(u_k))$  be the value of the solution to the restricted linear program in  $F$ , as following:

(WMF) maximum  $w^T x$   
 Subject to  
 1.  $x(\chi(B)) \leq \lfloor |B|/2 \rfloor$  for all  $B \in O(F)$ ,  
 2.  $x_e = 0$  for all  $e \in E(F)$ ,  $x_e = 0$  for  $e \notin E(F) \cup \{i,j,k\}$ .  
 $O(F) = \{B \subseteq V(F): |B| \text{ is odd and } |B| \geq 3\}$  and  $x(i), x(j), x(k)$  are given.

If no solution to a certain triple  $\{X(u_i), X(u_j), X(u_k)\}$ , then the corresponding value of  $W$  is set to be *NULL*. We consider an original vertex on the cycle  $C$  as a special fan. We define the function  $W$  to an original vertex  $u$  as following: Let  $i=uv_i, j=uv_j$  and  $k=uv_k$  such that  $v_j, v_k \in V(C)$ ,  $v_i \in V(H-C)$  and the direction of  $v_k, u, v_j$  is clockwise in the embed plane.

$W(u, -1, -1, -1) = 0, W(u, 1, 0, 0) = w(i), W(u, 0, 0, 1) = w(k), W(u, 0, 1, 0) = w(j)$ , others are all *NULL*.

Now we describe how to calculate the value of the function. Let  $F$  be a fan of  $H$ . Let  $u_r$  be the centre of  $F$  and  $u_1, u_2, \dots, u_n$  be the vertices on the cycle  $C$  in that order. Each  $u_i$  ( $1 \leq i \leq n$ ) can be either an original vertex or a pseudo-vertex where the function  $W$  of each  $u_i$  has been calculated already.

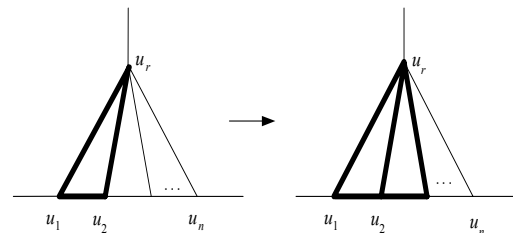


Figure.2 Extension of pseudo-fan

We denote the induced subgraph  $H[\{u_r, u_1, u_2, \dots, u_k\}]$  by pseudo-fan  $PF_k$  of  $F$  where  $2 \leq k \leq n$ . We examine the situation of  $k=2$  first. We use the following formulas (1)-(3) to compute the values to the function  $W$  of  $F$ . Let  $b, c \in \{-1, 0, 1\}$ .

Since there are four combinations of the values of  $b$  and  $c$ , formula (1) stands for nine formulas:  $W(PF_2, 0, -1, -1)$ ,  $W(PF_2, 0, -1, 0)$ ,  $W(PF_2, 0, 0, -1)$ ,  $W(PF_2, 0, 0, 0)$ ,  $W(PF_2, 0, 0, 1)$ ,  $W(PF_2, 0, 1, 0)$  and  $W(PF_2, 0, 1, 1)$ ,  $W(PF_2, 0, -1, 1)$ ,  $W(PF_2, 0, 1, -1)$ . So do formula (2) and (3). This abbreviation will be carried unless otherwise stated. Note that we don't add the value of the edge adjacent to  $u_r$  in its externally saturated situations, because we cannot determine which edge saturates it. To the pseudo-fan  $PF_k$ , each edge  $u_r u_m$  ( $k < m \leq n$ ) might be such an edge.

Once we have the values to the function  $W$  of  $PF_k$ , we can calculate the function  $W$  of  $PF_{k+1}$ . Similarly, we present the formulas (4)-(6) to the calculation as following:

Once the values to the function  $W$  of  $PF_n$  are obtained, we can compute the function  $W$  of  $F$ . The formulas are presented as follows:

$$W(F, 0, b, c) = W(PF_n, 0, b, c),$$

$$W(F,-1,b,c)=W(PF_n,-1,b,c),$$

$$W(F,1,b,c)=W(PF_n,1,b,c)+w(i).$$

In the calculation of the function  $W$ , we adopt a mechanism similar with “short-circuit evaluation”; that is, if a value used in a subformula is *NULL*, then the result is set to be *NULL*, directly. And we consider *NULL* as a minimum value in the comparison. Besides the situation of original vertex, the value to the function  $W$  might also be set to be *NULL* in some situations as to a fan.

When a fan is shrunk, the values to be the function  $W$  of the fan are stored into the function  $W$  of the pseudo-vertex. Moreover, we store the structure which gets the maximum value into the pseudo-vertex; that is, record the matching structure corresponding to the value to the function  $W$  in each input. If there are more than one situation obtain the maximum value, choose one arbitrarily.

$$W(PF_2,0,b,c)=\max \left\{ \begin{array}{l} W(u_1,0,b,0) + W(u_2,1,0,c), \\ W(u_1,0,b,0) + W(u_2,1,-1,c), \\ W(u_1,0,b,-1) + W(u_2,1,0,c), \\ W(u_1,-1,b,0) + W(u_2,1,0,c), \\ W(u_1,-1,b,0) + W(u_2,1,-1,c), \\ W(u_1,-1,b,-1) + W(u_2,1,0,c), \\ W(u_1,0,b,1) + W(u_2,1,1,c) - w(u_1,u_2), \\ W(u_1,-1,b,1) + W(u_2,1,1,c) - w(u_1,u_2), \\ W(u_1,1,b,0) + W(u_2,0,0,c), \\ W(u_1,1,b,0) + W(u_2,0,-1,c), \\ W(u_1,1,b,0) + W(u_2,-1,0,c), \\ W(u_1,1,b,0) + W(u_2,-1,-1,c), \\ W(u_1,1,b,-1) + W(u_2,-1,0,c), \\ W(u_1,1,b,-1) + W(u_2,0,0,c), \\ W(u_1,1,b,1) + W(u_2,0,1,c) - w(u_1,u_2)\}, \\ W(u_1,1,b,1) + W(u_2,-1,1,c) - w(u_1,u_2)\} \end{array} \right. \quad (1)$$

$$W(PF_2,1,b,c)=\max \left\{ \begin{array}{l} W(u_1,0,b,0) + W(u_2,0,0,c), \\ W(u_1,0,b,0) + W(u_2,0,-1,c), \\ W(u_1,0,b,0) + W(u_2,-1,0,c), \\ W(u_1,0,b,0) + W(u_2,-1,-1,c), \\ W(u_1,0,b,-1) + W(u_2,0,0,c), \\ W(u_1,0,b,-1) + W(u_2,-1,0,c), \\ W(u_1,-1,b,0) + W(u_2,0,0,c), \\ W(u_1,-1,b,0) + W(u_2,0,-1,c), \\ W(u_1,-1,b,-1) + W(u_2,0,0,c), \\ W(u_1,-1,b,-1) + W(u_2,-1,0,c), \\ W(u_1,0,b,1) + W(u_2,0,1,c) - w(u_1,u_2)\}, \\ W(u_1,-1,b,1) + W(u_2,0,1,c) - w(u_1,u_2)\}, \\ W(u_1,-1,b,1) + W(u_2,-1,1,c) - w(u_1,u_2)\} \end{array} \right. \quad (2)$$

$$W(PF_2,-1,b,c)=\max \left\{ \begin{array}{l} W(u_1,0,b,0) + W(u_2,0,0,c), \\ W(u_1,0,b,0) + W(u_2,0,-1,c), \\ W(u_1,0,b,-1) + W(u_2,0,0,c), \\ W(u_1,0,b,1) + W(u_2,0,1,c) - w(u_1,u_2)\} \end{array} \right. \quad (3)$$

$$W(PF_{k+1},0,b,c)=\max \left\{ \begin{array}{l} W(PF_k,1,b,1) + W(u_{k+1},1,1,c) - w(u_k,u_{k+1}), \\ W(PF_k,1,b,-1) + W(u_{k+1},1,0,c), \\ W(PF_k,0,b,-1) + W(u_{k+1},0,0,c), \\ W(PF_k,0,b,-1) + W(u_{k+1},-1,0,c), \\ W(PF_k,0,b,1) + W(u_{k+1},0,1,c) - w(u_k,u_{k+1}), \\ W(PF_k,0,b,1) + W(u_{k+1},-1,1,c) - w(u_k,u_{k+1}), \\ W(PF_k,1,b,0) + W(u_{k+1},1,0,c), \\ W(PF_k,1,b,0) + W(u_{k+1},1,-1,c), \\ W(PF_k,0,b,0) + W(u_{k+1},0,0,c), \\ W(PF_k,0,b,0) + W(u_{k+1},0,-1,c), \\ W(PF_k,0,b,0) + W(u_{k+1},-1,0,c), \\ W(PF_k,0,b,0) + W(u_{k+1},-1,-1,c) \end{array} \right. \quad (4)$$

$$W(PF_{k+1}, 1, b, c) = \max \left\{ \begin{array}{l} W(PF_k, 1, b, -1) + W(u_{k+1}, 0, 0, c), \\ W(PF_k, 1, b, -1) + W(u_{k+1}, -1, 0, c), \\ W(PF_k, 1, b, 1) + W(u_{k+1}, 0, 1, c) - w(u_k, u_{k+1}), \\ W(PF_k, 1, b, 1) + W(u_{k+1}, -1, 1, c) - w(u_k, u_{k+1}), \\ W(PF_k, 1, b, 0) + W(u_{k+1}, 0, 0, c), \\ W(PF_k, 1, b, 0) + W(u_{k+1}, 0, -1, c), \\ W(PF_k, 1, b, 0) + W(u_{k+1}, -1, 0, c), \\ W(PF_k, 1, b, 0) + W(u_{k+1}, -1, -1, c) \end{array} \right\} \quad (5)$$

$$W(PF_{k+1}, -1, b, c) = \max \left\{ \begin{array}{l} W(PF_k, -1, b, 0) + W(u_{k+1}, 0, 0, c), \\ W(PF_k, -1, b, 0) + W(u_{k+1}, 0, -1, c), \\ W(PF_k, -1, b, -1) + W(u_{k+1}, 0, 0, c), \\ W(PF_k, -1, b, 1) + W(u_{k+1}, 0, 1, c) - w(u_k, u_{k+1}) \end{array} \right\} \quad (6)$$

IV. PROCEDURE OF THE ALGORITHM

Algorithm to Find the Optimal Matching in Halin Graphs

- 1) Choose a non-leaf vertex of  $T$ , denoted by  $v_{root}$ , such that  $v_{root}$  is adjacent to a leaf of  $T$ , denoted by  $v_{leaf}$ .
  - 2) Perform a postorder scan of  $T$ , for each fan  $F$  has been found do
  - 3) If the centre of  $F \neq v_{root}$  then  
 Begin  
 Calculate the values to the function  $W$  of  $F$ ;  
 Shrink  $F$  to  $v_F$ ;  
 Store the values above into the function  $W$  of  $v_F$ ;  
 Store the corresponding structure of  $F$  to each value above.  
 End  
 Let  $H_w$  be the wheel we finally get and  $i$  be the edge joining  $v_{root}$  and  $v_{leaf}$ .  
 Let  $j, k$  be the two edges in  $C$  adjacent to  $v_{leaf}$  such that the direction of  $j, v_{leaf}, k$  is clockwise.  
 Let  $F_w = H_w - v_{leaf}$ .
  - 4) Calculate the values to the function  $W$  of  $F_w$ .
  - 5) Store the matching structure of  $F_w$  corresponding to each value.
  - 6) Shrink the fan  $F_w$  to the pseudo-vertex  $v_w$ .
  - 7) Cost :=  $\max \{ W(v_w, 1, 0, 0), W(v_w, 0, 1, 0), W(v_w, 0, 0, 1), W(v_w, 0, 0, 0) \}$ .
- Let the edge in  $\{i, j, k\}$ , the choice of which makes the maximum, be  $e_M$ .
- 8) Mark  $e_M$  belonging to the matching.
  - 9) While (there is pseudo-vertex in  $H$ ) do

10) Begin

Let  $v_F$  be a pseudo-vertex such that  $v_F$  is shrunk from the fan  $F$ .

- 11) According to the chosen situation of  $EC_3(F)$  and the information stored in  $v_F$ , we can obtain the corresponding matching structure in  $F$ .
- 12) Restore  $v_F$  to the fan  $F$ .
- 13) Mark the matching edges in  $F$ .
- 14) End
- 15) All the marked edges compose the maximum-weighted matching.

V. CORRECTNESS AND TIME COMPLEXITY

Let  $H$  be a Halin graph. Let  $F$  be a fan in  $H^* \in H^*$ . Let  $F_{ex}$  be the subgraph of  $H$  which  $F$  is fully restored to. Equivalently,  $F_{ex}$  is the component(shore) of  $H-EC_3(F)$  which  $F$  stands for.

**Theorem 1.** Let  $H$  be a Halin graph and  $F$  be a fan in  $H^* \in H^*$ . Let  $EC_3(F) = \{i, j, k\}$  be the same as in section 3). Then  $W(F, X(u_i), X(u_j), X(u_k))$  records the value of solutions to WMF of  $F_{ex}$  in the input  $X(u_i), X(u_j)$  and  $X(u_k)$ .

Proof. We apply induction on  $H^* = H$ .

Induction Basis: We need to prove the proposition holds for any fan  $F$  in  $H$ . Let  $u_r$  be the centre of  $F$  and let  $u_1, u_2, \dots, u_n$  ( $n \geq 2$ ) be the vertices of  $F$  on the cycle  $C$  (in anti-clockwise order). Since  $u_1, u_2, \dots, u_n$  are original vertices, the function  $W$  of  $u_1, u_2, \dots, u_n$  have been calculated. Table 1 shows that the formulas (1), (2) and (3) enumerate all the solutions to WMF of pseudo-fan  $PF_2$ . Since they contain at least one function whose value is  $NULL$ , the values of them are all  $NULL$ .

Table 1. Sub-formulas of  $PF_2$  in  $H$

$x(u, u_1)$	$x(u, u_2)$	$x(u_1, u_2)$	Sub-formula
0	0	0	$W(u_1, 0, 1, 0) + W(u_2, 0, 0, 1)$
0	0	1	$W(u_1, 0, 1, 0) + W(u_1, 0, 2, 1, 0) - w(u_1, u_2)$
0	1	0	$W(u_1, 0, 1, 0) + W(u_2, 1, 0, 0)$
1	0	0	$W(u_1, 1, 0, 0) + W(u_2, 0, 0, 1)$
1	0	0	$W(u_1, 1, 0, 0) + W(u_2, -1, -1, -1)$
0	1	0	$W(u_1, -1, -1, -1) + W(u_2, 1, 0, 0)$
0	0	0	$W(u_1, -1, -1, -1) + W(u_2, 0, 0, 1)$
0	0	0	$W(u_1, 0, 1, 0) + W(u_2, -1, -1, -1)$

The formulas (4)-(6) compute the solutions to WMF of  $PF_{k+1}$  ( $k < n$ ), basing on the function of  $PF_k$ . It is application of dynamic programming. Table 2 illustrates each matching structure and its corresponding formula. Let  $IS, ES$  and  $NS$  be the abbreviation of internally saturated, externally saturated and unsaturated. The saturated situations of  $u_r, u_1, u_k$  are those in  $PF_k$  (not in  $F$ ). Similarly, the values to the sub-formulas not appearing in Table 2 are  $NULL$ . Hence, we have our induction basis.

Table 2. Sub-formulas of  $PF_{k+1}$  in  $H$

$u_r$	$u_1$	$u_k$	Sub-formula
$IS$	$IS/NS/ES$	$IS$	$W(PF_k, 0, b, 0) + W(u_{k+1}, 0, 0, 1)$

			$W(u_{k+1,-1,-1,-1})$
IS	IS/NS/ES	NS	$W(PF_{k,0,b,-1})+W(u_{k+1,0,0,1})$
ES	IS/NS/ES	IS	$W(PF_{k,1,b,0})+W(u_{k+1,0,0,1})/$ $W(u_{k+1,-1,-1,-1})/ W(u_{k+1,1,0,0})$
ES	IS/NS/ES	NS	$W(PF_{k,1,b,-1})+W(u_{k+1,0,0,1})/$ $W(u_{k+1,1,0,0})$
NS	IS/ES	IS	$W(PF_{k,-1,b,0})+W(u_{k+1,0,0,1})$
IS	IS/NS/ES	ES	$W(PF_{k,0,b,1})+W(u_{k+1,0,1,0})-$ $w(u_k, u_{k+1})$
ES	IS/NS/ES	ES	$W(PF_{k,1,b,1})+W(u_{k+1,0,1,0})-$ $w(u_k, u_{k+1})$
NS	IS/ES	ES	$W(PF_{k,-1,b,1})+W(u_{k+1,0,1,0})-$ $w(u_k, u_{k+1})$
NS	IS/ES	NS	$W(PF_{k,-1,b,-1})+W(u_{k+1,0,0,1})$

Induction Step: Suppose the proposition holds in  $H_1 \in H^*$ . We need to prove it also holds in  $H_1 \times F$ , for any fan  $F$  in  $H_1$ . Let  $v_F$  be the pseudo-vertex which the fan  $F$  is shrunk to and  $H_2 \in (H_1)^*$  such that  $H_2$  contains a fan  $F_2$  which  $v_F$  lies in. It is sufficient to prove that  $W(F_2, X(u_i), X(u_j), X(u_k))$  records the value of solution to WMF of  $(F_2)_{ex}$ . Equivalently, we need to prove that if the function  $W$  of the vertices in  $F_2$  holds the proposition, then the function  $W$  of  $F_2$  also holds it.

In the proof of induction basis, we consider only original vertices in the calculated of  $PF_2$  and  $PF_k$ . When pseudo-vertices are taken into account, there will be more situations of the matching structure. We present all the additional situations not mentioned above and their corresponding formulas in Table 3 and Table 4.

Table 3. Sub-formulas of  $PF_2$  in  $H$

$x(u, u_1)$	$x(u, u_2)$	$x(u_1, u_2)$	Sub-formula
0	1	0	$W(u_1, 0, b, 0) + W(u_2, 1, 0, c) /$ $W(u_2, 1, -1, c)$
0	1	0	$W(u_1, 0, b, -1) + W(u_2, 1, 0, c)$
0	1	0	$W(u_1, -1, b, 0) + W(u_2, 1, 0, c) /$ $W(u_2, 1, -1, c)$
0	1	0	$W(u_1, -1, b, -1) + W(u_2, 1, 0, c)$
0	1	1	$W(u_1, 0, b, 1) + W(u_2, 1, 1, c) -$ $w(u_1, u_2)$
0	1	1	$W(u_1, -1, b, 1) + W(u_2, 1, 1, c) -$ $w(u_1, u_2)$
1	0	0	$W(u_1, 1, b, 0) + W(u_2, 0, 0, c) /$ $W(u_2, 0, -1, c) / W(u_2, -1, 0, c) /$ $W(u_2, -1, -1, c)$
1	0	0	$W(u_1, 1, b, -1) + W(u_2, 0, 0, c) /$ $W(u_2, -1, 0, c)$
1	0	1	$W(u_1, 1, b, 1) + W(u_2, 0, 1, c) /$ $W(u_2, -1, 1, c) - w(u_1, u_2)$
0	0	0	$W(u_1, 0, b, 0) + W(u_2, 0, 0, c) /$ $W(u_2, 0, -1, c) / W(u_2, -1, 0, c) /$ $W(u_2, -1, -1, c)$
0	0	0	$W(u_1, 0, b, -1) + W(u_2, 0, 0, c) /$ $W(u_2, -1, 0, c)$
0	0	0	$W(u_1, -1, b, 0) + W(u_2, 0, 0, c) /$ $W(u_2, 0, -1, c)$
0	0	0	$W(u_1, -1, b, -1) + W(u_2, 0, 0, c) /$ $W(u_2, -1, 0, c)$
0	0	1	$W(u_1, 0, b, 1) + W(u_2, 0, 1, c) -$ $w(u_1, u_2)$
0	0	1	$W(u_1, -1, b, 1) + W(u_2, 0, 1, c) -$ $w(u_1, u_2)$

0	0	1	$W(u_{1,-1,b,1})+W(u_{2,-1,1,c})-$ $w(u_1, u_2)$
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Let  $u_r$  be the centre of  $F_2$  and  $EC_3(F_2) = \{i, j, k\}$  be the same as in section 3).

Table 4. Sub-formulas of  $PF_{k+1}$  in  $H$

$x(u_k, u_{k+1})$	$x(u_r, u_{k+1})$	Sub-formula
0	0	$W(PF_{k,0,b,0})+W(u_{k+1,0,0,c})/$ $W(u_{k+1,0,-1,c})/W(u_{k+1,-1,0,c})/$ $W(u_{k+1,-1,-1,c})$
0	0	$W(PF_{k,0,b,-1})+W(u_{k+1,0,0,c})/$ $W(u_{k+1,-1,0,c})$
1	0	$W(PF_{k,0,b,1})+W(u_{k+1,0,1,c})-$ $w(u_k, u_{k+1})$
1	0	$W(PF_{k,0,b,1})+W(u_{k+1,-1,1,c})-$ $w(u_k, u_{k+1})$
0	1	$W(PF_{k,1,b,0})+ W(u_{k+1,1,0,c})/$ $W(u_{k+1,1,-1,c})$
1	1	$W(PF_{k,1,b,1})+W(u_{k+1,1,1,c})-$ $w(u_k, u_{k+1})$
0	1	$W(PF_{k,1,b,-1})+ W(u_{k+1,1,0,c})$
0	0	$W(PF_{k,1,b,0})+W(u_{k+1,0,0,c})/$ $W(u_{k+1,0,-1,c})/W(u_{k+1,-1,0,c})/$ $W(u_{k+1,-1,-1,c})$
0	0	$W(PF_{k,1,b,-1})+W(u_{k+1,0,0,c})/$ $W(u_{k+1,-1,0,c})$
1	0	$W(PF_{k,1,b,1})+W(u_{k+1,0,1,c})-w(u_k, u_{k+1})$
1	0	$W(PF_{k,1,b,1})+W(u_{k+1,-1,1,c})-$ $w(u_k, u_{k+1})$
0	0	$W(PF_{k,-1,b,0})+W(u_{k+1,0,0,c})/$ $W(u_{k+1,0,-1,c})$
0	0	$W(PF_{k,-1,b,-1})+W(u_{k+1,0,0,c})$
1	0	$W(PF_{k,-1,b,1})+W(u_{k+1,0,1,c})-$ $w(u_k, u_{k+1})$

Since our formulas enumerate all the situations of structure in  $PF_{k+1}$ , the function  $W$  of  $PF_{k+1}$  records the value of solution to WMF of  $(PF_{k+1})_{ex}$ . When we shrink the fan  $F_2$ , we calculate the function  $W$  of  $PF_k (2 \leq k \leq |V(F_2)| - 1)$ . Finally, we obtain the function  $W$  of  $F_2$  which also holds the proposition.

In the algorithm, the section 1) needs  $O(1)$  time. The section 3) is the operation of shrinking fans in  $H$ . If a fan  $F$  contains  $r+1$  vertices, then it is verified that the time of the shrinking operation is  $O(r)$ . Moreover, shrinking  $F$  reduces the number of vertices of the graph by  $r$ . Thus, the total time of the shrinking operation is  $O(|V|)$ . The time of the postorder scan without shrinking is bounded by  $O(|V|)$ . The section 4)-8) need  $O(1)$  time. The time of section 9)-14) is the same as the section 3). Therefore, the total time of the algorithm is  $O(|V|)$ .

## VI. EXTENSIONS

Our algorithm can be amended to search the maximum-weighted matching in the following four conditions: (1) If a Halin graph has even order, we can amend the algorithm to search the maximum-weighted perfect matching. (WPMF) maximum  $w^T x$ .

Subject to

1.  $x(\delta(u))=1$  for all  $u \in V(F)$ ,
  2.  $x(\chi(B)) \leq \lfloor |B|/2 \rfloor$  for all  $B \in O(F)$ ,
  3.  $x_e \geq 0$  for all  $e \in E(F)$ ,  $x_e = 0$  for all  $e \notin E(F) \cup \{i,j,k\}$
- where  $O(F) = \{ B \subseteq V(F) : |B| \text{ is odd and } |B| \geq 3 \}$

$W(u,0,0,0)=NULL$  ,  $W(u,0,0,1)=w(k)$  ;  
 $W(u,0,1,0)=w(j)$  ,  $W(u,0,1,1)=NULL$  ;  
 $W(u,1,0,0)=w(i)$  ,  $W(u,1,0,1)=NULL$  ;  
 $W(u,1,1,0)=NULL$  ,  $W(u,1,1,1)=NULL$

$$W(PF_2,0,b,c) = \max \left\{ \begin{array}{l} W(u_1,0,b,0) + W(u_2,1,0,c), \\ W(u_1,0,b,1) + W(u_2,1,1,c) - w(u_1, u_2), \\ W(u_1,1,b,0) + W(u_2,0,0,c), \\ W(u_1,1,b,1) + W(u_2,0,1,c) - w(u_1, u_2) \end{array} \right\} \quad (1)$$

$$W(PF_2,1,b,c) = \max \left\{ \begin{array}{l} W(u_1,0,b,0) + W(u_2,0,0,c), \\ W(u_1,0,b,1) + W(u_2,0,1,c) - w(u_1, u_2) \end{array} \right\} \quad (2)$$

$$W(PF_{k+1},0,b,c) = \max \left\{ \begin{array}{l} W(PF_k,0,b,0) + W(u_{k+1},0,0,c), \\ W(PF_k,0,b,1) + W(u_{k+1},0,1,c) - w(u_k, u_{k+1}), \\ W(PF_k,1,b,0) + W(u_{k+1},1,0,c), \\ W(PF_k,1,b,1) + W(u_{k+1},1,1,c) - w(u_k, u_{k+1}) \end{array} \right\} \quad (3)$$

$$W(PF_{k+1},1,b,c) = \max \left\{ \begin{array}{l} W(PF_k,1,b,0) + W(u_{k+1},0,0,c), \\ W(PF_k,1,b,1) + W(u_{k+1},0,1,c) - w(u_k, u_{k+1}) \end{array} \right\} \quad (4)$$

$W(F,0,b,c) = W(PF_n,0,b,c)$   
 $W(F,1,b,c) = W(PF_n,1,b,c) + w(i)$

Cost :=  $\max \{ W(v_w,1,0,0), W(v_w,0,1,0), W(v_w,0,0,1) \}$

(2) If a Halin graph has odd order, we can amend the algorithm to search the maximum-weighted near-perfect matching. Since the near-perfect matching doesn't saturate exactly one vertex of the graph. We augment the dimension of the function  $W$ . Use one more input to constrain the fan (the pseudo-vertex): 1 indicates that it misses one vertex; 0 means not. Hence,  $W(F, x(i), x(j), x(k), 0)$  is the same as the function  $W(F, x(i), x(j), x(k))$  above and  $W(F, x(i), x(j), x(k), 1)$  be the value of solution to the restricted linear program in  $F$ , as following:

WNPMF maximize  $w^T x$ .

Subject to:

1.  $\sum_{u \in F} x(\delta(u)) = |V(F)| - 1$
  2.  $x(\delta(u)) \leq 1$  for all  $u \in V(F)$
  3.  $x(\chi(B)) \leq \lfloor |B|/2 \rfloor$  for all  $B \in O(F)$ ,
  4.  $x_e \geq 0$  for all  $e \in E(F)$ ,  $x_e = 0$  for all  $e \notin E(F) \cup \{i,j,k\}$ .
- $O(F) = \{ B \subseteq V(F) : |B| \text{ is odd and } |B| \geq 3 \}$

$$W(PF_2,0,b,c,0) = \max \left\{ \begin{array}{l} W(u_1,0,b,0,0) + W(u_2,1,0,c,0), \\ W(u_1,0,b,1,0) + W(u_2,1,1,c,0) - w(u_1, u_2), \\ W(u_1,1,b,0,0) + W(u_2,0,0,c,0), \\ W(u_1,1,b,1,0) + W(u_2,0,1,c,0) - w(u_1, u_2) \end{array} \right\} \quad (1)$$

$$W(PF_2,0,b,c,1) = \max \left\{ \begin{array}{l} W(u_1,0,0,0,1) + W(u_2,1,0,0,0), \\ W(u_1,0,0,0,0) + W(u_2,1,0,0,1), \\ W(u_1,0,b,1,1) + W(u_2,1,1,c,0) - w(u_1, u_2), \\ W(u_1,0,b,1,0) + W(u_2,1,1,c,1) - w(u_1, u_2), \\ W(u_1,1,b,0,1) + W(u_2,0,0,c,0), \\ W(u_1,1,b,0,0) + W(u_2,0,0,c,1), \\ W(u_1,1,b,1,1) + W(u_2,0,1,c,0) - w(u_1, u_2), \\ W(u_1,1,b,1,0) + W(u_2,0,1,c,1) - w(u_1, u_2) \end{array} \right\} \quad (2)$$

$$W(PF_2,1,b,c,0) = \max \left\{ \begin{array}{l} W(u_1,0,b,0,0) + W(u_2,0,0,c,0), \\ W(u_1,0,b,1,0) + W(u_2,0,1,c,0) - w(u_1, u_2) \end{array} \right\} \quad (3)$$

$$W(PF_2,1,b,c,1) = \max \left\{ \begin{array}{l} W(u_1,0,b,0,1) + W(u_2,0,0,c,0), \\ W(u_1,0,b,0,0) + W(u_2,0,0,c,1), \\ W(u_1,0,b,1,1) + W(u_2,0,1,c,0) - w(u_1, u_2), \\ W(u_1,0,b,1,0) + W(u_2,0,1,c,1) - w(u_1, u_2) \end{array} \right\} \quad (4)$$

$$W(PF_{k+1},0,b,c,0) = \max \left\{ \begin{array}{l} W(PF_k,0,b,0,0) + W(u_{k+1},0,0,c,0), \\ W(PF_k,0,b,1,0) + W(u_{k+1},0,1,c,0) - w(u_k, u_{k+1}), \\ W(PF_k,1,b,0,0) + W(u_{k+1},1,0,c,0), \\ W(PF_k,1,b,1,0) + W(u_{k+1},1,1,c,0) - w(u_k, u_{k+1}) \end{array} \right\} \quad (5)$$

$$W(PF_{k+1},0,b,c,1) = \max \left\{ \begin{array}{l} W(PF_k,0,b,0,1) + W(u_{k+1},0,0,c,0), \\ W(PF_k,0,b,0,0) + W(u_{k+1},0,0,c,1), \\ W(PF_k,0,b,1,1) + W(u_{k+1},0,1,c,0) - w(u_k, u_{k+1}), \\ W(PF_k,0,b,1,0) + W(u_{k+1},0,1,c,1) - w(u_k, u_{k+1}), \\ W(PF_k,1,b,0,1) + W(u_{k+1},1,0,c,0), \\ W(PF_k,1,b,0,0) + W(u_{k+1},1,0,c,1), \\ W(PF_k,1,b,1,1) + W(u_{k+1},1,1,c,0) - w(u_k, u_{k+1}), \\ W(PF_k,1,b,1,0) + W(u_{k+1},1,1,c,1) - w(u_k, u_{k+1}) \end{array} \right\} \quad (6)$$

$$W(PF_{k+1}, 1, b, c, 0) = \max \left\{ \begin{array}{l} W(PF_k, 1, b, 0, 0) + W(u_{k+1}, 0, 0, c, 0), \\ W(PF_k, 1, b, 1, 0) + W(u_{k+1}, 0, 1, c, 0) - w(u_k, u_{k+1}) \end{array} \right\} \quad (7)$$

$$W(PF_{k+1}, 1, b, c, 1) = \max \left\{ \begin{array}{l} W(PF_k, 1, b, 0, 1) + W(u_{k+1}, 0, 0, c, 0), \\ W(PF_k, 1, b, 0, 0) + W(u_{k+1}, 0, 0, c, 1), \\ W(PF_k, 1, b, 1, 1) + W(u_{k+1}, 0, 1, c, 0) - w(u_k, u_{k+1}), \\ W(PF_k, 1, b, 1, 0) + W(u_{k+1}, 0, 1, c, 1) - w(u_k, u_{k+1}) \end{array} \right\} \quad (8)$$

$$\begin{aligned} W(u, 0, 0, 0, 0) &= NULL, & W(u, 0, 0, 0, 1) &= 0; \\ W(u, 0, 0, 1, 0) &= w(k), & W(u, 0, 0, 1, 1) &= NULL \\ W(u, 0, 1, 0, 0) &= w(j), & W(u, 0, 1, 0, 1) &= NULL \\ W(u, 1, 0, 0, 0) &= w(i), & W(u, 1, 0, 0, 1) &= NULL \\ W(u, 0, 1, 1, 0) &= W(u, 0, 1, 1, 1) &= NULL, \\ W(u, 1, 0, 1, 0) &= W(u, 1, 0, 1, 1) &= NULL \\ W(u, 1, 1, 0, 0) &= W(u, 1, 1, 0, 1) &= NULL, \\ W(u, 1, 1, 1, 0) &= W(u, 1, 1, 1, 1) &= NULL \end{aligned}$$

$$\text{Cost} = \max \{ W(v_w, 1, 0, 0, 1), W(v_w, 0, 1, 0, 1), W(v_w, 0, 0, 1, 1), W(v_w, 0, 0, 0, 0) \}$$

(3) Let  $H$  be a Halin graph and  $H=(V,E)$  and  $V \subseteq V$ . We want to find the maximum-weighted matching covering all the vertices in  $V$ .  $\forall v \in V$ , the vertex  $v$  must be covered by  $M$ . So deleting all the situations that  $v$  is unsaturated in the formulas (1)-(6).

(4) Let  $H$  be a Halin graph and  $H=(V,E)$  and  $E \subseteq E$ . We want to find the maximum-weighted matching containing all the edges in  $E$ .  $\forall e \in E$ , the edge  $e$  must be contained in  $M$ . For each edge  $uv \in E$ , let let  $S = \delta(u) \cup \delta(v)$ .  $\forall e \in S$ , change  $w(e)$  to  $-n * d_{max}$  ( $d_{max} = \max \{ w(e) | e \in E(H) \}$ ,  $n = |V(H)|$ ).

## VII CONCLUSION

In this paper, we have proposed a linear time algorithm to solve maximum-weighted matching problem in Halin graphs. Our method can be extended to search maximum-weighted matching in some other conditions.

Many hard problems become linear solvable for special classes of graphs. For example, the minimum TSP which is NP-hard is solved in  $O(|V|)$ , in [2]. It is well-known that many optimization problems on graphs can also be formulated as integer programs. Hence, if the linear system, which is associated with the integer program, can use the divide-and-conquer strategy, then a more efficient algorithm will be obtained. For instance, TSP in graph with 3-edge cutsets has been studied in [3]. If more special structure, which makes the linear system dividable, is found out, then many hard problems will become much easier on the graphs with that structure.

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