Comparative Study between Linear and Graphical Methods in Solving Optimization Problems

Mona M. Abd El-Kareem

***Abstract*— The main target of this paper is to establish a comparative study between the performance of linear and graphical methods in solving optimization problems. Thus, in this paper, different case studies were presented to illustrate the computations of the minimax and maximin values of a game. Backward induction is a technique used for solving a game of perfect information. Firstly, it considers the moves of the last in the game, and determines the best move for the player in each case. Then, taking these as given future actions, it proceeds backwards in time, again determining the best move for the respective player, until the beginning of the game is reached. A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.**

***Index Terms*— Game theory, Optimization problems.**

I. INTRODUCTION

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms, or any combination of these. The concepts of game theory provide a language to formulate structure, analyze, and understand strategic scenarios. The object of study in game theory is the game, which is a formal model of an interactive situation. It typically involves several players; a game with only one player is usually called a decision problem. The formal definition lays out the players, their preferences, their information, the strategic actions available to them, and how these influence the outcome. Games can be described formally at various levels of detail. A coalitional (or Cooperative) game is a high-level description, specifying only what payoffs each potential group, or coalition, can obtain by the cooperation of its members. What is not made explicit is the process by which the coalition forms. As an example, the players may be several parties in parliament. Each party has a different strength, based upon the number of seats occupied by party members. The game describes which coalitions of parties can form a majority, but does not delineate, for example, the negotiation process through which an agreement to vote en bloc is achieved. Cooperative game theory investigates such coalitional games with respect to the relative amounts of power held by various players, or how a successful coalition should divide its proceeds. This is most naturally applied to situations arising in political science or international relations, where concepts like power are most important. For example, Nash proposed a solution for the division of gains from agreement in a bargaining problem which depends solely on the relative strengths of the two parties' bargaining position [5]. The amount of power a side has is determined by the usually inefficient outcome that results when negotiations break down. Nash's model fits within the cooperative framework in that it does not delineate a specific timeline of offers and counteroffers, but rather focuses solely on the outcome of the bargaining process. In contrast, noncooperative game theory is concerned with the analysis of strategic choices. The paradigm of noncooperative game theory is that the details of the ordering and timing of players' choices are crucial to determining the outcome of a game. In contrast to Nash's cooperative model, a noncooperative model of bargaining would posit a specific process in which it is prespecified who gets to make an offer at a given time. The term "noncooperative" means this branch of game theory explicitly models the process of players making choices out of their own interest. Cooperation can, and often does, arise in noncooperative models of games, when players find it in their own best interests.

Mona M. Abd El-Kareem is with the Mathematics and Computer Science Department, Faculty of Science, Suez Canal University, Ismailia 41522 ,Egypt (Phone +20-129744759, Fax +20-643206267, e-mail: monaia_21@yahoo.com) .

(Advance online publication: 1 February 2010)

II. OPTIMAL SOLUTION OF TWO-PERSON ZERO-SUM GAMES

The selection of a criterion for solving a decision problem depends largely on the available information. Games represent the ultimate case of lack of information in which intelligent opponents are working in a conflicting environment. The result is that a very conservative criterion, called the minimax-maximin criterion, is usually proposed for solving two-person zero-sum games. To accommodate the fact that each opponent is working against the other's interest, the minimax criterion selects each player's (mixed or pure) strategy, which yields the best of the worst possible outcomes. An optimal solution is said to be reached if neither player finds it beneficial to alter his strategy. In this case, the game matrix is said to be stable or in a state of equilibrium. The game matrix is usually expressed in terms of the payoff to player A (whose strategies are represented by the rows). The criterion calls for A to select the strategy (mixed or pure) that maximizes his minimum gains, the minimum being taken over all the strategies of player B. By the same reasoning, player B selects the strategy that minimizes his maximum losses. Again, the maximum is taken over all A's strategies [4]. The following case study illustrates the computations of the minimax and maximin values of a game.

Case study 1

Consider the following payoff matrix, which represents player A's gain. The computations of the minimax and maximin values are shown on the matrix.

$$\begin{array}{lcccccl} & & \text{Player B} & & & & \text{Row Minimum} \\ & \left[\begin{array}{c}8\\6\\7\end{array}\right. & \begin{array}{c}2\\5\\3\end{array} & \begin{array}{c}9\\7\\-4\end{array} & \left.\begin{array}{c}5\\18\\10\end{array}\right] & & \begin{array}{l}2\\ \mathbf{5} \leftarrow \text{Maximin}\\ -4\end{array} \\ \text{Player A} & & & & & & \\ \text{Column Maximum} & 8 & \mathbf{5} & 9 & 18 & & \\ & & \uparrow \text{Minimax} & & & & \end{array}$$

When player A plays his first strategy, he may gain 8, 2, 9, or 5 depending on player B's selected strategy. He can guarantee, however, a gain of at least min {8,2,9,5} =2 regardless of B's selected strategy. Similarly, if A plays his second strategy, he is guaranteed an income of at least min {6,5,7,18} = 5, and if he plays his third strategy, he is guaranteed an income of at least min {7,3,-4,10} = -4. Thus the minimum value in each row represents the minimum gain guaranteed A if he plays his pure strategies. These are indicated in the matrix above by "row minimum ". Now, player A, by selecting his second strategy, is maximizing his minimum gain. This gain is given by max {2,5,-4}=5, player A's selection is called the maximin strategy and his corresponding gain is called the maximin (or lower) value of the game. Player B, on the other hand, wants to minimize his losses. He realizes that, if he plays his first pure strategy, he can lose no more than max {8,6,7}=8 regardless of A's selections. A similar argument can also be applied to the three remaining strategies. The corresponding results are thus indicated in the matrix above by "column maximum" player B will then select the strategy that minimizes his maximum losses. This is given by the second strategy and his corresponding loss is given by min {8,5,9,18}=5 player B's selection is called minmax (or upper) value of the game. In the case where the equality holds, that is, minimax value = maximin value, the corresponding pure strategies are called "optimal" strategies and the game is said to have a saddle point. The value of the game, given by the common entry of the optimal pure strategies, is equal to the maximin and the minimax values. "Optimality" here signifies that neither player is tempted to change his strategy, since his opponent can counteract by selecting another strategy yielding less attractive payoff. In general, the value of the game must satisfy the inequality

Maximin (lower) value ≤ value of the game ≤ minimax (upper) value

In the example above, maximin value = minimax value = 5. This implies that the game has a saddle point. The value of the game is thus equal to 5.

III. MIXED STRATEGIES

If no saddle point is found in a game there is no single safest strategy for each player. In this case a mixture of strategies is used. The opponent cannot discover the strategy if, instead of a player using a single strategy, he chooses a probability distribution over the set of strategies, a situation which combines optimization and probability. A mixed strategy for X is a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Where xi, the probability of selecting the i_{th} strategy, satisfy xi ≥ 0, i=1… n and

$$\sum_{i=1}^{n} x_i = 1$$

A mixed strategy for Y is a vector y, which is similarly defined. Let A be the payoff matrix and let X' be the transpose of X. Then the payoff to X from strategy X is easily shown to be $X'AY$ [1,3]. The failure of the minimax-maximin (pure) strategies, in general, to give an optimal solution to the game has led to the idea of using mixed strategies. Each player, instead of selecting a pure strategy only, may play all his strategies according to a predetermined set of probabilities. Let $x_1, x_2 \ldots x_m$ and $y_1, y_2, \ldots, y_n$ be the row and column probabilities by which A and B, respectively, select their pure strategies. Then

$$\sum_{i=1}^{x} x_i = \sum_{j=1}^{x} y_j = 1,$$

$x_i, y_j \geq 0$ for all i and j. Thus, if a_{ij} represents the (i,j) entry of the game matrix , x_i and y_i will appear as in the following matrix

Player B

		y_1	y_2	...	y_n
Player A	x_1	a_{11}	a_{12}	...	a_{1n}
	x_2	a_{21}	a_{22}	...	a_{2n}
		.			
		.			
		.			
	x_m	a_{m1}	a_{m2}	...	a_{mn}

The payoff matrix

The solution of the mixed strategy problem is based also on the minimax-criterion. The only difference is that A selects x_i that maximize the smallest expected payoff in a column, whereas B selects y_j that minimize the largest expected payoff in a row. Mathematically, the minimax criterion for a mixed strategy case is given as follow. Player A selects x_i($x_i \geq 0$, $\sum_{i=1}^{m} x_i = 1$) that will yield.

$$\underset{x_i}{\text{Max}}\{\min\,(\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \ldots, \sum_{i=1}^{m} a_{in}x_i)\} \quad (1)$$

And player B selects y_j($y_j \geq 0$, $\sum_{j=1}^{n} y_j = 1$) that will yield

$$\underset{y_j}{\text{Max}}\{\max\,(\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \ldots, \sum_{j=1}^{n} a_{mj}y_j)\} \quad (2)$$

These values are referred to as the maximin and minimax expected payoffs, respectively. As in the pure strategies case, the relationship

Minimax expected payoff ≥ maximin expected payoff

When x_i and y_j correspond to the optimal solution. The equality holds and the resulting values become equal to the (optimal) expected value of the game. There are several methods for solving two- person zero-sum games for the optimal values of x_i and y_i , [1,3,4].

A. OPTIMAL STRATEGY IN TWO-PERSON ZERO-SUM GAMES WITH 2X2 MATRICES

Let a two-person zero-sum 2 by 2 game be represented as

X \ Y	y_1	y_2
x_1	a_{11}	a_{12}
x_2	a_{21}	a_{22}

To solve this game begins by looking for a saddle point solution. If there is none, then to obtain X's optimum mixed strategies the second column of the payoff matrix is subtracted from the first. The resulting column is

$$\begin{bmatrix} a_{11} - a_{12} \\ a_{21} - a_{22} \end{bmatrix}$$

Then X's optimum mixed strategy is $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,where

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

and $x_2 = \dfrac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$ (3)

Where $a_{11}+a_{22}-a_{12}-a_{21} \neq 0$, and $x_1+x_2=1$

For Y's mixed strategy the second row of the payoff matrix is subtracted from the first.

This gives $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where

$$y_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

$$\text{and } y_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad (4)$$

Where $a_{11}+a_{22}-a_{12}-a_{21} \neq 0$, and $y_1+y_2=1$

The expected payoff E of the game corresponding to these optimal strategies is

$$E = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad (5)$$

Case study 2

For the game matrix $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ determine the optimal strategies for player A and player B, and find the value of the game.

Using formula (3), we have

$$x_1 = \frac{3-(-2)}{1+3-(-1)-(-2)} = \frac{5}{7} \quad \text{and}$$

$$x_2 = \frac{1-(-1)}{1+3-(-1)-(-2)} = \frac{2}{7}$$

Thus, player A's optimal strategy is to select row 1with probability 5/7 and row 2 with probability 2/7. Also from formula (4), player B's optimal strategy is

$$y_1 = \frac{3-(-1)}{1+3-(-1)-(-2)} = \frac{4}{7} \quad \text{and}$$

$$y_2 = \frac{1-(-2)}{1+3-(-1)-(-2)} = \frac{3}{7}$$

Player B's optimal strategy is to select column 1 with probability 4/7 and column 2 with probability 3/7. The value of the game is

$$E = \frac{(1)(3)-(-1)(-2)}{1+3-(-1)-(-2)} = \frac{1}{7}$$

B. OPTIMAL STRATEGY IN OTHER TWO-PERSON ZERO-SUM GAMES USING GRAPHICAL METHODS

Graphical solutions are only applicable to games in which at least one of the players has two strategies only, [1,4]. Consider the following (2x2) game

$$\begin{array}{lcc} & & \text{Player B} \\ & & y_1 \;\; y_2 \;\; \cdots \;\; y_n \\ \text{Player A} & \begin{array}{r} x_1 \\ x_2 = 1 - x_1 \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \end{array}$$

It is assumed that the game does not have a saddle point. Since A has two strategies, it follows that $x_2=1-x_1$; $x_1 \geq 0$, $x_2 \geq 0$. His expected payoffs corresponding to the pure strategies of B are given by

Table1 expected payoffs corresponding to the pure strategies of B

B's pure strategy	A's expected payoff
1	$(a_{11}-a_{21})x_1+a_{21}$
2	$(a_{12}-a_{22})x_1+a_{22}$
.	.
.	.
.	.
n	$(a_{1n}-a_{2n})x_1+a_{2n}$

This shows that A's average payoff varies linearly with x_1. According to the minimax criterion for mixed-strategy games, player A should select the value of x_1 that maximizes his minimum expected payoffs. This may be done by plotting the straight lines above as functions of x_1 [4].

Case Study 3

Consider the following (2 x 4) game.

$$\begin{array}{lcc} & & \text{Player B} \\ & & 1 \quad 2 \quad 3 \quad 4 \\ \text{Player A} & \begin{array}{r} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

This game does not have a saddle point. Thus A's expected payoffs corresponding to B's pure strategies are given by:

Table 2 A's expected payoffs corresponding to B's pure strategies

B's pure strategy	A's expected payoff
1	$-2x_1+4$
2	$-x_1+3$
3	x_1+2
4	$-7x_1+6$

These four straight lines are then plotted as function of x_1 as shown in figure (1). The maximin occurs at $x_1^*=1/2$. This is the point of intersection of any two of the lines 2,3, and 4 . Consequently, A's optimal strategy is ($x_1^*=1/2, x_2^*=1/2$)and the value of the game is obtained by substituting for x_1 in the equation of any of the lines passing through the maximin point. This gives

$$V^* = \begin{cases} -1/2+3=5/2 \\ 1/2+2=5/2 \\ -7(1/2)+6=5/2 \end{cases}$$

To determine B's optimal strategies, it should be noticed that three lines pass through the maximin point. This is an indication that B can mix all three strategies. Any two lines having opposite signs for their slopes define an alternative optimum solution. Thus, of the three combinations (2,3) , (2,4), and (3,4), the combination (2,4) must be excluded a nonoptimal.

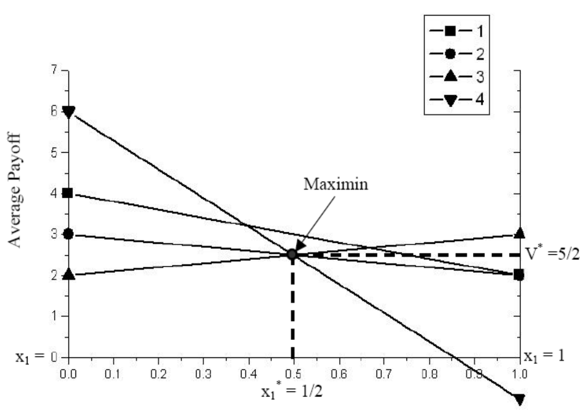

Fig.1 graphical solution of the (2*4) game

In the first combination (2,3) implies that $y_1^*=y_4^*=0$. Consequently $y_3=1-y_2$ and B's average payoffs corresponding to A's pure strategies are given by :

Table 3 B's average payoffs corresponding to A's pure strategies

B's pure strategy	A's expected payoff
1	$-y_1+4$
2	y_1+2

Thus y_2^* (corresponding to the minmax point) can be determined from $-y_2^*+3 = y_2^*+2$ this gives $y_2^*=1/2$. Notice that by substituting $y_2^* = 1/2$ in B's expected payoffs given above, the minmax value is 5/2 , which equals the value of the game V*,as should be expected .

C. Optimal Strategy In Other Two-Person Zero-Sum Games Using Linear Programming

Game theory bears a strong relationship to linear programming since every finite two-person zero-sum game can be expressed as a linear program and, conversely every linear program can be represented as a game. Linear programming problems must have three elements: objective function, constraints and nonnegativity conditions. These three elements also exist in a two-person zero-sum game. A two-person zero-sum game can be converted into an equivalent linear programming problem, in a two-person zero-sum game the objective of one player is to maximize his expected gain while the other player tries to minimize his expected loss. In other words the aim of the players in game theory is either to maximize or minimize gains. In short, the objective of the game is a linear function of the decision variables, [2,4]. In linear programming the players wish to optimize their gain subject to given constraints and the variables must be always non-negative. When both players select the optimal strategies in a two-person zero-sum game, one player's highest expected gain is equal to the other player's lowest expected loss, [2]. Therefore the value of the maximization problem is exactly the same as that of the minimization problem. This is the same as the primal/dual relationship in linear programming. The optimal solution to a game problem may be selected by formulating it as a linear programming problem, [2,3,4]. This section illustrates the solution of game problems by linear programming. It is especially useful for games with large matrices. Player A's optimum mixed strategies satisfy

$$\max_{x_i}\{\min\left(\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \dots, \sum_{i=1}^{m} a_{in}x_i\right)\} \quad (6)$$

Subject to the constraints xi ≥0, i=1,…,m and $\sum_{i=1}^{m} x_i = 1$. This problem can be put in the linear programming form as follows.

Let

$$V = \min\left(\sum_{i=1}^{m} a_{i1}x_i \sum_{i=1}^{m} a_{i2}x_i, \ldots, \sum_{i=1}^{m} a_{in}x_i\right) \quad (7)$$

Then the problem becomes maximize Z=V Subject to

$$\sum_{i=1}^{m} a_{i1}\, x_i \geq V \,, j = 1,2, \ldots, n$$

xi ≥0 , i=1,…,m and $\sum_{i=1}^{m} x_i = 1$

V represents the value of the game in this case.

If player B wants to adopt B1, then A's strategy must be such that

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + \cdots + a_{m1}x_m \geq V$$

Similarly if player B uses B2, then to guarantee V, A must have

$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + \cdots + a_{m2}x_m \geq V$$

A similar condition holds for any strategy B may play. Hence the linear programming problem for A is :Maximize V Subject to

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + \cdots + a_{m1}x_m \geq V$$
$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + \cdots + a_{m2}x_m \geq V$$

.

. (8)

.

$$a_{1n}x_1 + a_{2n}x_2 + a_{3n}x_3 + \cdots + a_{mn}x_m \geq V$$
$$x_1 + x_2 + x_3 + \cdots + x_m = 1$$

all $x_i \geq 0$

The solution of this problem gives the equilibrium mixed strategy $(x_1,x_2,\ldots,x_m)$ for player A and the value of the game V. From formula (5.4) assuming that V>0, the constraints of the linear program becomes

$$a_{11}\frac{x_1}{V} + a_{21}\frac{x_2}{V} + \cdots + a_{m1}\frac{x_m}{V} \geq 1$$

$$a_{12}\frac{x_1}{V} + a_{22}\frac{x_2}{V} + \cdots + a_{m2}\frac{x_m}{V} \geq 1$$

.

. (9)

.

$$a_{1n}\frac{x_1}{V} + a_{2n}\frac{x_2}{V} + \cdots + a_{mn}\frac{x_m}{V} \geq 1$$
$$\frac{x_1}{V} + \frac{x_2}{V} + \cdots + \frac{x_m}{V} = \frac{1}{V}$$

all xi ≥ 0

Let X_i =, i=1,2,…,m since
Max V = min $\frac{1}{V}$ = min$\{X_1+ X_2+\ldots+X_m\}$

The problem becomes

Minimize Z= $X_1+ X_2+\ldots+X_m$

Subject to

$a_{11}X_1+a_{21}X_2+\ldots+a_{m1}X_m \geq 1$
$a_{12}X_1+a_{22}X_2+\ldots+a_{m2}X_m \geq 1$

.

. (10)

.

$a_{1n}X_1+a_{2n}X_2+\ldots+a_{mn}X_m \geq 1$

all Xi ≥ 0 for i =1,2,…,m

where Z= $\frac{1}{V}$, X_i=, i=1,2,…,m

Player B's problem is given by

$$\min_{y_j}\{(\max\left(\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \ldots, \sum_{j=1}^{n} a_{mj}y_j\right)\} \quad (11)$$

Subject to the constraints y_j ≥0 , j=1,…,n and $\sum_{j=1}^{n} y_j = 1$

This can also be expressed as a linear program as follows

Maximize W= $Y_1+ Y_2+\ldots+Y_n$

Subject to

$$
\begin{aligned}
&a_{11}Y_1+a_{12}Y_2+\ldots+a_{1n}Y_n \le 1\\
&a_{21}Y_1+a_{22}Y_2+\ldots+a_{2n}Y_n \le 1\\
&.\\
&. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12)\\
&.\\
&a_{m1}Y_1+a_{m2}Y_2+\ldots+a_{mn}Y_n \le 1
\end{aligned}
$$

all $Y_i \ge 0$ for $i=1,2,\ldots,n$

Where $W=\frac{1}{V}$, $Y_j=\frac{y_j}{V}$, $j=1,2\ldots n$

Noticing that B's problem is actually the dual of A's problem, thus the optimal solution of one problem automatically yields the optimal solution to the other. Player B's problem can be solved by the regular simplex method, and player A's problem is solved by the dual simplex method, [2, 3, 4].

Case study 4

Consider the following (3x3) game

$$
\text{Player A}\quad
\begin{array}{c} \\ 1\\ 2\\ 3 \end{array}
\overset{\text{Player B}}{\begin{array}{ccc} 1 & 2 & 3 \\ \hline \end{array}}
$$

$$
\text{Player A} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{bmatrix}
$$

B's linear programming is thus given as

Maximize $W= Y_1+ Y_2+Y_3$

Subject to

$8Y_1+4Y_2+2Y_3 \le 1$

$2Y_1+8Y_2+4Y_3 \le 1$

$Y_1+2Y_2+8Y_3 \le 1$

$Y_1,Y_2,Y_3 \ge 0$

The optimal strategy for B is obtained from the solution to the problem above

$$W=\frac{45}{196}, Y_1=\frac{1}{14}, Y_2=\frac{11}{196}, and\ Y_3=\frac{5}{49}$$

$$V=\frac{1}{W}, y_1=\frac{Y_1}{W}, y_2=\frac{Y_2}{W}, and\ y_3=\frac{Y_3}{W}$$

Then $V=\frac{196}{45}$, $y_1=\frac{14}{45}$, $y_2=\frac{11}{45}$, and $y_3=\frac{20}{45}$

Hence, the optimal strategies for A are obtained from the dual solution to the problem above. This is given by

$$Z=W=\frac{45}{196}, X_1=\frac{5}{49}, X_2=\frac{11}{196}, and\ X_3=\frac{1}{14}$$

$$V=\frac{1}{Z}, x_1=\frac{X_1}{Z}, x_2=\frac{X_2}{Z}, and\ x_3=\frac{X_3}{Z}$$

Then $V=\frac{196}{45}, x_1=\frac{20}{45}, x_2=\frac{11}{45}, and\ x_3=\frac{14}{45}$

IV. CONCLUSION

This paper is interested in solving some linear programming problems by solving systems of differential equations using game theory. First of all, the linear programming problem must be a classical constraints problem which means that a maximization/minimization problem should be described in the canonical form with all the coefficients (from objective function, constraints matrix and right sides) positive. We notice that in linear programming the players wish to optimize their gain subject to given constraints and the variables must be always non-negative. When both players select the optimal strategies in a two-person zero-sum game, one player's highest expected gain is equal to the other player's lowest expected loss. Therefore the value of the maximization problem is exactly the same as that of the minimization problem

REFERENCES

[1] Mizrahi, A. , and M. Sullivan , " Finite Mathematics With Application " , John Wiley & sons , 1988.

[2] Özkan, B. , H. V. Akcaöz , " Game Theory And Its Application To Field Crops In Antalya Province " , Turk J Agric for 26 , pp. 303 – 309, 2002.

[3] Saaty, T. L. , " Lectures On Methods Of Optimization " , Memo. No. 391, pp. 183 – 216 , INP, Cairo,1964.

[4] Taha, H. A. , " Operations Research An Introduction " , Macmillan publishing co., Inc. United States , 1982.

[5] Wu, M. D. , Y. H. Liao , and C. T. Sun , " Network Tournament Pedagogical Approach Involving Game Playing In Artificial Intelligence", Journal of information science and engineering , Vol. 19 , pp. 589 – 603 , 2003.

Mona M. Abd El-Kareem received the Ms.c Degree in 2009 from the Mathematics Department, Faculty of Science, Suez Canal University, Egypt. From 2008until now, she was an Instructor in University Education Development Center, Suez Canal University, Ismailia, Egypt. Currently, Her research interests include: Computional intelligent techniques, Genetic Algorithms, Intelligent system, image processing and various simulations techniques .She published 10 articles in international refereed journals and conferences also she joined a lot of international journal as a reviwers of articles related to the above topics.