An Algorithm to Find the Optimal Matching in Halin Graphs

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Abstract—**This paper deals with the problem of searching the maximum-weighted matching in Halin graphs. We give an $O(|V|)$ algorithm.**

Keywords—**maximum-weighted, matching, Halin graph, algorithm**

I. INTRODUCTION

Finding a matching with the maximum total weight is the wellkonwn assignment problem of assigning people to jobs and maximize the profits.

We consider the maximum-weighted matching problem in graphs. That is, given an undirected graph $G=(V,E)$ and let w be a nonnegative real-valued function on its edges; we wish to find a matching with the maximum total weight. This is the wellknown assignment problem of assigning people to jobs and maximum the profits. Let M be a matching of graph G and $v\in V(G)$. If v is incident with an edge in M, then v is called saturated, otherwise v is said to be unsaturated.

A Halin graph $H=T\cup C$ is obtained by embedding a tree T having no nodes of degree 2 in the plane, and then adding a cycle C to join the leaves of T in such a way that the resulting graph is planar. If T is a star, that is, a single vertex joining to the other vertices, then H is called a wheel. Suppose T has at least two nonleaves. Let w be a nonleaf of T which is adjacent to only one nonleaf of T. Then the set of leaves of T adjacent to w, which we denote by $C(w)$, comprises a consecutive subsequence of the cycle C. We call the subgraph of H induced by $\{w\}\cup C(w)$ a fan and call w the centre of the fan. In Fig. 1, the black vertices are the centers of the fans.

A solution of this problem can be computed in $O(|V|^3)$ time by Edmonds' method [4] with the modifications proposed by Gabow [5] and Lawler [7]. Gail, Micali and Gabow developed an $O(|V||E|log|V|)$ algorithm in [6]. We restrict the problem to Halin graphs and present a linear time algorithm. For the terminology and notation not defined in this paper, reader can refer to [1].

Let $G=(V,E)$ be a graph. For any $S\subseteq V(G)$, let $\delta(S)$ denote the set of edges with one end-vertex in S and the other in $V(G)$-S; that is $\delta(S)=\{uv\in E: u\in S, v\notin S\}$. For $v\in V$, we abbreviate $\delta(\{v\})$ by $\delta(v)$. The edges of G having both ends in $S\subseteq V$ are denoted by $\gamma(S)=\{uv\in E: u\in S \text{ and } v\in S\}$.

Let F be a fan. We use $H\times F$ to denote the graph obtained from H by shrinking F to a new 'pseudo-vertex', say v_F. That is $V(H\times F)=\{v_F\}\cup\{V(H)\backslash V(F)\}$ and the edges of $H\times F$ are defined as follows:

1. An edge with both ends in F is deleted;
2. An edge with both ends in H-F remains unchanged;
3. An edge of $\delta(F)$ now joins the incident vertex of H-F and v_F

Denote the closure of H under the operation of "Shrinking Fan" by H^*.

A graph G is called a wheel if G consists of a cycle every vertex of which is joined to a single common vertex by an edge.

An edge cutset of a connected graph $G=(V,E)$ is a set of edges whose removal leaves a disconnected graph. If it consists of exactly k edges, then we call it a k-edge cutset. The shores of an edge cutest J are the vertices of the (two) components of G-J. Given a fan F of H, the three edges connecting $V(F)$ to $V(H$-$F)$ compose a 3-edge cutset of H. We denote the 3-edge cutest by $EC_3(F)$.

This paper is organized as follows: in section 2) we review the lemmas which are basic for the method of "Shrinking Fan" in Halin graphs, in section 3) we reveal the feasibility of finding the maximum-weighted matching in Halin graphs by means of "Shrinking Fans", in section 4)-5) we present the algorithm to find the maximum-weighted matching in Halin graphs and in section 6) extensions of the algorithm are provided.

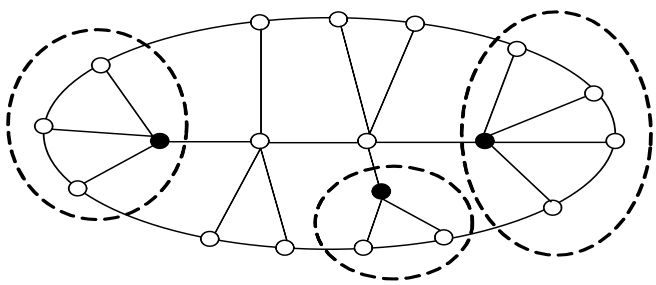

Figure.1 A Halin graph

II. PRELIMINARY RESULTS

Lemma 1 (Cornuejols ,Naddef and Pulleyblank[2]). A Halin graph $H=T\cup C$ which is not a wheel has at least two fans.

Lemma 2 (Cornuejols, Naddef and Pulleyblank [2]). If F is a fan in a Halin graph H, then $H\times F$ is a Halin graph.

The $O(n)$ algorithm for the TSP developed in [2] uses Lemma 1 and Lemma 2 recursively to reduce the Halin graph into a wheel. After each reduction step, the edge weights are modified so that the optimal objective function value of the TSP on the original graph is the same as that on the reduced graph.

III. Analysis

Let $G=(V,E)$ be a general graph with edge weights w: $E\rightarrow R$ and M is a matching of G. An incidence vector x is associated with the edges of G: x_e is set to 1 if $e\in M$; otherwise, x_e is set to 0. For any subset $S\subseteq E$, we define $x(S)=\Sigma x_e\ (e\in S)$. Moreover, let O consist of all non-singleton odd cardinality subsets of V: $O=\{B\subseteq V: |B| \text{ is odd and } |B|\quad 3\}$.

The maximum-weighted matching problem of G with weighted function w can be formulated as :

(WM) maximum w^Tx.

Subject to

1. $x(\gamma(B))\quad \lfloor |B|/2 \rfloor$ for all $B\in O$,
2. $x_e\quad 0$ for all $e\in E$.

(the edges which are set to 1 are not adjacent and there don't exit two adjacent vertices which are both unsaturated).

Now we investigate the method of "Shrinking Fan". It is a variant usage to the idea of shrinking a fan in [2]. Cornueljols, Naddef and Pulleyblank described a linear system for Travelling Salesman Polytope $P(H)$ for a Halin graph H. Morever, they proved the following lemma:

Lemma 3 (Cornuejols, Naddef and Pulleyblank[2]). Let K be a 3-edge cutest in a hamiltonian graph $G=(V,E)$ and let S be a shore of the cut. Then a linear system for $P(G)$ can be obtained by taking the union of linear system for $P(G\times S)$ and $P(G\times(V - S))$.

It is an application of "divide and conquer". In the light of this, we provide a system to solve the WM problem in Halin graphs. Firstly, we introduce the basic function of our system.

Let F be a fan of H, M be a matching of H and $EC_3(F)=\{i,j,k\}$. Let $i=u_iv_i$, $j=u_jv_j$ and $k=u_kv_k$ such that $u_i,u_j,u_k\in V(F)$, $v_i,v_j,v_k\in V(H\text{-}F)$, $j,k\in$ C and the direction v_j,u_j,u_k,v_k is anti-clockwise in the cycle C. The vertex u_i has three situations: 1) u_i is unsaturated; 2) If $i\in M$, we say the vertex u_i is externally saturated in F; 3) otherwise, internally. The denotation is the same to u_j and u_k. Only the situation of $\{X(u_i),X(u_j),X(u_k)\}$ affects the situation of $M\cap(H\text{-}F)$. Hence, we use a function W to store the total weight of maximum-weighted matching to a certain fan F, corresponding to each chosen situation of $EC_3(F)$. We use $X(v)$ to denote the state of the vertex v. $X(v)=1$ means v is externally saturated. $X(v)=0$ means v is internally saturated. $X(v)=-1$ means v is unsaturated. Let $W(F, X(u_i),X(u_j),X(u_k))$ be the value of the solution to the restricted linear program in F, as following:

(WMF) maximum w^Tx

Subject to

1. $x(\gamma(B))\quad \lfloor |B|/2 \rfloor$ for all $B\in O(F)$,
2. $x_e\quad 0$ for all $e\in E(F)$, $x_e = 0$ for $e\notin E(F)\cup\{i,j,k\}$.

$O(F)=\{B\subseteq V(F): |B| \text{ is odd and } |B|\quad 3\}$ and $x(i),x(j),x(k)$ are given.

If no solution to a certain triple $\{X(u_i),X(u_j),X(u_k)\}$, then the corresponding value of W is set to be *NULL*.We consider an original vertex on the cycle C as a special fan. We define the function W to an original vertex u as following: Let $i=uv_i$, $j=uv_j$ and $k=uv_k$ such that $v_j,v_k\in V(C)$, $v_i\in V(H\text{-}C)$ and the direction of v_k,u,v_j is clockwise in the embed plane.

$W(u,-1,-1,-1)=0, W(u,1,0,0)=w(i), W(u,0,0,1)=w(k), W(u,0,1,0)=w(j)$, others are all *NULL*.

Now we describe how to calculate the value of the function. Let F be a fan of H. Let u_r be the centre of F and $u_1,u_2\ldots u_n$ be the vertices on the cycle C in that order. Each u_i $(1\leq i\leq n)$ can be either an original vertex or a pseudo-vertex where the function W of each u_i has been calculated already.

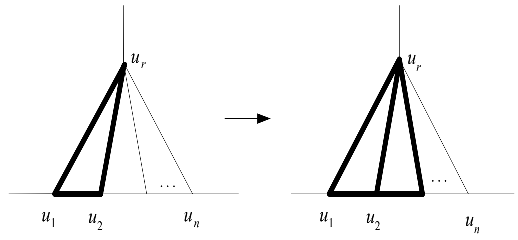

Figure.2 Extention of pseudo-fan

We denote the induced subgraph $H[\{u_r,u_1,u_2,\ldots,u_k\}]$ by pseudo-fan PF_k of F where $2\leq k\leq n$. We examine the situation of $k=2$ first. We use the following formulas (1)-(3) to compute the values to the function W of F. Let $b,c\in\{-1,0,1\}$.

Since there are four combinations of the values of b and c, formula (1) stands for nine formulas: $W(PF_2,0,-1,-1)$, $W(PF_2,0,-1,0)$, $W(PF_2,0,0,-1)$, $W(PF_2,0,0,0)$, $W(PF_2,0,0,1)$, $W(PF_2,0,1,0)$ and $W(PF_2,0,1,1)$, $W(PF_2,0,-1,1)$, $W(PF_2,0,1,-1)$. So do formula (2) and (3). This abbreviation will be carried unless otherwise stated. Note that we don't add the value of the edge adjacent to u_r in its externally saturated situations, because we cannot determine which edge saturates it. To the pseudo-fan PF_k, each edge $u_ru_m(k<m\leq n)$ might be such an edge.

Once we have the values to the function W of PF_k, we can calculate the function W of PF_{k+1}. Similarly, we present the formulas (4)-(6) to the calculation as following:

Once the values to the function W of PF_n are obtained, we can compute the function W of F. The formulas are presented as follows:

$W(F,0,b,c)=W(PF_n,0,b,c)$,

$W(F,-1,b,c)=W(PF_n,-1,b,c)$,
$W(F,1,b,c)= W(PF_n,1,b,c)+w(i)$.

In the calculation of the function W, we adopt a mechanism similar with "short-circuit evaluation"; that is, if a value used in a subformula is *NULL*, then the result is set to be *NULL*, directly. And we consider *NULL* as a minimum value in the comparison. Besides the situation of orginal vertex, the value to the function W might also be set to be *NULL* in some situations as to a fan.

When a fan is shrunk, the values to be the function W of the fan are stored into the function W of the pseudo-vertex. Moreover, we store the structure which gets the maximum value into the pseudo-vertex; that is, record the matching structure corresponding to the value to the function W in each input. If there are more than one situation obtain the maximum value, choose one arbitrarily.

$$W(PF_2,0,b,c)=\max\left\{\begin{array}{l} W(u_1,0,b,0)+W(u_2,1,0,c),\\ W(u_1,0,b,0)+W(u_2,1,-1,c),\\ W(u_1,0,b,-1)+W(u_2,1,0,c),\\ W(u_1,-1,b,0)+W(u_2,1,0,c),\\ W(u_1,-1,b,0)+W(u_2,1,-1,c),\\ W(u_1,-1,b,-1)+W(u_2,1,0,c),\\ W(u_1,0,b,1)+W(u_2,1,1,c)-w(u_1,u_2),\\ W(u_1,-1,b,1)+W(u_2,1,1,c)-w(u_1,u_2),\\ W(u_1,1,b,0)+W(u_2,0,0,c),\\ W(u_1,1,b,0)+W(u_2,0,-1,c),\\ W(u_1,1,b,0)+W(u_2,-1,0,c),\\ W(u_1,1,b,0)+W(u_2,-1,-1,c),\\ W(u_1,1,b,-1)+W(u_2,-1,0,c),\\ W(u_1,1,b,-1)+W(u_2,0,0,c),\\ W(u_1,1,b,1)+W(u_2,0,1,c)-w(u_1,u_2)\},\\ W(u_1,1,b,1)+W(u_2,-1,1,c)-w(u_1,u_2)\} \end{array}\right\} \quad (1)$$

$$W(PF_2,1,b,c)=\max\left\{\begin{array}{l} W(u_1,0,b,0)+W(u_2,0,0,c),\\ W(u_1,0,b,0)+W(u_2,0,-1,c),\\ W(u_1,0,b,0)+W(u_2,-1,0,c),\\ W(u_1,0,b,0)+W(u_2,-1,-1,c),\\ W(u_1,0,b,-1)+W(u_2,0,0,c),\\ W(u_1,0,b,-1)+W(u_2,-1,0,c),\\ W(u_1,-1,b,0)+W(u_2,0,0,c),\\ W(u_1,-1,b,0)+W(u_2,0,-1,c),\\ W(u_1,-1,b,-1)+W(u_2,0,0,c),\\ W(u_1,-1,b,-1)+W(u_2,-1,0,c),\\ W(u_1,0,b,1)+W(u_2,0,1,c)-w(u_1,u_2)\},\\ W(u_1,-1,b,1)+W(u_2,0,1,c)-w(u_1,u_2)\}\\ W(u_1,-1,b,1)+W(u_2,-1,1,c)-w(u_1,u_2)\} \end{array}\right\} \quad (2)$$

$$W(PF_2,-1,b,c)=\max\left\{\begin{array}{l} W(u_1,0,b,0)+W(u_2,0,0,c),\\ W(u_1,0,b,0)+W(u_2,0,-1,c),\\ W(u_1,0,b,-1)+W(u_2,0,0,c),\\ W(u_1,0,b,1)+W(u_2,0,1,c)-w(u_1,u_2)\} \end{array}\right\} \quad (3)$$

$$W(PF_{k+1},0,b,c)=\max\left\{\begin{array}{l} W(PF_k,1,b,1)+W(u_{k+1},1,1,c)-w(u_k,u_{k+1}),\\ W(PF_k,1,b,-1)+W(u_{k+1},1,0,c),\\ W(PF_k,0,b,-1)+W(u_{k+1},0,0,c),\\ W(PF_k,0,b,-1)+W(u_{k+1},-1,0,c)\\ W(PF_k,0,b,1)+W(u_{k+1},0,1,c)-w(u_k,u_{k+1}),\\ W(PF_k,0,b,1)+W(u_{k+1},-1,1,c)-w(u_k,u_{k+1}),\\ W(PF_k,1,b,0)+W(u_{k+1},1,0,c),\\ W(PF_k,1,b,0)+W(u_{k+1},1,-1,c),\\ W(PF_k,0,b,0)+W(u_{k+1},0,0,c),\\ W(PF_k,0,b,0)+W(u_{k+1},0,-1,c),\\ W(PF_k,0,b,0)+W(u_{k+1},-1,0,c),\\ W(PF_k,0,b,0)+W(u_{k+1},-1,-1,c) \end{array}\right\} \quad (4)$$

$$W(PF_{k+1},1,b,c)=\max\begin{cases} W(PF_k,1,b,-1)+W(u_{k+1},0,0,c), \\ W(PF_k,1,b,-1)+W(u_{k+1},-1,0,c), \\ W(PF_k,1,b,1)+W(u_{k+1},0,1,c)-w(u_k,u_{k+1}), \\ W(PF_k,1,b,1)+W(u_{k+1},-1,1,c)-w(u_k,u_{k+1}), \\ W(PF_k,1,b,0)+W(u_{k+1},0,0,c), \\ W(PF_k,1,b,0)+W(u_{k+1},0,-1,c), \\ W(PF_k,1,b,0)+W(u_{k+1},-1,0,c), \\ W(PF_k,1,b,0)+W(u_{k+1},-1,-1,c) \end{cases} \quad (5)$$

$$W(PF_{k+1},-1,b,c)=\max\begin{cases} W(PF_k,-1,b,0)+W(u_{k+1},0,0,c), \\ W(PF_k,-1,b,0)+W(u_{k+1},0,-1,c), \\ W(PF_k,-1,b,-1)+W(u_{k+1},0,0,c), \\ W(PF_k,-1,b,1)+W(u_{k+1},0,1,c)-w(u_k,u_{k+1}) \end{cases} \quad (6)$$

IV. PROCEDURE OF THE ALGORITHM

Algorithm to Find the Optimal Matching in Halin Graphs

1) Choose a non-leaf vertex of T, denoted by v_{root}, such that v_{root} is adjacent to a leaf of T, denoted by v_{leaf}.
2) Perform a postorder scan of T, for each fan F has been found do
3) If the centre of $F \neq v_{root}$ then
   Begin
   Calculate the values to the function W of F;
   Shrink F to v_F;
   Store the values above into the function W of v_F;
   Store the corresponding structure of F to each value above.
   End

Let H_w be the wheel we finally get and i be the edge joining v_{root} and v_{leaf}.
Let j,k be the two edges in C adjacent to v_{leaf} such that the direction of j, v_{leaf}, k is clockwise.
Let $F_w=H_w - v_{leaf}$.

4) Calculate the values to the function W of F_w.
5) Store the matching structure of F_w corresponding to each value.
6) Shrink the fan F_w to the pseudo-vertex v_w.
7) Cost:=max{$W(v_w,1,0,0)$, $W(v_w,0,1,0)$, $W(v_w,0,0,1)$, $W(v_w,0,0,0)$}.

Let the edge in $\{i,j,k\}$, the choice of which makes the maximum, be e_M.

8) Mark e_M belonging to the matching.
9) While (there is pseudo-vertex in H) do
10) Begin
    Let v_F be a pseudo-vertex such that v_F is shrunk from the fan F.
11) According to the chosen situation of $EC_3(F)$ and the information stored in v_F, we can obtain the corresponding matching structure in F.
12) Restore v_F to the fan F.
13) Mark the matching edges in F.
14) End
15) All the marked edges compose the maximum-weighted matching.

V. CORRECTNESS AND TIME COMPLEXITY

Let H be a Halin graph. Let F be a fan in $H' \in H^*$. Let F_{ex} be the subgraph of H which F is fully restored to. Equivalently, F_{ex} is the component(shore) of H-$EC_3(F)$ which F stands for.

Theorem 1. Let H be a Halin graph and F be a fan in $H' \in H^*$. Let $EC_3(F)=\{i,j,k\}$ be the same as in section 3). Then $W(F, X(u_i),X(u_j), X(u_k))$ records the value of solutions to WMF of F_{ex} in the input $X(u_i)$,$X(u_j)$ and $X(u_k)$.

Proof. We apply induction on $H'=H$.

Induction Basis: We need to prove the proposition holds for any fan F in H. Let u_r be the centre of F and let $u_1,u_2\ldots u_n$ ($n\geq 2$) be the vertices of F on the cycle C(in anti-clockwise order). Since $u_1,u_2\ldots u_n$ are original vertices, the function W of $u_1,u_2\ldots u_n$ have been calculated. Table 1 shows that the formulas (1),(2) and (3) enumerate all the solutions to WMF of pseudo-fan PF_2. Since they contain at least one function whose value is $NULL$, the values of them are all $NULL$.

Table 1. Sub-formulas of PF_2 in H

$x(u_r u_1)$	$x(u_r u_2)$	$x(u_1 u_2)$	Sub-formula
0	0	0	$W(u_1,0,1,0)+W(u_2,0,0,1)$
0	0	1	$W(u_1,0,1,0)+$ $W(u_1,0,1,0)-w(u_1,u_2)$
0	1	0	$W(u_1,0,1,0)+W(u_2,1,0,0)$
1	0	0	$W(u_1,1,0,0)+W(u_2,0,0,1)$
1	0	0	$W(u_1,1,0,0)+W(u_2,-1,-1,-1)$
0	1	0	$W(u_1,-1,-1,-1)+W(u_2,1,0,0)$
0	0	0	$W(u_1,-1,-1,-1)+W(u_2,0,0,1)$
0	0	0	$W(u_1,0,1,0)+W(u_2,-1,-1,-1)$

The formulas (4)-(6) compute the solutions to WMF of PF_{k+1} ($k<n$), basing on the function of PF_k. It is application of dynamic programming. Table 2 illustrates each matching structure and its corresponding furmula. Let IS ,ES and NS be the abbreviation of internally saturated, externally saturated and unsaturated. The saturated situations of u_r,u_1,u_k are those in PF_k(not in F). Similarly, the values to the sub-formulas not appearing in Table 2 are $NULL$. Hence, we have our induction basis.

Table 2. Sub-formulas of PF_{k+1} in H

u_r	u_1	u_k	Sub-formula
IS	$IS/NS/ES$	IS	$W(PF_k,0,b,0)+W(u_{k+1},0,0,1)/$

			$W(u_{k+1},-1,-1,-1)$
IS	*IS/NS/ES*	*NS*	$W(PF_k,0,b,-1)+W(u_{k+1},0,0,1)$
ES	*IS/NS/ES*	*IS*	$W(PF_k,1,b,0)+W(u_{k+1},0,0,1)/$ $W(u_{k+1},-1,-1,-1)/$ $W(u_{k+1},1,0,0)$
ES	*IS/NS/ES*	*NS*	$W(PF_k,1,b,-1)+W(u_{k+1},0,0,1)/$ $W(u_{k+1},1,0,0)$
NS	*IS/ES*	*IS*	$W(PF_k,-1,b,0)+W(u_{k+1},0,0,1)$
IS	*IS/NS/ES*	*ES*	$W(PF_k,0,b,1)+W(u_{k+1},0,1,0)-$ $w(u_k,u_{k+1})$
ES	*IS/NS/ES*	*ES*	$W(PF_k,1,b,1)+W(u_{k+1},0,1,0)-$ $w(u_k,u_{k+1})$
NS	*IS/ES*	*ES*	$W(PF_k,-1,b,1)+W(u_{k+1},0,1,0)-$ $w(u_k,u_{k+1})$
NS	*IS /ES*	*NS*	$W(PF_k,-1,b,-1)+W(u_{k+1},0,0,1)$

Induction Step: Suppose the proposition holds in $H_1\in H^*$. We need to prove it also holds in $H_1\times F$, for any fan F in H_1. Let v_F be the pseudo-vertex which the fan F is shrunk to and $H_2\in (H_1)^*$ such that H_2 contains a fan F_2 which v_F lies in. It is sufficient to prove that $W(F_2, X(u_i),X(u_j),X(u_k))$ records the value of solution to WMF of $(F_2)_{ex}$. Equivalently, we need to prove that if the function W of the vertices in F_2 holds the proposition, then the function W of F_2 also holds it.

In the proof of induction basis, we consider only original vertices in the calculated of PF_2 and PF_k. When pseudo-vertices are taken into account, there will be more situations of the matching structure. We present all the additional situations not mentioned above and their corresponding formulas in Table 3 and Table 4.

Table 3. Sub-formulas of PF_2 in H

$x(u_ru_1)$	$x(u_ru_2)$	$x(u_1u_2)$	Sub-formula
0	1	0	$W(u_1,0,b,0)+W(u_2,1,0,c)/$ $W(u_2,1,-1,c)$
0	1	0	$W(u_1,0,b,-1)+W(u_2,1,0,c)$
0	1	0	$W(u_1,-1,b,0)+W(u_2,1,0,c)/$ $W(u_2,1,-1,c)$
0	1	0	$W(u_1,-1,b,-1)+W(u_2,1,0,c)$
0	1	1	$W(u_1,0,b,1)+W(u_2,1,1,c)-$ $w(u_1, u_2)$
0	1	1	$W(u_1,-1,b,1) + W(u_2,1,1,c)-$ $w(u_1, u_2)$
1	0	0	$W(u_1,1,b,0)+W(u_2,0,0,c)/$ $W(u_2,0,-1,c)/W(u_2,-1,0,c)/$ $W(u_2,-1,-1,c)$
1	0	0	$W(u_1,1,b,-1)+W(u_2,0,0,c)/$ $W(u_2,-1,0,c)$
1	0	1	$W(u_1,1,b,1)+W(u_2,0,1,c)/$ $W(u_2,-1,1,c)-w(u_1, u_2)$
0	0	0	$W(u_1,0,b,0)+W(u_2,0,0,c)/$ $W(u_2,0,-1,c)/W(u_2,-1,0,c)/$ $W(u_2,-1,-1,c)$
0	0	0	$W(u_1,0,b,-1)+W(u_2,0,0,c)/$ $W(u_2,-1,0,c)$
0	0	0	$W(u_1,-1,b,0)+W(u_2,0,0,c)/$ $W(u_2,0,-1,c)$
0	0	0	$W(u_1,-1,b,-1)+W(u_2,0,0,c)/$ $W(u_2,-1,0,c)$
0	0	1	$W(u_1,0,b,1)+W(u_2,0,1,c)-$ $w(u_1, u_2)$
0	0	1	$W(u_1,-1,b,1)+W(u_2,0,1,c)-$ $w(u_1,u_2)$
0	0	1	$W(u_1,-1,b,1)+W(u_2,-1,1,c)-$ $w(u_1, u_2)$

Let u_r be the centre of F_2 and $EC_3(F_2)=\{i,j,k\}$ be the same as in section 3).

Table 4. Sub-formulas of PF_{k+1} in H

$x(u_ku_{k+})$	$x(u_ru_{k+1})$	Sub-formula
0	0	$W(PF_k,0,b,0)+W(u_{k+1},0,0,c)/$ $W(u_{k+1},0,-1,c)/W(u_{k+1},-1,0,c)/$ $W(u_{k+1},-1,-1,c)$
0	0	$W(PF_k,0,b,-1)+W(u_{k+1},0,0,c)/$ $W(u_{k+1},-1,0,c)$
1	0	$W(PF_k,0,b,1)+W(u_{k+1},0,1,c)-$ $w(u_ku_{k+1})$
1	0	$W(PF_k,0,b,1)+W(u_{k+1},-1,1,c)-$ $w(u_ku_{k+})$
0	1	$W(PF_k,1,b,0)+ W(u_{k+1},1,0,c)/$ $W(u_{k+1},1,-1,c)$
1	1	$W(PF_k,1,b,1)+W(u_{k+1},1,1,c)-$ $w(u_ku_{k+1})$
0	1	$W(PF_k,1,b,-1)+ W(u_{k+1},1,0,c)$
0	0	$W(PF_k,1,b,0)+W(u_{k+1},0,0,c)/$ $W(u_{k+1},0,-1,c)/W(u_{k+1},-1,0,c)/$ $W(u_{k+1},-1,-1,c)$
0	0	$W(PF_k,1,b,-1)+W(u_{k+1},0,0,c)/$ $W(u_{k+1},-1,0,c)$
1	0	$W(PF_k,1,b,1)+W(u_{k+1},0,1,c)-w(u_ku_{k+1})$
1	0	$W(PF_k,1,b,1)+W(u_{k+1},-1,1,c)-$ $w(u_ku_{k+1})$
0	0	$W(PF_k,-1,b,0)+W(u_{k+1},0,0,c)/$ $W(u_{k+1},0,-1,c)$
0	0	$W(PF_k,-1,b,-1)+W(u_{k+1},0,0,c)$
1	0	$W(PF_k,-1,b,1)+W(u_{k+1},0,1,c)-$ $w(u_ku_{k+1})$

Since our formuals enumerate all the situations of structure in PF_{k+1}, the function W of PF_{k+1} records the value of solution to WMF of $(PF_{k+1})_{ex}$. When we shrink the fan F_2, we calculate the function W of $PF_k(2\le k\le|V(F_2)|-1)$. Finally, we obtain the function W of F_2 which also holds the proposition.

In the algorithm, the section 1) needs $O(1)$ time. The section 3) is the operation of shrinking fans in H. If a fan F contains r+1 vertices, then it is verified that the time of the shrinking operation is $O(r)$. Moreover, shrinking F reduces the number of vertices of the graph by r. Thus, the total time of the shrinking operation is $O(|V|)$. The time of the postorder scan without shrinking is bounded by $O(|V|)$. The section 4)-8) need $O(1)$ time. The time of section 9)-14) is the same as the section 3). Therefore, the total time of the algorithm is $O(|V|)$.

VI. EXTENSIONS

Our algorithm can be amended to search the maximum-weighted matching in the following four conditions:
(1) If a Halin graph has even order, we can amend the algorithm to search the maximum-weighted perfect matching.
(WPMF) maximum w^Tx.
Subject to

1. $x(\delta(u))$=1 for all $u\in V(F)$,
2. $x(\gamma(B))$ $\lfloor |B|/2 \rfloor$ for all $B\in O(F)$,
3. x_e 0 for all $e\in E(F)$, x_e=0 for all $e\notin E(F)\cup\{i,j,k\}$

where $O(F)=\{ B\subseteq V(F)$: $|B|$ is odd and $|B|\geq 3\}$

$W(u,0,0,0)=NULL$， $W(u,0,0,1)=w(k)$；
$W(u,0,1,0)=w(j)$， $W(u,0,1,1)=NULL$；
$W(u,1,0,0)=w(i)$， $W(u,1,0,1)=NULL$；
$W(u,1,1,0)=NULL$， $W(u,1,1,1)=NULL$

$$W(PF_2,0,b,c)=\max\begin{cases} W(u_1,0,b,0)+W(u_2,1,0,c), \\ W(u_1,0,b,1)+W(u_2,1,1,c)-w(u_1,u_2), \\ W(u_1,1,b,0)+W(u_2,0,0,c), \\ W(u_1,1,b,1)+W(u_2,0,1,c)-w(u_1,u_2) \end{cases} \quad (1)$$

$$W(PF_2,1,b,c)=\max\begin{cases} W(u_1,0,b,0)+W(u_2,0,0,c), \\ W(u_1,0,b,1)+W(u_2,0,1,c)-w(u_1,u_2) \end{cases} \quad (2)$$

$$W(PF_{k+1},0,b,c)=\max\begin{cases} W(PF_k,0,b,0)+W(u_{k+1},0,0,c), \\ W(PF_k,0,b,1)+W(u_{k+1},0,1,c)-w(u_k,u_{k+1}), \\ W(PF_k,1,b,0)+W(u_{k+1},1,0,c), \\ W(PF_k,1,b,1)+W(u_{k+1},1,1,c)-w(u_k,u_{k+1}) \end{cases} \quad (3)$$

$$W(PF_{k+1},1,b,c)=\max\begin{cases} W(PF_k,1,b,0)+W(u_{k+1},0,0,c), \\ W(PF_k,1,b,1)+W(u_{k+1},0,1,c)-w(u_k,u_{k+1}) \end{cases} \quad (4)$$

$W(F,0,b,c)= W(PF_n,0,b,c)$
$W(F,1,b,c)= W(PF_n,1,b,c)+w(i)$

Cost:=max{$W(v_w,1,0,0)$, $W(v_w,0,1,0)$, $W(v_w,0,0,1)$ }

(2) If a Halin graph has odd order, we can amend the algorithm to search the maximum-weighted near -perfect matching. Since the near-perfect matching doesn't saturate exactly one vertex of the graph. We augment the dimension of the function W. Use one more input to constrain the fan (the pseudo-vertex): 1 indicates that it misses one vertex; 0 means not. Hence, $W(F, x(i), x(j),x(k),0)$ is the same as the function $W(F, x(i), x(j),x(k))$ above and $W(F, x(i), x(j),x(k),1)$ be the value of solution to the restricted linear program in F, as following:

WNPMF maximize $w^T x$.

Subject to:

1. $\Sigma_{u\in F}\, x(\delta(u))= |V(F)|-1$
2. $x(\delta(u)) \leq 1$ for all $u\in V(F)$
3. $x(\gamma(B)) \leq \lfloor |B|/2 \rfloor$ for all $B\in O(F)$,
4. $x_e \geq 0$ for all $e\in E(F)$, $x_e = 0$ for all $e\notin E(F)\cup\{i,j,k\}$.

$O(F)=\{B\subseteq V(F)$: $|B|$ is odd and $|B|\geq 3\}$

$$W(PF_2,0,b,c,0)=\max\begin{cases} W(u_1,0,b,0,0)+W(u_2,1,0,c,0), \\ W(u_1,0,b,1,0)+W(u_2,1,1,c,0)-w(u_1,u_2), \\ W(u_1,1,b,0,0)+W(u_2,0,0,c,0), \\ W(u_1,1,b,1,0)+W(u_2,0,1,c,0)-w(u_1,u_2) \end{cases} \quad (1)$$

$$W(PF_2,0,b,c,1)=\max\begin{cases} W(u_1,0,0,0,1)+W(u_2,1,0,0,0), \\ W(u_1,0,0,0,0)+W(u_2,1,0,0,1), \\ W(u_1,0,b,1,1)+W(u_2,1,1,c,0)-w(u_1,u_2), \\ W(u_1,0,b,1,0)+W(u_2,1,1,c,1)-w(u_1,u_2), \\ W(u_1,1,b,0,1)+W(u_2,0,0,c,0), \\ W(u_1,1,b,0,0)+W(u_2,0,0,c,1), \\ W(u_1,1,b,1,1)+W(u_2,0,1,c,0)-w(u_1,u_2), \\ W(u_1,1,b,1,0)+W(u_2,0,1,c,1)-w(u_1,u_2) \end{cases} \quad (2)$$

$$W(PF_2,1,b,c,0)=\max\begin{cases} W(u_1,0,b,0,0)+W(u_2,0,0,c,0), \\ W(u_1,0,b,1,0)+W(u_2,0,1,c,0)-w(u_1,u_2) \end{cases} \quad (3)$$

$$W(PF_2,1,b,c,1)=\max\begin{cases} W(u_1,0,b,0,1)+W(u_2,0,0,c,0), \\ W(u_1,0,b,0,0)+W(u_2,0,0,c,1), \\ W(u_1,0,b,1,1)+W(u_2,0,1,c,0)-w(u_1,u_2), \\ W(u_1,0,b,1,0)+W(u_2,0,1,c,1)-w(u_1,u_2) \end{cases} \quad (4)$$

$$W(PF_{k+1},0,b,c,0)=\max\begin{cases} W(PF_k,0,b,0,0)+W(u_{k+1},0,0,c,0), \\ W(PF_k,0,b,1,0)+W(u_{k+1},0,1,c,0)-w(u_k,u_{k+1}), \\ W(PF_k,1,b,0,0)+W(u_{k+1},1,0,c,0), \\ W(PF_k,1,b,1,0)+W(u_{k+1},1,1,c,0)-w(u_k,u_{k+1}) \end{cases} \quad (5)$$

$$W(PF_{k+1},0,b,c,1)=\max\begin{cases} W(PF_k,0,b,0,1)+W(u_{k+1},0,0,c,0), \\ W(PF_k,0,b,0,0)+W(u_{k+1},0,0,c,1), \\ W(PF_k,0,b,1,1)+W(u_{k+1},0,1,c,0)-w(u_k,u_{k+1}), \\ W(PF_k,0,b,1,0)+W(u_{k+1},0,1,c,1)-w(u_k,u_{k+1}) \\ W(PF_k,1,b,0,1)+W(u_{k+1},1,0,c,0), \\ W(PF_k,1,b,0,0)+W(u_{k+1},1,0,c,1), \\ W(PF_k,1,b,1,1)+W(u_{k+1},1,1,c,0)-w(u_k,u_{k+1}), \\ W(PF_k,1,b,1,0)+W(u_{k+1},1,1,c,1)-w(u_k,u_{k+1}) \end{cases} \quad (6)$$

$$W(PF_{k+1},1,b,c,0)=\max\begin{cases} W(PF_k,1,b,0,0)+W(u_{k+1},0,0,c,0), \\ W(PF_k,1,b,1,0)+W(u_{k+1},0,1,c,0)-w(u_k,u_{k+1}) \end{cases} \quad (7)$$

$$W(PF_{k+1},1,b,c,1)=\max\begin{cases} W(PF_k,1,b,0,1)+W(u_{k+1},0,0,c,0), \\ W(PF_k,1,b,0,0)+W(u_{k+1},0,0,c,1), \\ W(PF_k,1,b,1,1)+W(u_{k+1},0,1,c,0)-w(u_k,u_{k+1}), \\ W(PF_k,1,b,1,0)+W(u_{k+1},0,1,c,1)-w(u_k,u_{k+1}) \end{cases} \quad (8)$$

$W(u,0,0,0,0)=NULL$， $W(u,0,0,0,1)=0$；
$W(u,0,0,1,0)=w(k)$, $W(u,0,0,1,1)=NULL$
$W(u,0,1,0,0)=w(j)$, $W(u,0,1,0,1)=NULL$
$W(u,1,0,0,0)= w(i)$, $W(u,1,0,0,1)=NULL$
$W(u,0,1,1,0)= W(u,0,1,1,1)=NULL$,
$W(u,1,0,1,0)= W(u,1,0,1,1) =NULL$
$W(u,1,1,0,0)= W(u,1,1,0,1)=NULL$,
$W(u,1,1,1,0)= W(u,1,1,1,1) =NULL$

Cost:=max$\{W(v_w,1,0,0,1), W(v_w,0,1,0,1), W(v_w,0,0,1,1), W(v_w,0,0,0,0)\}$

(3) Let H be a Halin graph and $H=(V,E)$ and $V'\subseteq V$. We want to find the maximum-weighted matching covering all the vertices in V'. $\forall v\in V'$, the vertex v must be covered by M. So deleting all the situations that v is unsaturated in the formulas (1)-(6).

(4) Let H be a Halin graph and $H=(V,E)$ and $E'\subseteq E$. We want to find the maximum-weighted matching containing all the edges in E'. $\forall e\in E'$, the edge e must be contained in M. For each edge $uv\in E'$, let let $S=\delta(u)\cup\delta(v)$. $\forall e\in S$, change $w(e)$ to $-n*d_{max}$ (d_{max}=max$\{w(e)|e\in E(H)\}$, $n=|V(H)|$).

VII CONCLUSION

In this paper, we have proposed a linear time algorithm to solve maximum-weighted matching problem in Halin graphs. Our method can be extended to search maximum-weighted matching in some other conditions.

Many hard problems become linear solvable for special classes of graphs. For example, the minimum TSP which is NP-hard is solved in O(|V|), in [2]. It is well-known that many optimization problems on graphs can also be formulated as integer programs. Hence, if the linear system, which is associated with the integer program, can use the divide-and-conquer strategy, then a more efficient algorithm will be obtained. For instance, TSP in graph with 3-edge cutsets has been studied in [3]. If more special structure, which makes the linear system dividable, is found out, then many hard problems will become much easier on the graphs with that structure.

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